Adaptive H_∞ synchronization of chaotic systems via linear and nonlinear feedback control

Fu Shi-Hui(付士慧)[a)†], Lu Qi-Shao(陆启韶)[b)], and Du Ying(杜 莹)[c)]

[a)] *Department of Mathematics, Zhengzhou University, Zhengzhou* 450001, *China*
[b)] *School of Aeronautics Science and Engineering, Beijing University of Aeronautics and Astronautics, Beijing* 100191, *China*
[c)] *School of Science, East China University of Science and Technology, Shanghai* 200237, *China*

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Adaptive H_∞ synchronization of chaotic systems via linear and nonlinear feedback control is investigated. The chaotic systems are redesigned by using the generalized Hamiltonian systems and observer approach. Based on Lyapunov's stability theory, linear and nonlinear feedback control of adaptive H_∞ synchronization is established in order to not only guarantee stable synchronization of both master and slave systems but also reduce the effect of external disturbance on an H_∞-norm constraint. Adaptive H_∞ synchronization of chaotic systems via three kinds of control is investigated with applications to Lorenz and Chen systems. Numerical simulations are also given to identify the effectiveness of the theoretical analysis.

Keywords: chaotic system, H_∞ synchronization, generalized Hamiltonian system, Lyapunov's stability theory, adaptive synchronization, Lorenz system, Chen system

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1. Introduction

Chaos synchronization has attracted increasing attention in recent years. It has many applications, especially in biologyl, secure communication, economics, and engineering.[1] The concept of chaos synchronization involves making two chaotic systems oscillate in a synchronous manner. Many effective control methods have been presented for synchronizing identical chaotic systems, such as linear state observer design,[2] nonlinear control,[3] active control,[4,5] impulsive control,[6,7] and sliding mode control.[8]

There are always noises or disturbances in physical systems, which may cause instability and destroy the synchronization. How to reduce their effect on chaotic synchronization has become an important issue. Hou *et al.* firstly introduced the H_∞ synchronization control of chaotic systems,[9] and H_∞ synchronization of chaotic systems via linear control was studied.[10,11] Lee *et al.* also investigated H_∞ synchronization of chaotic systems by using a new stabilizing control design.[12] This control method consists of two correlative parts, that is, linear dynamic feedback control and a nonlinear static one.

Because some or all of the parameters are unknown in real systems, it is important to study adaptive synchronization for chaotic systems with uncertain parameters.[13] For this reason, Choon *et al.* studied adaptive H_∞ synchronization of uncertain chaotic systems.[14] The system consisted of four parts: the linear function vector; the nonlinear function vector satisfying the global Lipschitz condition; the nonlinear function vector not satisfying the global Lipschitz condition; and the part including the unknown constant parameter vector. Whether the control includes the nonlinear part is determined by the nonlinear function vector not satisfying the global Lipschitz condition. If the nonlinear function vector not satisfying the global Lipschitz condition exists, the control includes the nonlinear part; otherwise, the control is linear. In this paper, adaptive H_∞ synchronization of chaotic systems via linear and nonlinear feedback control is investigated in a general way, where the linear feedback control is irrelevant with the nonlinear one. Hence, adaptive H_∞ synchronization of chaotic systems can be studied via three kinds of control schemes.

In 2001, Hebertt[15] firstly investigated synchronization of chaotic systems from the perspective of generalized Hamiltonian systems. The generalized Hamiltonian structure of most well-known chaotic systems allows one to clearly decide the nature of the synchronous signal (output) on the basis of the system dissipation and conservative energy managing struc-

†Corresponding author. E-mail: fshtp@163.com

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ture and a need to eliminate the destabilizing vector field at the receiver end, locally or globally. Furthermore, Gámez-Guzmán[16] and Mu[17] extended the approach to synchronization of Chua's circuits with multi-scroll attractors and chaotic systems with unknown parameters, respectively.

The organization of this paper is as follows. In Section 2, the chaotic system is redesigned based on the generalized Hamiltonian system and observer approach, and its locally destabilizing vector field is eliminated in order to make the error systems simpler. In Section 3, the feedback control scheme and adaptive law for chaotic systems with uncertain parameters are designed to guarantee realizing adaptive H_∞ synchronization by means of Lyapunov's stability theory. Numerical simulation and conclusions are given in Sections 4 and 5, respectively.

2. Chaotic systems redesigning

Firstly, we redesign chaotic systems based on the generalized Hamiltonian system and observer approach. A chaotic system is given as follows:

$$\dot{x} = f(x), \quad x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in R^n, \tag{1}$$

where $f \in R^n$ is continuously differential. The chaotic system (1) can be transformed into the following form with the output map y:[15]

$$\begin{cases} \dot{x} = J(y)\dfrac{\partial H}{\partial x} + (I+S)\dfrac{\partial H}{\partial x} + F(y) + F_1(y)P, \\ y = C\dfrac{\partial H}{\partial x}, \end{cases} \tag{2}$$

where $H(x)$ denotes an energy function and is globally positive-definite in R^n, the column gradient vector $\partial H/\partial x$ of $H(x)$ is assumed to exist everywhere; $J(x) + J^{\mathrm{T}}(x) = 0$, I is a constant skew symmetric, S is a constant symmetric matrix, $F(x)$ represents a locally destabilizing vector field, $P \in R^n$ is an unknown parameter vector, and C is a constant matrix. Equation (2) is regarded as the master system. Especially, if $H = 1/2x^{\mathrm{T}}Mx$ (M is a constant symmetric positive-definite matrix), we have $\partial H/\partial x = Mx$.

The corresponding slave system with linear and nonlinear feedback control is given by

$$\begin{cases} \dot{\xi} = J(y)\dfrac{\partial H}{\partial \xi} + (I+S)\dfrac{\partial H}{\partial \xi} + F(y) \\ \qquad + F_1(y)\,\widehat{P} + U(t) + \alpha(t) + Gd(t), \\ \eta = C\dfrac{\partial H}{\partial \xi}, \end{cases} \tag{3}$$

where ξ, η, and $\widehat{P}$ are the estimation of the state vectors x, y, and parameter vector P, respectively; $\partial H/\partial \xi$ is naturally the gradient of $H(\xi)$; $U(t) \in R^n$ is the linear feedback control and $\alpha(t) \in R^n$ is the nonlinear feedback control; $G \in R^{n\times k}$ is a constant matrix and $d(t) \in R^k$ is the external disturbance.

Let $e(t) = x(t) - \xi(t)$, $e_y = y - \eta$, and $\tilde{P} = \widehat{P} - P$. The error system is written as

$$\begin{cases} \dot{e} = J(y)\dfrac{\partial H}{\partial e} + (I+S)\dfrac{\partial H}{\partial e} + F_1(y)\tilde{P} \\ \qquad + U(t) + \alpha(t) + Gd(t), \\ e_y = C\dfrac{\partial H}{\partial e}, \qquad e_y \in R^m. \end{cases} \tag{4}$$

Since the locally destabilizing vector field $F(x)$ of the master system (2) is eliminated by that of the slave system (3), it does not exist in the error system (4). Here $J(x)$ and I are skew symmetric and represent the conservative part of the system (4), but S depicts nonconservative (or dissipative) part of the system (4). Thus the design based on the generalized Hamiltonian system and observer approach makes the error system (4) in a simple way.

Definition 1 (Adaptive asymptotic synchronization) The error system (4) is adaptively asymptotically synchronized if the synchronization error $e(t)$ satisfies

$$\lim_{t\to\infty} e(t) = 0 \tag{5}$$

under the control $U(t)$, $\alpha(t)$, and the adaptive law $\tilde{P}$ while the disturbance disappears.

Definition 2 (Adaptive H_∞ synchronization) With zero initial condition and a given level $\gamma > 0$, the error system (4) is adaptively H_∞ synchronized if the synchronization error $e(t)$ satisfies

$$\int_0^\infty e^{\mathrm{T}}(t)S_1 e(t)\,\mathrm{d}t < \gamma^2 \int_0^\infty d^{\mathrm{T}}(t)d(t)\,\mathrm{d}t \tag{6}$$

under the control $U(t)$, $\alpha(t)$, and the adaptive law $\tilde{P}$, where S_1 is a positive symmetric matrix. The parameter γ is called the H_∞-norm bound or the disturbance attenuation level.

Let $H(t) = \int_0^t e^{\mathrm{T}}(\sigma)S_1 e(\sigma)\,\mathrm{d}\sigma / \int_0^t d^{\mathrm{T}}(\sigma)d(\sigma)\,\mathrm{d}\sigma$, the condition (6) can be represented by

$$H(\infty) < \gamma^2. \tag{7}$$

In what follows, the feedback control $U(t)$, $\alpha(t)$, and adaptive law $\tilde{P}$ will be designed to guarantee the adaptive H_∞ synchronization if the external disturbance $d(t)$ exists. At the same time, they will be shown to guarantee adaptive asymptotic synchronization if the external disturbance $d(t)$ disappears.

3. Main results

Let

$$\dot{\tilde{P}} = -F_1^{\mathrm{T}}(y)Me(t), \tag{8}$$
$$U(t) = K_1(\eta(t) - y(t)) = K_1Ce(t), \tag{9}$$
$$\alpha(t) = K_2(g(\eta(t)) - g(y(t))), \tag{10}$$

where K_1 and K_2 are constant matrices, $g(x)$ is a nonlinear function vector satisfying the global Lipschitz condition with the Lipschitz constant $\lambda > 0$.

Theorem 1 In the error system (4), let the adaptive law $\tilde{P}$, the linear feedback control $U(t)$, and the nonlinear feedback control $\alpha(t)$ be given by Eqs. (8)–(10), respectively. For given $\gamma > 0$ and S_1, if there exist K_1, K_2, and λ such that the following matrix inequality holds:

$$2MSM + MK_1C + C^{\mathrm{T}}K_1^{\mathrm{T}}M + \lambda^2 \left\|K_2\right\|^2 MC^{\mathrm{T}}CM + M^2 + (1/\gamma^2)MGG^TM + S_1 < 0, \tag{11}$$

where $\|K_2\|$ is the matrix norm of K_2, then the adaptive H_∞ synchronization under the disturbance $d(t)$ is achieved.

Proof Let the Lyapunov function be

$$v(t) = 2H(e(t)) + \left|\tilde{P}\right|^2,$$

where $\left|\tilde{P}\right|$ is the norm of $\tilde{P}$, then $v(t)$ is a positive-definite function. The derivative of the Lyapunov function with respect to t along the trajectory of the error system (4) is calculated as

$$\begin{aligned}\dot{v}(t) &= \dot{e}^{\mathrm{T}}(t)Me(t) + e^{\mathrm{T}}(t)M\dot{e}(t) + 2\tilde{P}^{\mathrm{T}}\dot{\tilde{P}}\\ &= e^{\mathrm{T}}(t)M^{\mathrm{T}}(J^{\mathrm{T}}(y) + I^{\mathrm{T}})Me(t) + e^{\mathrm{T}}(t)\\ &\quad \times M(J(y) + I)Me(t) + e^{\mathrm{T}}(t)M^{\mathrm{T}}S^{\mathrm{T}}Me(t)\\ &\quad + e^{\mathrm{T}}(t)MSMe(t) + \tilde{P}^{\mathrm{T}}F_1^{\mathrm{T}}(y)Me(t)\\ &\quad + e^{\mathrm{T}}(t)MF_1(y)\tilde{P} + U^{\mathrm{T}}(t)Me(t) + e^{\mathrm{T}}(t)MU(t)\\ &\quad + \alpha^{\mathrm{T}}(t)Me(t) + e^{\mathrm{T}}(t)M\alpha(t)\\ &\quad + d^{\mathrm{T}}(t)G^{\mathrm{T}}Me(t) + e^{\mathrm{T}}(t)MGd(t) + 2\tilde{P}^{\mathrm{T}}\dot{\tilde{P}}\\ &= 2e^{\mathrm{T}}(t)MSMe(t) + 2\tilde{P}^{\mathrm{T}}F_1^{\mathrm{T}}(y)Me(t)\\ &\quad + U^{\mathrm{T}}(t)Me(t) + e^{\mathrm{T}}(t)MU(t) + \alpha^{\mathrm{T}}(t)Me(t)\\ &\quad + e^{\mathrm{T}}(t)M\alpha(t) + d^{\mathrm{T}}(t)G^{\mathrm{T}}Me(t)\\ &\quad + e^{\mathrm{T}}(t)MGd(t) + 2\tilde{P}^{\mathrm{T}}\dot{\tilde{P}}.\end{aligned}$$

According to Eq. (10), we obtain

$$\begin{aligned}\dot{v}(t) &= 2e^{\mathrm{T}}(t)MSMe(t) + U^{\mathrm{T}}(t)Me(t)\\ &\quad + e^{\mathrm{T}}(t)MU(t) + \alpha^{\mathrm{T}}(t)Me(t) + e^{\mathrm{T}}(t)M\alpha(t)\\ &\quad + d^{\mathrm{T}}(t)G^{\mathrm{T}}Me(t) + e^{\mathrm{T}}(t)MGd(t).\end{aligned}$$

According to Eq. (8), we obtain

$$\begin{aligned}\dot{v}(t) &= 2e^{\mathrm{T}}(t)MSMe(t) + e^{\mathrm{T}}(t)C^{\mathrm{T}}K_1^{\mathrm{T}}Me(t)\\ &\quad + e^{\mathrm{T}}(t)MK_1Ce(t) + \alpha^{\mathrm{T}}(t)Me(t)\\ &\quad + e^{\mathrm{T}}(t)M\alpha(t) + d^{\mathrm{T}}(t)G^{\mathrm{T}}Me(t) + e^{\mathrm{T}}(t)MGd(t)\\ &= e^{\mathrm{T}}(t)(MSM + C^{\mathrm{T}}K_1^{\mathrm{T}}M + MK_1C)e(t)\\ &\quad + \alpha^{\mathrm{T}}(t)Me(t) + e^{\mathrm{T}}(t)M\alpha(t)\\ &\quad + d^{\mathrm{T}}(t)G^{\mathrm{T}}Me(t) + e^{\mathrm{T}}(t)MGd(t).\end{aligned}$$

Taking account of $X^{\mathrm{T}}Y + Y^{\mathrm{T}}X \leqslant X^{\mathrm{T}}\Lambda X + Y^{\mathrm{T}}\Lambda^{-1}Y$ (where $X \in R^{n\times m}$, $Y \in R^{n\times m}$, $\Lambda \in R^{n\times n}$, and Λ is positive-definite) and Eq. (10), we obtain

$$\begin{aligned}&\alpha^{\mathrm{T}}(t)Me(t) + e^{\mathrm{T}}(t)M\alpha(t)\\ &\leqslant \alpha^{\mathrm{T}}(t)\alpha(t) + e^{\mathrm{T}}(t)MMe(t)\\ &= \|\alpha(t)\|^2 + e^{\mathrm{T}}(t)MMe(t)\\ &= \|K_2(g(\eta(t)) - g(y(t)))\|^2 + e^{\mathrm{T}}(t)MMe(t)\\ &\leqslant \|K_2\|^2\lambda^2\|Ce(t)\|^2 + e^{\mathrm{T}}(t)MMe(t)\\ &= \|K_2\|^2\lambda^2 e^{\mathrm{T}}(t)MC^{\mathrm{T}}CMe(t) + e^{\mathrm{T}}(t)MMe(t)\\ &= e^{\mathrm{T}}(t)(\|K_2\|^2\lambda^2MC^{\mathrm{T}}CM + MM)e(t).\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&d^{\mathrm{T}}(t)G^{\mathrm{T}}Me(t) + e^{\mathrm{T}}(t)MGd(t)\\ &\leqslant \gamma^2d^{\mathrm{T}}(t)d(t) + (1/\gamma^2)e^{\mathrm{T}}(t)MGG^{\mathrm{T}}Me(t),\end{aligned}$$

where $\gamma > 0$ is a constant. Then we have

$$\begin{aligned}\dot{v}(t) &\leqslant e^{\mathrm{T}}(t)(2MSM + C^{\mathrm{T}}K_1^{\mathrm{T}}M + MK_1C\\ &\quad + \|K_2\|^2\lambda^2MC^{\mathrm{T}}CM + MM\\ &\quad + (1/\gamma^2)MGG^{\mathrm{T}}M)e(t) + \gamma^2d^{\mathrm{T}}(t)d(t).\end{aligned}$$

From the assumption (11), it is known that the following matrix is negative, i.e.

$$2MSM + C^{\mathrm{T}}K_1^{\mathrm{T}}M + MK_1C + \|K_2\|^2\lambda^2MC^{\mathrm{T}}CM + MM + (1/\gamma^2)MGG^{\mathrm{T}}M + S_1 < 0,$$

we have

$$\dot{v}(t) < -e^{\mathrm{T}}(t)S_1e(t) + \gamma^2d^{\mathrm{T}}(t)d(t). \tag{12}$$

Integrating both sides of Eq. (12) from 0 to ∞, we obtain

$$\begin{aligned}v(\infty) - v(0) < &-\int_0^\infty e^{\mathrm{T}}(t)S_1e(t)\,\mathrm{d}t\\ &+\gamma^2\int_0^\infty d^{\mathrm{T}}(t)d(t)\,\mathrm{d}t.\end{aligned}$$

Since $v(\infty) \geqslant 0$ and $v(0) = 0$, the condition (6) for adaptive H_∞ synchronization is satisfied. The proof is completed.

Corollary 1 Under the assumptions of Theorem 1 for the error system (4), the adaptive asymptotic

synchronization without the external disturbance $d(t)$ is achieved.

Proof When $d(t) = 0$, from Eq. (12) we obtain

$$\dot{v}(t) < -e^{\mathrm{T}}(t)S_1e(t) \leqslant 0. \tag{13}$$

This guarantees $\lim\limits_{t\to\infty} e(t) = 0$ from Lyapunov's stability theory and then adaptive asymptotic synchronization is achieved. The proof is completed.

Corollary 2 In the error system (4), let the adaptive law $\tilde{P}$, the linear feedback control $U(t)$, and the nonlinear feedback control $\alpha(t)$ be given by Eqs. (8)–(10), respectively. For given $\gamma > 0$ and S_1, if there exists K_1 such that

$$\begin{aligned} &2MSM + MK_1C + C^{\mathrm{T}}K_1^{\mathrm{T}}M \\ &\quad + (1/\gamma^2)MGG^{\mathrm{T}}M + S_1 < 0. \end{aligned} \tag{14}$$

Then the adaptive H_∞ synchronization under the external disturbance $d(t)$ is achieved. Furthermore, the adaptive asymptotic synchronization without the external disturbance $d(t)$ is achieved.

The proof can be referred to Theorem 1 and Corollary 1.

Corollary 3 In the error system (4), let the adaptive law $\tilde{P}$, the linear feedback control $U(t)$, and the nonlinear feedback control $\alpha(t)$ be given by Eqs. (8)–(10), respectively. For given $\gamma > 0$ and S_1, if there exist K_2 and λ such that

$$\begin{aligned} &2MSM + \lambda^2 \left\|K_2\right\|^2 MC^{\mathrm{T}}CM + M^2 \\ &\quad + (1/\gamma^2)MGG^{\mathrm{T}}M + S_1 < 0, \end{aligned} \tag{15}$$

the adaptive H_∞ synchronization under the external disturbance $d(t)$ is achieved. Furthermore, the adaptive asymptotical synchronization without the external disturbance $d(t)$ is achieved.

The proof can be also referred to Theorem 1 and Corollary 1.

4. Numerical simulations

4.1. Lorenz system

The Lorenz system is shown as follows:

$$\begin{cases} \dot{x}_1 = \sigma(x_2 - x_1), \\ \dot{x}_2 = rx_1 - x_2 - x_1x_3, \\ \dot{x}_3 = x_1x_2 - bx_3, \end{cases} \tag{16}$$

where σ, and r, b are parameters. When $\sigma = 10$, $r = 28$, $b = 2.667$, the system is chaotic[15] with the phase diagram as given in Fig. 1. Various authors have investigated the chaotic synchronization of the Lorenz system, such as Wang, who analyzed the synchronization conditions of the fraction-order Lorenz system with activation feedback,[18] Tao investigated the dislocated feedback synchronization and the adaptive dislocated feedback synchronization methods based on the classical Lorenz system,[19] Yang achieved the exponential synchronization between two nearly identical Lorenz-like chaotic systems,[20] and Yahyazadeh studied synchronization of chaotic systems with known and unknown parameters by using a modified active sliding mode control.[21] To the best of our knowledge, there is no study on the H_∞ synchronization of a Lorenz system.

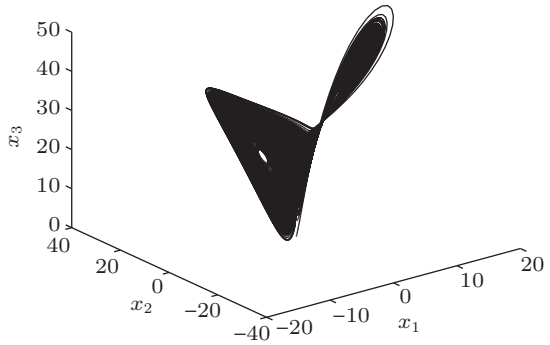


Fig. 1. The phase diagram of Lorenz system for $\sigma = 10$, $r = 28$, $b = 2.667$.

The master and slave systems corresponding to Lorenz system (16) are rewritten based on Eqs. (2) and (3), and their error system by Eq. (4) are also obtained. Let

$$J(y) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -x_1 \\ 0 & x_1 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 0 & \sigma/2 & 0 \\ -\sigma/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$S = \begin{bmatrix} -\sigma^2 & \sigma/2 & 0 \\ \sigma/2 & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \quad M = \begin{bmatrix} 1/\sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$F(y) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 \\ r \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} \sigma & 0 & 0 \end{bmatrix}, \quad x = (x_1, x_2, x_3)^{\mathrm{T}}, \quad H(x) = 1/2x^{\mathrm{T}}Mx.$$

In the following numerical simulation, let

$$G = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$

$$\gamma = 0.3, \quad d(t) = \cos(5t).$$

(i) When a Lorenz system has the adaptive law $\tilde{P}$, the linear feedback control $U(t)$ and the nonlinear feedback control $\alpha(t)$ given by Eqs. (8)–(10), respectively, we take $K_2 = 0.2$ and $g(x) = \begin{bmatrix} |x_1 + 1| & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ with the Lipschitz constant $\lambda = 1$. According to Eq. (11), we obtain $K_1 = \begin{bmatrix} -0.6818 & -0.1011 & -0.0011 \end{bmatrix}^{\mathrm{T}}$ by the linear matrix inequality (LMI).[22]

If a Lorenz system has no external disturbance $d(t)$, its master and slave systems can achieve the adaptive asymptotical synchronization according to Corollary 1. Figure 2 gives their synchronization errors which converge to zero exponentially. If a Lorenz system has the external disturbance $d(t)$, its master and slave systems can achieve the adaptive H_∞ synchronization according to Theorem 1. Figure 3 shows the plot of $H(t)$ versus time, which verifies $H(\infty) < \gamma^2 < 0.09$.

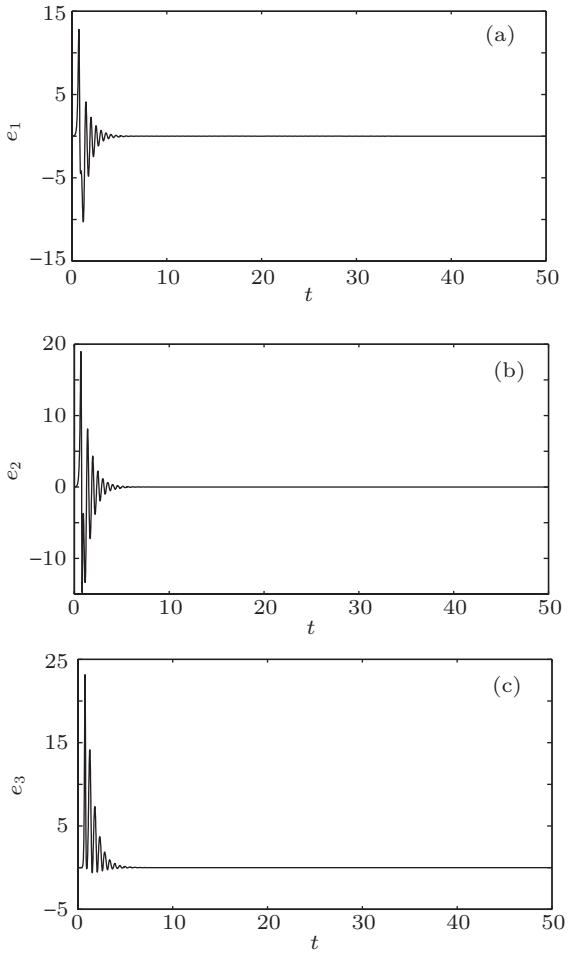


Fig. 2. The synchronization errors of a Lorenz system with the linear and nonlinear feedback controls $U(t)$, $\alpha(t)$.

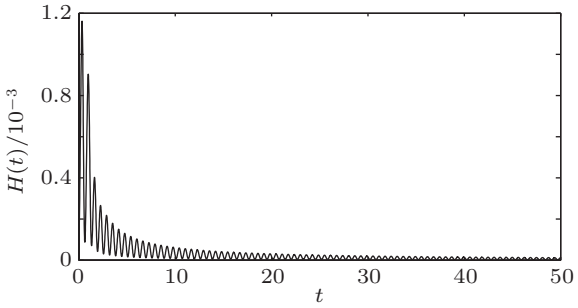


Fig. 3. The plot of $H(t)$ for a Lorenz system with the linear and nonlinear feedback controls $U(t)$ and $\alpha(t)$.

(ii) When a Lorenz system only has the linear feedback control $U(t)$ given by Eq. (9) and the adaptive law $\tilde{P}$ given by Eq. (8), we take $K_2 = 0$, i.e., $\alpha(t) = 0$. By using the linear matrix inequality (LMI), we obtain $K_1 = \begin{bmatrix} 7.8732 & -0.5511 & -0.0011 \end{bmatrix}^{\mathrm{T}}$ from Eq. (14).

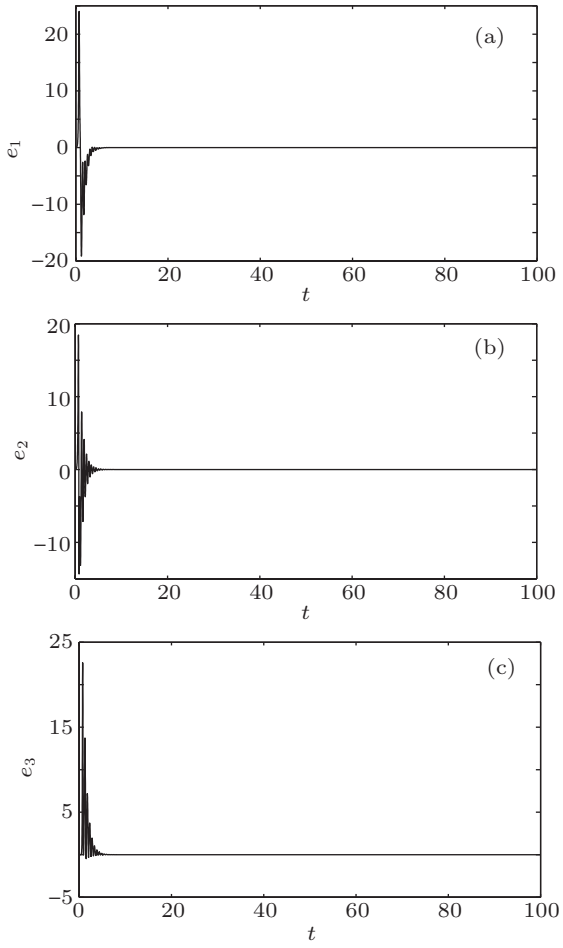


Fig. 4. The synchronization errors of a Lorenz system with linear feedback control $U(t)$ only.

The master and slave systems achieve the adaptive asymptotic synchronization without the external disturbance $d(t)$, while they achieve the adaptive H_∞ synchronization under the external disturbance $d(t)$

from Corollary 2. Figure 4 gives synchronization errors which converge to zero exponentially. Figure 5 shows the plot of $H(t)$ versus time, which verifies $H(\infty) < \gamma^2 < 0.09$.

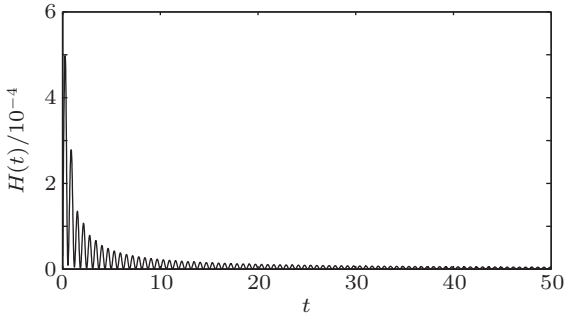


Fig. 5. The plot of $H(t)$ for a Lorenz system with linear feedback control $U(t)$ only.

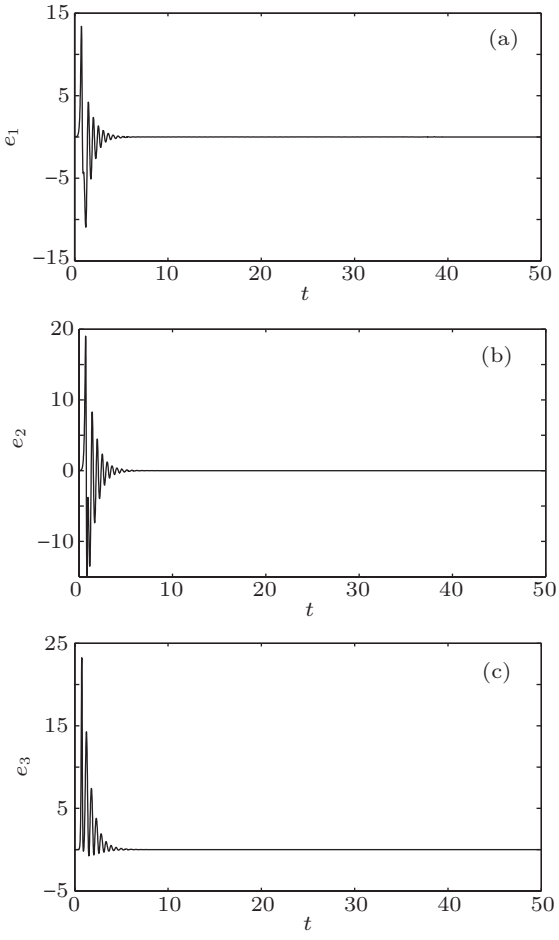


Fig. 6. The synchronization errors of a Lorenz system with the nonlinear feedback control $\alpha(t)$ only.

(iii) When a Lorenz system only has the nonlinear feedback control $\alpha(t)$ given by Eq. (10) and adaptive law $\tilde{P}$ given by Eq. (8), we take $K_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, i.e., $U(t) = 0$, $K_2 = 0.2$, and $g(x) = \begin{bmatrix} |x_1 + 1| & 0 & 0 \end{bmatrix}^{\mathrm{T}}$. The eigenvalues of $2MSM + \lambda^2 \|K_2\|^2 MC^{\mathrm{T}}CM + M^2 + (1/\gamma^2)MGG^{\mathrm{T}}M + S_1$ are calculated as $\lambda_1 = -4.1268$, $\lambda_2 = -2.3441$, $\lambda_3 = -0.1297$. So the matrix is negative-definite and satisfies the condition (15).

The master and slave systems achieve the adaptive asymptotic synchronization without the external disturbance $d(t)$, while achieve the adaptive H_∞ synchronization under the external disturbance $d(t)$ according to Corollary 3. Figure 6 gives the synchronization errors which converge to zero exponentially. Figur 7 shows the plot of $H(t)$ versus time, which verifies that $H(\infty) < \gamma^2 < 0.09$.

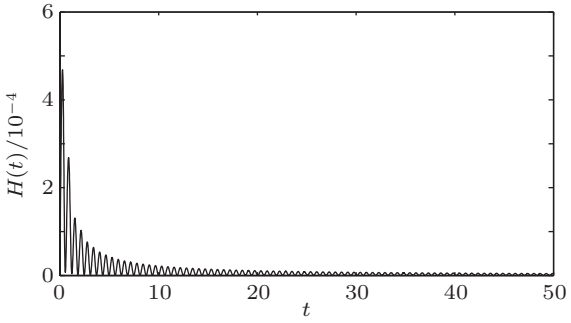


Fig. 7. The plot of $H(t)$ for a Lorenz system with nonlinear feedback control $\alpha(t)$ only.

4.2. Chen's chaotic attractor

The Chen system is described by

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1), \\ \dot{x}_2 = (c - a)x_1 - x_1x_3 + cx_2, \\ \dot{x}_3 = x_1x_2 - bx_3, \end{cases} \tag{17}$$

where a, b, and c are parameters. If $a = 35$, $b = 3$, $c = 28$, the system is chaotic[15] with the phase diagram given by Fig. 8. Wang *et al.* have extended the synchronization schemes in usual chaotic systems to chaotic fractional order Chen systems, successfully.[23] Wang addressed control via a single variable for the synchronization of chaotic Chen systems[24] and proposed a fuzzy model-based designs for chaotic stability and synchronization.[25] Asheghan

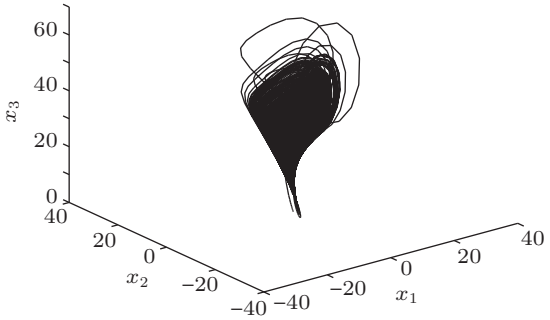


Fig. 8. The phase diagram of Chen's chaotic attractor.

achieved robust synchronization between two perturbed fractional-order Chen systems with different external disturbances.[26] However, there is no study on the H_∞ synchronization of Chen systems.

According to Eqs. (2) and (3), we rewrite the master and slave systems corresponding to Chen system (17) and obtain their error system from Eq. (4). Let

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad J(y) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -x_1 \\ 0 & x_1 & 0 \end{bmatrix},$$

$$I = \begin{bmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -b \end{bmatrix},$$

$$F(y) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \quad F_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_1 + x_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}},$$

$$P = \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$x = (x_1, x_2, x_3)^{\mathrm{T}}, \quad H(x) = 1/2 x^{\mathrm{T}} M x.$$

In the following numerical simulation, let

$$G = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$

$$\gamma = 0.3, \quad d(t) = \cos(5t).$$

(I) When Chen's system has the linear feedback control $U(t)$ given by Eq. (9), the nonlinear feedback control $\alpha(t)$ given by Eq. (10) and the adaptive law $\tilde{P}$ given by Eq. (8), we take $K_2 = 0.2$, $g(x) = \begin{bmatrix} |x_1 + 1| & 0 & 0 \end{bmatrix}^{\mathrm{T}}$. According to Eq. (11), we obtain

$$K_1 = \begin{bmatrix} 25.8982 & 0 \\ 0 & -3.5463 \\ 0 & 0 \end{bmatrix}$$

by the linear matrix inequality (LMI).

If Chen's system has no external disturbance $d(t)$, its master and slave systems achieve the adaptive asymptotical synchronization according to Corollary 1. Figure 9 gives their synchronization errors which converge to zero exponentially. If Chen's system is under the external disturbance $d(t)$, its master and slave systems achieve the adaptive H_∞ synchronization from Theorem 1. Figure 10 shows the plot of $H(t)$ versus time, which verifies that $H(\infty) < \gamma^2 < 0.09$.

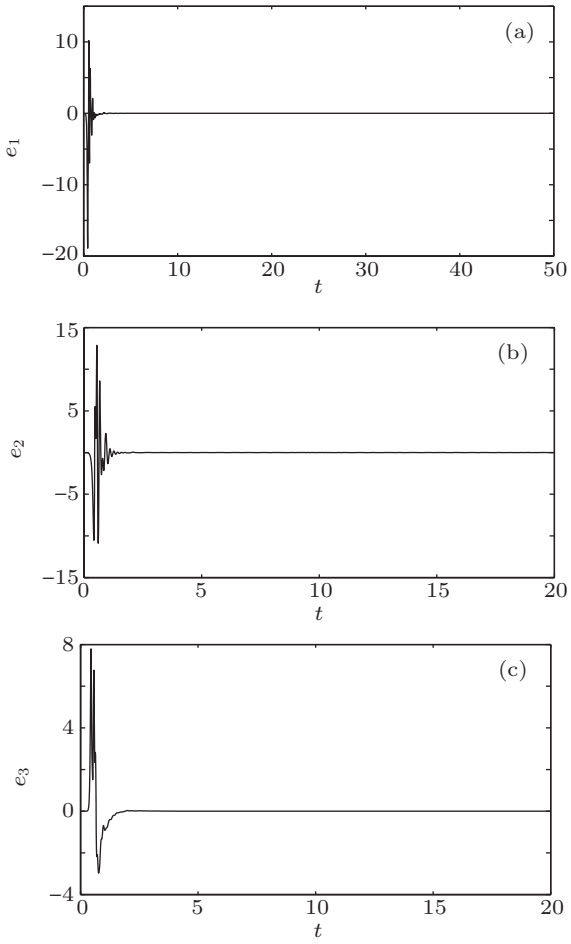


Fig. 9. The synchronization errors of Chen's system with the linear and nonlinear feedback controls $U(t)$ and $\alpha(t)$.

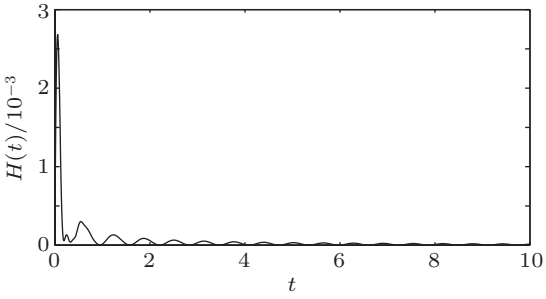


Fig. 10. The plot of $H(t)$ for Chen's system with the linear and nonlinear feedback controls $U(t)$ and $\alpha(t)$.

(II) When Chen's system only has the linear feedback control $U(t)$ given by Eq. (9) and the adaptive law $\tilde{P}$ given by Eq. (8), we take $K_2 = 0$, i.e., $\alpha(t) = 0$. According to Eq. (14), we obtain $K_1 = \begin{bmatrix} 25.9482 & 0 \\ 0 & -3.4963 \\ 0 & 0 \end{bmatrix}$ by the linear matrix inequality (LMI).

The master and slave systems of Chen's system achieve the adaptive asymptotical synchronization

without the external disturbance $d(t)$, while achieving the adaptive H_∞ synchronization under the external disturbance $d(t)$ from Corollary 2. Figure 11 gives the synchronization errors which converge to zero exponentially. Figure 12 shows the plot of $H(t)$ versus time, which verifies that $H(\infty) < \gamma^2 < 0.09$..

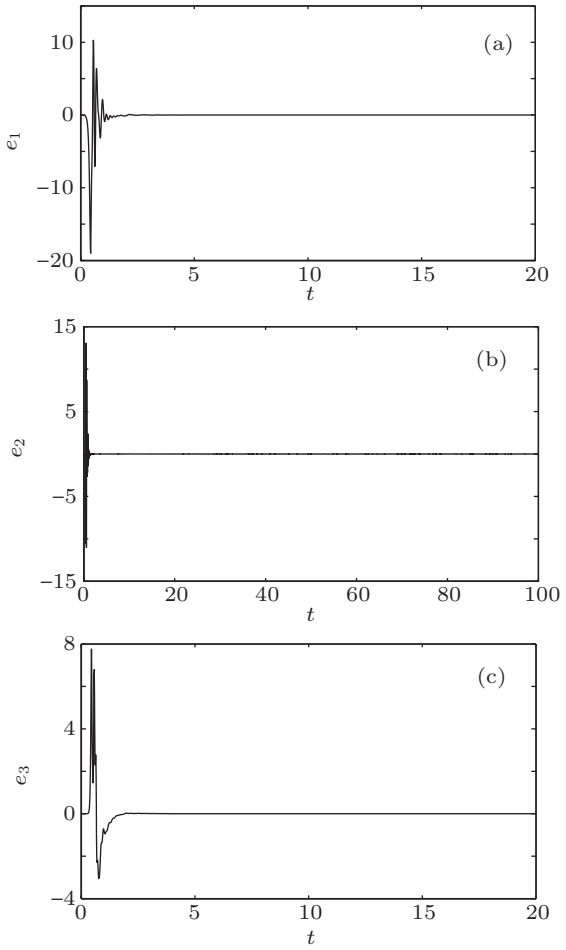


Fig. 11. The synchronization errors of Chen's system with linear feedback control $U(t)$ only.

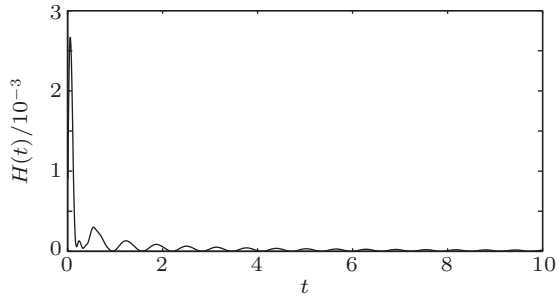


Fig. 12. The plot of $H(t)$ for Chen's system with linear feedback control $U(t)$ only.

(III) When Chen's system only has the nonlinear feedback control $\alpha(t)$ given by Eq. (10) and the adaptive law $\tilde{P}$ given by Eq. (8), we take $K_1 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, i.e., $U(t) = 0$, $K_2 = 0.2$, and $g(x) = \begin{bmatrix} |x_1 + 1| & 0 & 0 \end{bmatrix}^{\mathrm{T}}$. The eigenvalues of $2MSM + \lambda^2 \|K_2\|^2 MC^{\mathrm{T}}CM + M^2 + (1/\gamma^2)MGG^{\mathrm{T}}M + S_1$ are calculated as $\lambda_1 = -57.5889$, $\lambda_2 = -4.9$, $\lambda_3 = 1.3$. The matrix is not negative-definite and then does not satisfy the condition (15). However, Fig. 13 gives the synchronization errors without the external disturbance $d(t)$, which converge to zero exponentially, and then the adaptive asymptotic synchronization is achieved. Figure 14 shows the plot of $H(t)$ versus time under the external

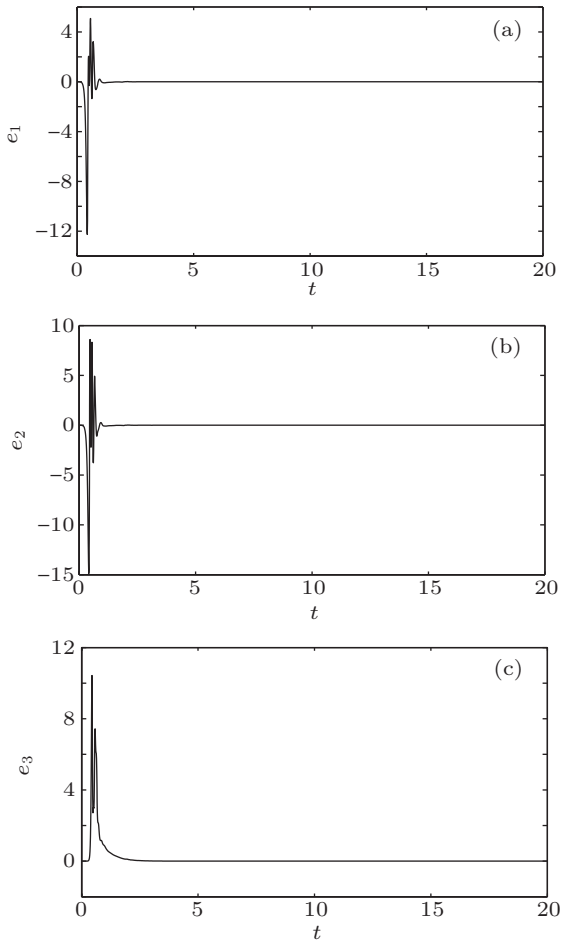


Fig. 13. The synchronization errors of Chen's system with nonlinear feedback control $\alpha(t)$ only.

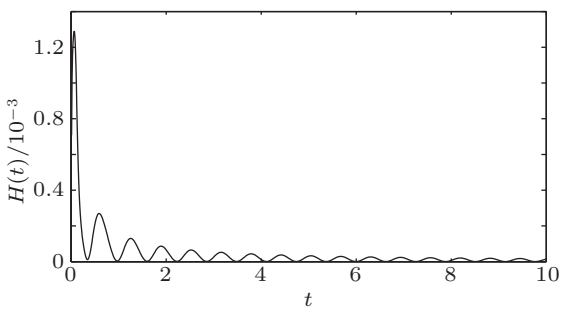


Fig. 14. The plot of $H(t)$ for Chen's system with nonlinear feedback control $\alpha(t)$ only.

disturbance $d(t)$. It verifies that $H(\infty) < \gamma^2 < 0.09$ and the adaptive H_∞ synchronization is also achieved. This shows that Eq. (15) is only sufficient but not necessary condition for synchronization.

5. Conclusion

In order to reduce the effects of disturbances to chaotic synchronization, adaptive H_∞ synchronization of chaotic systems with uncertain parameters via linear and nonlinear feedback controls is investigated. For simplicity, chaotic systems are redesigned in terms of the generalized Hamiltonian system and observer approach. The linear feedback control is independent of the nonlinear one, and the synchronization condition via the linear or nonlinear feedback control is also obtained. Finally, the method of adaptive H_∞ synchronization is applied to a Lorenz system and a Chen system, respectively. Further applications of this method to more complex systems will be considered in the future.

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