# Photon-number distribution of two-mode squeezed thermal states by entangled state representation*

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Using the entangled state representation, we convert a two-mode squeezed number state to a Hermite polynomial excited squeezed vacuum state. We first analytically derive the photon number distribution of the two-mode squeezed thermal states. It is found that it is a Jacobi polynomial; a remarkable result. This result can be directly applied to obtaining the photon number distribution of non-Gaussian states generated by subtracting from (adding to) two-mode squeezed thermal states.

**Keywords:** entangled state representation, Hermite polynomial excited state, squeezed thermal states, photon-number distribution

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## 1. Introduction

Among many topics in quantum optics, the squeezed state and the squeezing operators are of special importance. Since 1970, the squeezed states have been an important topic in quantum optics due to their interesting nonclassical behavior and their applications in quantum detection and quantum communication.[1,2] Usually, squeezed states are created experimentally through nonlinear optical processes such as four-wave mixing[3] and parametric down conversion.[4] In addition, squeezed states can be engineered in cavities.[5] Theoretically, squeezed states can be constructed by acting with the appropriate squeezing operator (one-mode or two-mode) on the vacuum state. On the other hand, it is often the case that we do not have enough information to specify completely the state of a system. In such a situation, the system can only be described by mixed states, such as thermal states. In this paper, we introduce a kind of squeezed state, generated by applying the two-mode squeezing operator onto two-mode thermal states. Then we analytically derive the photon-number distribution (PND), which is a key characteristic of every optical field and can be directly used to obtain the PNDs of non-Gaussian states generated by subtracting from (adding to) two-mode squeezed thermal states. We shall employ entangled state representation to realize our purpose.

## 2. Introducing two-mode squeezed thermal states

First, let us introduce the two-mode squeezed thermal states (TMSTSs), whose density operator is given by

$$\rho = S(r)\rho_{\rm th1} \otimes \rho_{\rm th2} S^{\dagger}(r), \tag{1}$$

where $S(r) = \exp[r(a_1 a_2 - a_1^{\dagger} a_2^{\dagger})]$ is the two-mode squeezing operator[1,2] with squeezing parameter $r$, and $\rho_{\rm th1,2}$ is a density operator of thermal state,

$$\rho_{{\rm th}j} = \sum_{n=0}^{\infty} \frac{\bar{n}_j^n}{(\bar{n}_j + 1)^{n+1}} |n\rangle \langle n|, \tag{2}$$

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where $\bar{n}_j$ $(j = 1, 2)$ being the average photon number of thermal state $\rho_{\rm th}$. For simplicity, we assume the average photon number of $\rho_{{\rm th}j}$ to be identical $\bar{n}_1 = \bar{n}_2 = \bar{n}$. In addition, the $P$-representation of density operator $\rho_{\rm th}$ can be expanded as[6]

$$\rho_{\rm th} = \frac{1}{\bar{n}} \int \frac{{\rm d}^2\alpha}{\pi} \exp\left(-\frac{1}{\bar{n}} |\alpha|^2\right) |\alpha\rangle \langle\alpha| , \qquad (3)$$

which is useful for later calculation and here $|\alpha\rangle$ is the coherent state.

## 3. Squeezed number state as a two-variable Hermite polynomial excited state

Theoretically, the two-mode squeezed number state can be obtained by applying the two-mode squeezing operator $S(r)$ on the number state $|m, n\rangle$, i.e., $S|m, n\rangle$. It is interesting to notice that the two-mode squeezed operator has a natural expression in terms of entangled state representation $\langle\xi|$,[7]

$$S(r) = \int \frac{{\rm d}^2\xi}{\pi\mu} \left|\frac{\xi}{\mu}\right\rangle \langle\xi| , \quad \mu = {\rm e}^r, \qquad (4)$$

where $|\xi\rangle$ is the common eigenstate of $Q_1+Q_2$ and $P_1-P_2$, where $Q_i$ and $P_i$ are coordinate and momentum operators, $(Q_1 + Q_2)|\xi\rangle = \sqrt{2}\xi_1 |\xi\rangle$, $(P_1 - P_2)|\xi\rangle = \sqrt{2}\xi_2 |\xi\rangle$, and

$$|\xi\rangle = \exp\left[-\frac{1}{2}|\xi|^2 + a_1^\dagger \xi + a_2^\dagger \xi^* - a_1^\dagger a_2^\dagger\right] |00\rangle . \qquad (5)$$

Here, $\xi = \xi_1 + {\rm i}\xi_2$ and $|00\rangle$ is the two-mode vacuum state, $[a_i, a_j^\dagger] = \delta_{i,j}$, $i, j = 1, 2$, $a_j = (Q_j + {\rm i}P_j)/\sqrt{2}$. Thus we can use the $\langle\xi|$ representation to demonstrate a two-mode squeezed number state

$$S(r)|m, n\rangle = \int \frac{{\rm d}^2\xi}{\pi\mu} \left|\frac{\xi}{\mu}\right\rangle \langle\xi|\, m, n\rangle . \qquad (6)$$

On the other hand, according to the generating function of two-variable Hermite polynomials ${\rm H}_{m,n}$,[8,9]

$$\sum_{m,n=0}^{\infty} \frac{\tau^m t^n}{m!n!} {\rm H}_{m,n}(x, y) = \exp\left[-\tau t + \tau x + ty\right] , \qquad (7)$$

we can rewrite $\langle\xi|$ in Eq. (5) as

$$\langle\xi| = \langle 00| \, {\rm e}^{-|\xi|^2/2} \sum_{m,n=0}^{\infty} \frac{a_1^m a_2^n}{m!n!} {\rm H}^*_{m,n}(\xi, \xi^*), \qquad (8)$$

so the inner product between $\langle\xi|$ and the two-mode Fock state is

$$\langle\xi|\, m, n\rangle = {\rm e}^{-|\xi|^2/2} {\rm H}^*_{m,n}(\xi, \xi^*)/\sqrt{m!n!}. \qquad (9)$$

Substituting Eq. (9) into Eq. (6) yields

$$\begin{aligned} & S(r)|m, n\rangle \\ &= \frac{1}{\sqrt{m!n!}} \int \frac{{\rm d}^2\xi}{\pi\mu} \left|\frac{\xi}{\mu}\right\rangle {\rm e}^{-|\xi|^2/2} {\rm H}^*_{m,n}(\xi, \xi^*) \\ &= \frac{\operatorname{sech} r}{\sqrt{m!n!}} \frac{\partial^{m+n}}{\partial\tau^m \partial t^n} \exp\Big[t\tau \tanh r + \tau a_1^\dagger \operatorname{sech} r \\ &\quad + ta_2^\dagger \operatorname{sech} r - a_1^\dagger a_2^\dagger \tanh r\Big] |00\rangle\,|_{t=\tau=0} \\ &= \frac{(-\tanh r)^n}{\sqrt{m!n!}\cosh r} {\rm H}_{m,n}\left[\frac{a_1^\dagger}{\cosh r}, \frac{-a_2^\dagger}{\sinh r}\right] \\ &\quad \times {\rm e}^{-a_1^\dagger a_2^\dagger \tanh r} |00\rangle , \end{aligned} \qquad (10)$$

where we have used the following formula

$${\rm H}_{m,n}(x, y) = \frac{\partial^{m+n}}{\partial\tau^m \partial t^n} \exp\left[-t\tau + \tau x + ty\right]\Big|_{t=\tau=0} , \qquad (11)$$

and

$$\int \frac{{\rm d}^2 z}{\pi} {\rm e}^{\zeta|z|^2 + \xi z + \eta z^*} = -\frac{1}{\zeta} {\rm e}^{-\xi\eta/\zeta}, \quad {\rm Re}(\zeta) < 0. \qquad (12)$$

Equation (10) indicates that the two-mode squeezed number state is just a two-variable Hermite polynomial excited squeezed vacuum state.[10,11] Notice that $S(r)|00\rangle = \operatorname{sech} r \, {\rm e}^{-a_1^\dagger a_2^\dagger \tanh r} |00\rangle$, thus Eq. (10) can be read as

$$\begin{aligned} & S(r)|m, n\rangle \\ &= \frac{(-\tanh r)^n}{\sqrt{m!n!}} {\rm H}_{m,n}\left[\frac{a_1^\dagger}{\cosh r}, \frac{-a_2^\dagger}{\sinh r}\right] S(r)|00\rangle . \end{aligned} \qquad (13)$$

## 4. Photon-number distribution of TMSTSs

Next, we apply the characteristic of the two-mode squeezed number state in Eq. (10) to directly derive the photon-number distribution (PND) of TMSTSs. For the two-mode case described by density operator $\rho$, the PND is defined by $\mathscr{P}(m, n) = \langle m, n|\, \rho\, |m, n\rangle$. Thus the PND of TMSTSs is given by

$$\mathscr{P}(m, n) = \langle m, n|\, S(r)\rho_{\rm th1} \otimes \rho_{\rm th2} S^\dagger(r)\, |m, n\rangle . \qquad (14)$$

Using the fact in Eq. (10) and the $P$-representation of $\rho_{\rm th}$ (Eq. (3)) as well as noting $S^\dagger(r) = S(-r)$, Eq. (14) can be directly written as

$$\begin{aligned} \mathscr{P}(m, n) = \frac{1}{\bar{n}^2} \frac{(\tanh r)^{2n}}{m!n! \cosh^2 r} \int \frac{{\rm d}^2\alpha\, {\rm d}^2\beta}{\pi^2} \\ \times \exp\left[-\frac{\bar{n}+1}{\bar{n}} \left(|\alpha|^2 + |\beta|^2\right)\right] \end{aligned}$$

$$\times \mathrm{H}_{m,n}\left[\frac{\alpha}{\cosh r}, \frac{\beta}{\sinh r}\right] \mathrm{e}^{\alpha\beta\tanh r} \times \mathrm{H}_{m,n}\left[\frac{\alpha^*}{\cosh r}, \frac{\beta^*}{\sinh r}\right] \mathrm{e}^{\alpha^*\beta^*\tanh r}, \quad (15)$$

where $|\alpha,\beta\rangle = |\alpha\rangle \otimes |\beta\rangle$. Further using Eqs. (11) and (12), Eq. (15) becomes

$$\begin{aligned}
&\mathscr{P}(m,n) \\
&= \frac{\tanh^{2n} r \operatorname{sech}^2 r}{A\bar{n}\left(\bar{n}+1\right) m!n!} \frac{\partial^{m+n}}{\partial\tau^m \partial t^n} \frac{\partial^{m+n}}{\partial s^m \partial v^n} \\
&\quad \times \exp\left[Bs\tau - C\left(sv+\tau t\right) + Dvt\right]_{s=v=t=\tau=0} \\
&= \frac{m!n!B^m\left(D\tanh^2 r\right)^n}{\bar{n}^2 + (2\bar{n}+1)\cosh^2 r} \\
&\quad \times \sum_{l=0}^{\min[m,n]} \frac{\left[C^2/(BD)\right]^l}{\left(l!\right)^2 (n-l)!\,(m-l)!}, \qquad (16)
\end{aligned}$$

where we have set

$$\begin{aligned}
A &= \frac{\bar{n}^2 \operatorname{sech}^2 r + 2\bar{n} + 1}{\bar{n}\left(\bar{n}+1\right)}, \\
B &= \frac{\operatorname{sech}^2 r}{A}, \quad D = \frac{\operatorname{csch}^2 r}{A}, \\
C &= \frac{2\bar{n}+1}{\bar{n}\left(\bar{n}+1\right)A}. \qquad (17)
\end{aligned}$$

Without losing the generality, assuming $m \leqslant n$ and comparing Eq. (16) with the standard expression of Jacobi polynomials[12]

$$P_m^{(\alpha,\beta)}(x) = \frac{(x-1)^m}{2^m} \sum_{k=0}^{m} \binom{m+\alpha}{k} \binom{m+\beta}{m-k} \times \left(\frac{x+1}{x-1}\right)^k, \qquad (18)$$

we have

$$P_m^{(n-m,0)}(\chi) = \frac{(\chi-1)^m}{2^m} \sum_{l=0}^{m} \frac{m!n!\left[(\chi+1)/(\chi-1)\right]^l}{\left(l!\right)^2 (n-l)!\,(m-l)!}, \qquad (19)$$

where

$$\chi = \frac{\left((2\bar{n}+1)\sinh 2r\right)^2 + \left(2\bar{n}\left(\bar{n}+1\right)\right)^2}{\left((2\bar{n}+1)\sinh 2r\right)^2 - \left(2\bar{n}\left(\bar{n}+1\right)\right)^2}, \qquad (20)$$

thus Eq. (16) can be put into the following form:

$$\mathscr{P}(m,n) = \frac{2^m B^{n+m+1}}{\bar{n}\left(\bar{n}+1\right)} \left(\chi-1\right)^{-m} P_m^{(n-m,0)}(\chi), \quad (21)$$

which indicates that the PND of TMSTSs is the Jacobi polynomials of $\chi$. In particular, when $\bar{n} \to 0$ leading to $\chi \to 1$, the corresponding two-mode squeezed vacuum Eq. (16) reduces to

$$\begin{aligned}
&\mathscr{P}_{\bar{n}\to 0}(m,n) \\
&= \lim_{\bar{n}\to 0} \frac{m!n!}{\left[\bar{n}^2 + (2\bar{n}+1)\cosh^2 r\right]^{m+n+1}} \\
&\quad \times \sum_{l=0}^{\min[m,n]} \frac{(2\bar{n}+1)^{2l}\left[\bar{n}\left(\bar{n}+1\right)\right]^{m+n-2l} \sinh^{2l} 2r}{2^{2l}\left(l!\right)^2 (n-l)!\,(m-l)!} \\
&= \frac{\tanh^{2m} r}{\cosh^2 r}\delta_{m,n}, \qquad (22)
\end{aligned}$$

which is just the PND of two-mode squeezed vacuum state.[13] On the other hand, when $r \to 0$, corresponding to the case of two-mode thermal state, leading to $\chi \to -1$, $A \to (\bar{n}+1)/\bar{n}$, $B \to \bar{n}/(\bar{n}+1)$, and noting $P_m^{(n-m,0)}(-1) = (-1)^m$, thus Eq. (21) becomes

$$\mathscr{P}_{r\to 0}(m,n) = \frac{\bar{n}^n}{\left(\bar{n}+1\right)^{n+1}} \frac{\bar{n}^m}{\left(\bar{n}+1\right)^{m+1}}, \qquad (23)$$

which is just the product of PNDs of two thermal fields, as expected. Here we should point out that this result can be applied directly to calculate the PND of some non-Gaussian states generated by subtracting from (adding to) two-mode squeezed thermal states, such as $a_1^{\dagger q} a_2^{\dagger p} \rho a_1^q a_2^p$, $a_1^q a_2^p \rho a_1^{\dagger q} a_2^{\dagger p}$, and $a_1^m a_2^{\dagger n} \rho_s a_2^n a_1^{\dagger m}$, where $q, p$ are two non-negative integers. For example, for photon-subtracted squeezed thermal states, $\bar{\rho} = N^{-1} a_1^q a_2^p \rho a_1^{\dagger q} a_2^{\dagger p}$, where $N$ is the normalization factor, by noting $a^{\dagger m}|n\rangle = \sqrt{(m+n)!/n!}\,|m+n\rangle$, the PND $\bar{\mathscr{P}}(m,n)$ of $\bar{\rho}$ is

$$\begin{aligned}
\bar{\mathscr{P}}(m,n) &= \frac{(m+q)!(n+p)!}{m!n!N} \\
&\quad \times \langle m+q, n+p|\,\rho\,|m+q, n+p\rangle \\
&= \frac{2^{m+q} B^{n+p+m+q+1}}{\bar{n}\left(\bar{n}+1\right)} \frac{(m+q)!(n+p)!}{m!n!N} \\
&\quad \times \left(\chi-1\right)^{-(m+q)} P_{m+q}^{(n+p-m-q,0)}(\chi). \qquad (24)
\end{aligned}$$

## 5. Conclusion

In summary, using the natural representation of a two-mode squeezing operator in terms of the entangled state, we reveal that the two-mode squeezed number state can be considered as a Hermite polynomial excited squeezed vacuum state. Then we first analytically derive the photon number distribution of two-mode squeezed thermal states. It is found that it is a Jacobi polynomial; a remarkable result. This result can be directly applied to obtain photon distribution of non-Gaussian states generated by subtracting from (adding to) two-mode squeezed thermal states. For more applications of entangled state representation, one can refer to Refs. [14] and [15].

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