A generalized Weyl–Wigner quantization scheme unifying *P–Q* and *Q–P* ordering and Weyl ordering of operators*

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By extending the usual Wigner operator to the s-parameterized one as

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}y\,\mathrm{d}u \exp\left[\mathrm{i}u\,(q-Q)+\mathrm{i}y\,(p-P)+\mathrm{i}\frac{s}{2}yu\right]$$

with s being a real parameter, we propose a generalized Weyl quantization scheme which accompanies a new generalized s-parameterized ordering rule. This rule recovers P–Q ordering, Q–P ordering, and Weyl ordering of operators in $s=1,-1,0$ respectively. Hence it differs from the Cahill–Glaubers' ordering rule which unifies normal ordering, anti-normal ordering, and Weyl ordering. We also show that in this scheme the s-parameter plays the role of correlation between two quadratures Q and P. The formula that can rearrange a given operator into its new s-parameterized ordering is presented.

Keywords: generalized Wigner operator, generalized operator ordering rule, bivariate normal distribution

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1. Introduction

Since the birth of quantum mechanics the relation between operators and their classical correspondence function has been a fundamental topic, which is essential to quantum phase-space techniques, the latter has found many useful applications in quantum statistics, quantum optics, and quantum information.[1–4] The Wigner function[5] is a quasi-probability distribution in phase space which is defined as the expectation value of the Wigner operator $\Delta\,(p,q)$ in quantum mechanical states,

$$\Delta(p,q)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{d}u\,\mathrm{e}^{\mathrm{i}u(q-Q)+\mathrm{i}y(p-P)}, \quad (1)$$

where $[Q,P]=\mathrm{i}$ ($\hbar=1$). Q, P are the coordinate and the momentum operators, respectively, or named the two quadratures in the context of quantum optics. An equivalent expression of $\Delta(p,q)$ by using the coordinate eigenvector $|q\rangle$ of Q is

$$\Delta(p,q)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{e}^{-\mathrm{i}py}\left|q-\frac{1}{2}y\right\rangle\left\langle q+\frac{1}{2}y\right|, \quad (2)$$

where $|q\rangle$ expressed in Fock space as

$$|q\rangle=\pi^{-1/4}\exp\left(-\frac{q^2}{2}+\sqrt{2}qa^{\dagger}-\frac{a^{\dagger 2}}{2}\right)|0\rangle\,, \quad (3)$$

$|0\rangle$ is the Fock vacuum state, annihilated by the Bose operator a, $\left[a,a^{\dagger}\right]=1$. Performing this integration with the technique of integration within normal product of operators,[6] we have the normally ordered form[7]

$$\Delta(p,q)=\frac{1}{\pi}\colon \mathrm{e}^{-(q-Q)^2-(p-P)^2}\colon. \quad (4)$$

The Wigner operator $\Delta(p,q)$ is also an integration kernel in Weyl quantization scheme[8]

$$H=\int \mathrm{d}q\mathrm{d}p\Delta(p,q)h(p,q), \quad (5)$$

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where $h(p,q)$ is the Weyl classical correspondence of the operator H. This quantization scheme naturally defines the Weyl ordering rule of operators, for instance,

$$\int \mathrm{d}q\mathrm{d}p\Delta(p,q)q^m p^r = \frac{1}{2^m}\sum_l \binom{m}{l} Q^l P^r Q^{m-l}, \quad (6)$$

so

$$q^m p^r \to \frac{1}{2^m}\sum_l \binom{m}{l} Q^l P^r Q^{m-l}$$

is named the Weyl ordered, denoted as

$$\frac{1}{2^m}\sum_l \binom{m}{l} Q^l P^r Q^{m-l} = \frac{1}{2^m}\vdots \sum_l \binom{m}{l} Q^l P^r Q^{m-l} \vdots = \vdots Q^m P^r \vdots , \quad (7)$$

where the symbol $\vdots \quad \vdots$ means Weyl ordering within which Bose operators can be permuted.[9] However, this quantization scheme is not unique, Cohen has studied its extension in some ways.[10] For every integration kernel $\Delta_g(p,q)$, where the subscript g implies a generalized Wigner operator, there corresponds to an ordering rule of operators. One possible generalization of the Wigner operator is

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}y\mathrm{d}u\exp\left[\mathrm{i}u(q-Q)+\mathrm{i}y(p-P)+\mathrm{i}\frac{s}{2}yu\right] \equiv \Omega_s(p,q), \quad (8)$$

where s is a real parameter, Ω_s composes a family of generalized Wigner operators for different s values. The reasons why we have dealt with it lie in two aspects: 1) we use it to propose a generalized Weyl quantization scheme which accompanies a new generalized s-parameterized ordering rule. 2) In the process of the generalized Weyl quantization the s parameter will play the role of correlation between two quadratures Q and P. This can be exposed by comparing the normally ordered form of $\Omega_s(p,q)$ with the standard form of Gaussian bivariate normal distribution of random variables in statistics (see Eq. (13) below). Using $|q\rangle$ representation we can reform Ω_s as

$$\begin{aligned}\Omega_s(p,q) &= \frac{1}{2\pi}\int \mathrm{d}y\,\mathrm{e}^{-\mathrm{i}py}\exp\left(\mathrm{i}\frac{s+1}{2}yP\right)\\ &\quad\times|q\rangle\langle q|\exp\left(\mathrm{i}\frac{1-s}{2}yP\right)\\ &= \frac{1}{2\pi}\int \mathrm{d}y\,\mathrm{e}^{-\mathrm{i}py}\left|q-\frac{s+1}{2}y\right\rangle\\ &\quad\times\left\langle q+\frac{1-s}{2}y\right|, \end{aligned} \quad (9)$$

when $s=0$, it reduces to Eq. (2), as expected. In this work we shall reveal that the following generalized Weyl correspondence

$$H = \int \mathrm{d}q\mathrm{d}p\,\Omega_s(p,q)\,h_s(p,q), \quad (10)$$

naturally accompanies a new s-parameter ordering rule of operators, which excludes P–Q ordering and Q–P ordering as well as the Weyl ordering as its three special cases, respectively for $s=1,-1,0$. Hence it differs from the Cahill–Glauber ordering rule[11] which unifies normal ordering, anti-normal ordering, and Weyl ordering. The formula that can rearrange a given operator into its new s-parameter ordering is presented, which can help us to derive many new operator identities.

2. Deriving $h_s(p,q)$ in Eq. (10) from the generalized Weyl correspondence

Now we discuss the new quantization scheme in Eq. (10). By using Eq. (9) we calculate

$$\begin{aligned}&\mathrm{Tr}\left[\Omega_s(p,q)\,\Omega_{-s}(p',q')\right]\\ =&\frac{1}{4\pi^2}\mathrm{Tr}\left\{\int \mathrm{d}y\,\mathrm{e}^{-\mathrm{i}py}\left|q-\frac{s+1}{2}y\right\rangle\left\langle q+\frac{1-s}{2}y\right|\int \mathrm{d}y'\,\mathrm{e}^{-\mathrm{i}p'y'}\left|q'-\frac{-s+1}{2}y'\right\rangle\left\langle q'+\frac{1+s}{2}y'\right|\right\}\\ =&\frac{1}{4\pi^2}\int \mathrm{d}y\,\mathrm{e}^{-\mathrm{i}py}\int \mathrm{d}y'\,\mathrm{e}^{-\mathrm{i}p'y'}\,\mathrm{Tr}\left|q-\frac{s+1}{2}y\right\rangle\left\langle q'+\frac{1+s}{2}y'\right|\delta\left(q+\frac{1-s}{2}y-q'+\frac{-s+1}{2}y'\right)\\ =&\frac{1}{4\pi^2}\int \mathrm{d}y\,\mathrm{e}^{-\mathrm{i}py}\int \mathrm{d}y'\,\mathrm{e}^{-\mathrm{i}p'y'}\delta\left[q'-q+\frac{1+s}{2}(y+y')\right]\delta\left[q-q'+\frac{1-s}{2}(y+y')\right]\\ =&\frac{1}{2\pi^2(1-s)}\int \mathrm{d}y\,\mathrm{e}^{-\mathrm{i}py}\exp\left\{\mathrm{i}p'\left[(q'-q)\frac{2}{1-s}+y\right]\right\}\delta\left[(q'-q)\frac{2}{1-s}\right]\\ =&\frac{1}{2\pi}\delta(q'-q)\,\delta(p'-p), \end{aligned} \quad (11)$$

we see

$$h_s(p,q) = 2\pi \,\mathrm{tr}\left[\Omega_{-s}(p,q)\, H\right], \tag{12}$$

so $h_s(p,q)$ is considered as a generalized Weyl classical correspondence of H. Using the technique of integration within an ordered product (IWOP) of operators[6,7] we can directly perform the integration in Eq. (8) within the normal ordering symbol $: \ :$,

$$\begin{aligned}
\Omega_s(p,q) &= \frac{1}{4\pi^2} : \int_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}u \exp\left[-\frac{1}{4}\left(y^2+u^2\right) + \mathrm{i}u(q-Q) + \mathrm{i}y(p-P) + \mathrm{i}\frac{s}{2}yu\right] : \\
&= \frac{1}{2\pi\sqrt{\pi}} : \int \mathrm{d}u \exp\left[-\frac{1}{4}u^2 + \mathrm{i}u(q-Q) - \left(p-P+\frac{s}{2}u\right)^2\right] : \\
&= \frac{1}{2\pi\sqrt{\pi}} : \int \mathrm{d}u \exp\left\{-\frac{1}{4}u^2\left(1+s^2\right) + \mathrm{i}u\left[q-Q+\mathrm{i}s(p-P)\right] - (p-P)^2\right\} : \\
&= \frac{1}{\pi\sqrt{(1+s^2)}} : \exp\left\{-\frac{1}{1+s^2}[(q-Q)^2 + 2\mathrm{i}s(q-Q)(p-P) + (p-P)^2]\right\} : ,
\end{aligned} \tag{13}$$

this is the normally ordered form of $\Omega_s(p,q)$. The s parameter plays the role of correlation between two quadratures Q and P. When $s=0$

$$\Omega_{s=0}(p,q) = \frac{1}{\pi} : \mathrm{e}^{-(q-Q)^2-(p-P)^2} : , \tag{14}$$

which is just the normally ordered explicit form of $\Delta(p,q)$.[8]

3. The explicit Weyl ordered form of $\Omega_s(p,q)$

By noticing the operator $\mathrm{e}^{\mathrm{i}u(q-Q)+\mathrm{i}y(p-P)}$ itself is in Weyl ordering $\vdots \ \vdots$

$$\mathrm{e}^{\mathrm{i}u(q-Q)+\mathrm{i}y(p-P)} = \vdots \ \mathrm{e}^{\mathrm{i}u(q-Q)+\mathrm{i}y(p-P)} \vdots , \tag{15}$$

we know

$$\begin{aligned}
\Omega_{s=0}(p,q) &= \Delta(p,q) \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}u \vdots \ \mathrm{e}^{\mathrm{i}u(q-Q)+\mathrm{i}y(p-P)} \vdots \\
&= \vdots \ \delta(q-Q)\,\delta(p-P) \vdots ,
\end{aligned} \tag{16}$$

then the Weyl ordering form of $\Omega_s(p,q)$ is

$$\begin{aligned}
\Omega_s(p,q) &\equiv \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}u \vdots \ \exp\left[\mathrm{i}u(q-Q) \right. \\
&\quad \left. + \mathrm{i}y(p-P) + \mathrm{i}\frac{s}{2}yu\right] \vdots \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}u \vdots \ \mathrm{e}^{\mathrm{i}u(q-Q)}\delta\left(p-P+\frac{s}{2}u\right) \vdots \\
&= \frac{1}{\pi s} \vdots \ \exp\left[\frac{-2\mathrm{i}}{s}(q-Q)(p-P)\right] \vdots .
\end{aligned} \tag{17}$$

It is interesting to notice that when $s=-1$,

$$\Omega_{s=-1}(p,q) = \frac{-1}{\pi} \vdots \ \mathrm{e}^{2\mathrm{i}(q-Q)(p-P)} \vdots . \tag{18}$$

On the other hand, from Eq. (8) we see

$$\begin{aligned}
\Omega_{s=-1}(p,q) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}u\, \mathrm{e}\left[\mathrm{i}u(q-Q) \right. \\
&\quad \left. + \mathrm{i}y(p-P) - \mathrm{i}\frac{1}{2}yu\right] \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}u\, \mathrm{e}^{\mathrm{i}u(q-Q)}\, \mathrm{e}^{\mathrm{i}y(p-P)} \\
&= \delta(q-Q)\,\delta(p-P),
\end{aligned} \tag{19}$$

so we have

$$\begin{aligned}
&\delta(q-Q)\,\delta(p-P) \\
&= \frac{-1}{\pi} \vdots \ \mathrm{e}^{2\mathrm{i}(q-Q)(p-P)} \vdots \\
&= \frac{-1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}p'\mathrm{d}q' \Delta(p',q')\ \mathrm{e}^{2\mathrm{i}(q-q')(p-p')},
\end{aligned} \tag{20}$$

this is the integration transform for converting $\Delta(p',q')$ to $\delta(q-Q)\,\delta(p-P)$. Its reciprocal transform is

$$\begin{aligned}
&\frac{-1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}p\mathrm{d}q\delta(q-Q)\,\delta(p-P)\ \mathrm{e}^{-2\mathrm{i}(q-q')(p-p')} \\
&= \Delta(p',q').
\end{aligned} \tag{21}$$

On the other hand, when $s=1$,

$$\begin{aligned}
\Omega_{s=1}(p,q) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}u \exp\left[\mathrm{i}u(q-Q) \right. \\
&\quad \left. + \mathrm{i}y(p-P) + \mathrm{i}\frac{1}{2}yu\right] \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}u\, \mathrm{e}^{\mathrm{i}y(p-P)}\, \mathrm{e}^{\mathrm{i}u(q-Q)}
\end{aligned}$$

$$= \delta(p-P)\,\delta(q-Q), \tag{22}$$

so

$$\delta(p-P)\,\delta(q-Q) = \frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}p'\,\mathrm{d}q'\,\Delta(p',q')\,\mathrm{e}^{-2\mathrm{i}(q-q')(p-p')}, \tag{23}$$

its reciprocal transform is

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \mathrm{d}p'\,\mathrm{d}q'\,\mathrm{e}^{2\mathrm{i}(q-q')(p-p')}\delta(p-P)\,\delta(q-Q) = \Delta(p',q'). \tag{24}$$

From Eq. (17) we know that the classical Weyl correspondence of $\Omega_s(p,q)$ is

$$\frac{1}{\pi s}\exp\left[\frac{-2\mathrm{i}}{s}(q-q')(p-p')\right], \tag{25}$$

so from Eqs. (4) and (5) we also have

$$\begin{aligned}\Omega_s(p,q) &= \frac{1}{\pi^2 s}\int \mathrm{d}p'\,\mathrm{d}q' \exp\left[\frac{-2\mathrm{i}}{s}(q-q')(p-p')\right] :\, \mathrm{e}^{-(q'-Q)^2-(p'-P)^2} :\\ &= \frac{1}{\pi\sqrt{(1+s^2)}} :\, \exp\left\{-\frac{1}{1+s^2}[(q-Q)^2 + 2\mathrm{i}s(q-Q)(p-P) + (p-P)^2]\right\} : .\end{aligned} \tag{26}$$

4. Explicit form of $\Omega_s(\alpha)$ and unifying P–Q, Q–P, and Weyl ordering

From another aspect, let $z = (u+\mathrm{i}y)/\sqrt{2}$, $\alpha = (q+\mathrm{i}p)/\sqrt{2}$ and using the IWOP technique we have

$$\begin{aligned}\Omega_s(\alpha) &= \frac{1}{2\pi^2}\int \mathrm{d}^2 z \exp\left[\mathrm{i}z^*(\alpha-a) - \mathrm{i}z\left(a^\dagger-\alpha^*\right) + \frac{s}{4}\left(z^2-z^{*2}\right)\right]\\ &= \frac{1}{2\pi^2}\int \mathrm{d}^2 z :\, \exp\left[-\frac{1}{2}|z|^2 - \mathrm{i}z\left(a^\dagger-\alpha^*\right) + \mathrm{i}z^*(\alpha-a) + \frac{s}{4}\left(z^2-z^{*2}\right)\right] :\\ &= \frac{1}{\pi\sqrt{1+s^2}} :\, \exp\left\{\frac{1}{1+s^2}\left[-2(a-\alpha)\left(a^\dagger-\alpha^*\right) + s\left(a^\dagger-\alpha^*\right)^2 - s(a-\alpha)^2\right]\right\} :\\ &= \frac{1}{\pi\sqrt{1+s^2}}\exp\left[\frac{s}{1+s^2}\left(a^\dagger-\alpha^*\right)^2\right] :\, \exp\left\{-2(a-\alpha)\left(a^\dagger-\alpha^*\right)\frac{1}{1+s^2}\right\} :\, \exp\left[\frac{-s}{1+s^2}(a-\alpha)^2\right],\end{aligned} \tag{27}$$

where

$$:\, \exp\left\{-(a-\alpha)\left(a^\dagger-\alpha^*\right)\frac{2}{1+s^2}\right\} :\, = \exp\left[\left(a^\dagger-\alpha^*\right)(a-\alpha)\ln\frac{s^2-1}{1+s^2}\right]. \tag{28}$$

From Eq. (12) we know that the s-parameterized classical correspondence function of the coherent state projector is

$$\begin{aligned}2\pi\langle z|\,\Omega_{-s}(\alpha)\,|z\rangle &= 2\pi\,\mathrm{Tr}\left[|z\rangle\langle z|\,\Omega_{-s}(\alpha)\right]\\ &= \frac{2}{\sqrt{1+s^2}}\exp\left\{\frac{s[(z-\alpha)^2-(z^*-\alpha^*)^2]}{1+s^2} - \frac{2(z-\alpha)(z^*-\alpha^*)}{1+s^2}\right\}\\ &= \frac{2}{\sqrt{(1+s^2)}}\exp\left\{-\frac{1}{1+s^2}[(q-u)^2 - 2\mathrm{i}s(q-u)(p-y) + (p-y)^2]\right\},\end{aligned} \tag{29}$$

thus equation (10) tells us

$$|z\rangle\langle z| = \frac{2}{\sqrt{(1+s^2)}}\int \mathrm{d}p\mathrm{d}q\Omega_s(\alpha)\exp\left\{-\frac{1}{1+s^2}[(q-u)^2 - 2\mathrm{i}s(q-u)(p-y) + (p-y)^2]\right\}, \tag{30}$$

so when $s=1$, $\Omega_{s=1}(\alpha) = \delta(p-P)\,\delta(q-Q)$, substituting it into Eq. (30) yields

$$|z\rangle\langle z| = \sqrt{2}\mathfrak{P}\exp\left\{-\frac{1}{2}[(q-Q)^2 - 2\mathrm{i}(q-Q)(p-P) + (p-P)^2]\right\}\mathfrak{P}, \tag{31}$$

where the symbol $\mathfrak{P}\cdots\mathfrak{P}$ indicates P–Q ordering; when $s=-1$, $\Omega_{s=-1}(\alpha)=\delta(q-Q)\,\delta(p-P)$, substituting it into Eq. (30) leads to

$$|z\rangle\langle z| = \sqrt{2}\mathfrak{Q}\exp\left\{-\frac{1}{2}[(q-Q)^2 + 2\mathrm{i}(q-Q)(p-P)+(p-P)^2]\right\}\mathfrak{Q}, \quad (32)$$

where $\mathfrak{Q}\cdots\mathfrak{Q}$ indicates Q–P ordering; When $s=0$

$$|z\rangle\langle z| = 2\vdots\ \exp\left\{-\left[(q-Q)^2+(p-P)^2\right]\right\}\vdots\ , \quad (33)$$

as expected.[9] Especially, when $z=0$, equation (32) becomes to

$$|0\rangle\langle 0| = 2\vdots\ \exp\left\{-Q^2-P^2\right\}\vdots = 2\vdots\ \exp\left\{-2a^\dagger a\right\}\vdots\ . \quad (34)$$

Since equation (32) is in Q–P ordering, so we obtain the Q–P ordered form of $|0\rangle\langle 0|$

$$\begin{aligned}|0\rangle\langle 0| =& \sqrt{2}\exp\left(-\frac{1}{2}Q^2\right)\mathfrak{Q}\left[\exp(-\mathrm{i}QP)\right]\mathfrak{Q}\\ &\times\exp\left(-\frac{1}{2}P^2\right)\\ =&\sqrt{2}\,\mathrm{e}^{-1/2Q^2}\sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n Q^n P^n}{n!}\exp\left(-\frac{1}{2}P^2\right), \quad (35)\end{aligned}$$

which can be confirmed by checking the overlap

$$\begin{aligned}\langle q|0\rangle\langle 0|p\rangle =& \sqrt{2}\langle q|\exp\left(-\frac{1}{2}Q^2\right)\\ &\times\sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n Q^n P^n}{n!}\exp\left(-\frac{1}{2}P^2\right)|p\rangle\\ =&\sqrt{2}\exp\left(-\frac{1}{2}p^2-\frac{1}{2}q^2\right)\\ &\times\exp\{-\mathrm{i}qp\}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}pq}\\ =&\frac{1}{\sqrt{\pi}}\exp\left(-\frac{1}{2}p^2-\frac{1}{2}q^2\right), \quad (36)\end{aligned}$$

as it should be.

5. General formula that can rearrange a given operator into its new s-parameterized ordering

We now try to derive the s-ordering form of generalized Weyl quantization scheme. From Eqs. (30)–(32) we conclude that the s-ordering form of $|z\rangle\langle z|$ is

$$\begin{aligned}|z\rangle\langle z| =& \frac{2}{\sqrt{(1+s^2)}}\mathfrak{S}\exp\left\{-\frac{1}{1+s^2}[(q-Q)^2\right.\\ &\left.+\ 2\mathrm{i}s(q-Q)(p-P)+(p-P)^2]\right\}\mathfrak{S}, \quad (37)\end{aligned}$$

where $\mathfrak{S}\cdots\mathfrak{S}$ indicates the generalized s-ordering within which Bose operators are commutable, or

$$\begin{aligned}|z\rangle\langle z| =& \frac{2}{\sqrt{1+s^2}}\mathfrak{S}\exp\left\{\frac{s\left[(z-a)^2-\left(z^*-a^\dagger\right)^2\right]}{1+s^2}\right.\\ &\left.-\frac{2\,(z-a)\left(z^*-a^\dagger\right)}{1+s^2}\right\}\mathfrak{S}. \quad (38)\end{aligned}$$

Using the Glauber–Sudarshan P-representation formula for a density operator[12]

$$\rho = \int\frac{\mathrm{d}^2 z}{\pi}P(z)\,|z\rangle\langle z|, \quad (39)$$

where[13]

$$P(z) = \mathrm{e}^{|z|^2}\int\frac{\mathrm{d}^2\beta}{\pi}\langle-\beta|\,\rho\,|\beta\rangle\ \mathrm{e}^{|\beta|^2+\beta^* z-\beta z^*}, \quad (40)$$

and Eq. (38) we have

$$\begin{aligned}\rho =& \frac{2}{\sqrt{1+s^2}}\int\frac{\mathrm{d}^2 z}{\pi}\mathrm{e}^{|z|^2}\int\frac{\mathrm{d}^2\beta}{\pi}\langle-\beta|\,\rho\,|\beta\rangle\ \mathrm{e}^{|\beta|^2+\beta^* z-\beta z^*}\\ &\times\mathfrak{S}\exp\left\{\frac{s\left[(z-a)^2-\left(z^*-a^\dagger\right)^2\right]}{1+s^2}-\frac{2\,(z-a)\left(z^*-a^\dagger\right)}{1+s^2}\right\}\mathfrak{S}\\ =&\frac{2}{\sqrt{1+s^2}}\int\frac{\mathrm{d}^2\beta}{\pi}\langle-\beta|\,\rho\,|\beta\rangle\ \mathrm{e}^{|\beta|^2}\int\frac{\mathrm{d}^2 z}{\pi}\exp\left(\frac{s^2-1}{s^2+1}|z|^2+\beta^* z-\beta z^*\right)\\ &\times\mathfrak{S}\exp\left\{\frac{s\left[(z-a)^2-\left(z^*-a^\dagger\right)^2\right]}{1+s^2}\frac{2za^\dagger+2az^*-2a^\dagger a}{1+s^2}\right\}\mathfrak{S}, \quad (41)\end{aligned}$$

this integration can be done by using the integration technique within the new s-ordered operators product, and by virtue of the formula

$$\int \frac{\mathrm{d}^2 z}{\pi} \exp\left(\zeta |z|^2 + \xi z + \lambda z^* + f z^2 + g z^{*2}\right) = \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left[\frac{-\zeta\xi\lambda + \xi^2 g + \lambda^2 f}{\zeta^2 - 4fg}\right], \quad (42)$$

with the convergent condition

$$\mathrm{Re}\,(\zeta \pm f \pm g) < 0, \qquad \mathrm{Re}\left(\frac{\zeta^2 - 4fg}{\zeta \pm f \pm g}\right) < 0, \quad (43)$$

and after a straightforward but lengthy calculation we obtain

$$\rho = \frac{2}{\sqrt{1+s^2}} \mathfrak{S} \int \frac{\mathrm{d}^2\beta}{\pi} \langle -\beta|\rho|\beta\rangle \times \exp\Big\{\frac{1}{1+s^2}[2s^2|\beta|^2 + s\beta^2 - s\beta^{*2} + 2(\beta^* - s\beta)a - 2(\beta + s\beta^*)a^\dagger + 2a^\dagger a - sa^{\dagger 2} + sa^2]\Big\}\mathfrak{S}. \quad (44)$$

This is the formula that can rearrange a given operator into its new s-parameter ordering. When $s = 0$, equation (44) reduces to the Weyl ordering formula[11,14]

$$\rho = \frac{2}{\sqrt{1+s^2}} \vdots \int \frac{\mathrm{d}^2\beta}{\pi} \langle -\beta|\rho|\beta\rangle \times \exp\{2\beta^* a - 2\beta a^\dagger + 2a^\dagger a\} \vdots . \quad (45)$$

In particular, when ρ is a unit operator, $\langle -\beta|1|\beta\rangle = \mathrm{e}^{-2|\beta|^2}$, substituting it into Eq. (44) we do have

$$\frac{2}{\sqrt{1+s^2}} \int \frac{\mathrm{d}^2\beta}{\pi} \mathfrak{S} \times \exp\Big\{\frac{1}{1+s^2}[-2|\beta|^2 + s\beta^2 - s\beta^{*2} + 2(\beta^* - s\beta)a - 2(\beta + s\beta^*)a^\dagger + 2a^\dagger a - sa^{\dagger 2} + sa^2]\Big\}\mathfrak{S} = 1, \quad (46)$$

which confirms the correctness of Eq. (44).

In summary, based on the s-parameterized generalized Wigner operator Ω_s we have analyzed the accompanied new generalized s-parameterized ordering rule. This rule recovers P–Q ordering, Q–P ordering, and Weyl ordering of operators for $s = 1, -1, 0$ respectively. Hence it differs from the Cahill–Glauber ordering rule which unifies normal ordering, anti-normal ordering, and Weyl ordering. The formula which can rearrange a given operator into its new s-parameterized ordering is presented. The generalized quantization scheme with use of this special Ω_s is also completed. Many operator identities can then be derived using this formula.

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