Nonsingular terminal sliding mode approach applied to synchronize chaotic systems with unknown parameters and nonlinear inputs

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This paper deals with the design of a novel nonsingular terminal sliding mode controller for finite-time synchronization of two different chaotic systems with fully unknown parameters and nonlinear inputs. We propose a novel nonsingular terminal sliding surface and prove its finite-time convergence to zero. We assume that both the master's and the slave's system parameters are unknown in advance. Proper adaptation laws are derived to tackle the unknown parameters. An adaptive sliding mode control law is designed to ensure the existence of the sliding mode in finite time. We prove that both reaching and sliding mode phases are stable in finite time. An estimation of convergence time is given. Two illustrative examples show the effectiveness and usefulness of the proposed technique. It is worthwhile noticing that the introduced nonsingular terminal sliding mode can be applied to a wide variety of nonlinear control problems.

Keywords: nonsingular terminal sliding mode, finite-time synchronization, uncertain parameter, nonlinear input

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1. Introduction

Chaotic systems are highly complex dynamical systems that exhibit several special characteristics, such as extraordinary sensitivity to initial conditions and system parameter variations, broad Fourier transform spectra, and fractal properties of the motion in the phase space. Due to these particular features and various efficient applications in secure communication, biological systems, economical systems, power convertors, information processing, and chemical reactions, synchronization of chaotic systems has attracted the interest of many researchers in recent years. Up to now, many different useful control methodologies have been proposed for chaos synchronization.[1−11] However, most of the previous works have concentrated on the study of synchronizing chaotic systems with fully known parameters. While, in practical situations the values of the system parameters are unknown in advance. The effects of these uncertainties can destroy the synchronization and even break it. Therefore, some researchers have paid attention to this problem and developed some adaptive controllers[12−25] for synchronization of chaotic systems with uncertain parameters.

Furthermore, when the controller is implemented in real world systems, physical limitations of actuators cause nonlinearities in the control input. Since chaotic systems are extremely sensitive to any system parameter variation, the nonlinearities in the control input may lead the perturbed chaotic system to unpredictable results.[26] Therefore, the effects of the nonlinearities in the control input cannot be neglected in controller design for chaos synchronization. Some works have focused on synchronizing chaotic systems with complete known parameters and input nonlinearities.[26,27] Also, in Refs. [28]–[31], the authors have reported chaos synchronization between two identical chaotic systems with fully (or partially) unknown parameters and nonlinear inputs. However, synchronization of two different chaotic systems is not straightforward and the previously proposed methods in Refs. [27]–[31] cannot be easily applied to the chaos synchronization between two different chaotic systems with unknown parameters and nonlinear inputs.

On the other hand, most of the aforementioned methods have been introduced for asymptotic stabi-

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lization of the synchronization error system, which means the states of the response system can converge to the states of the drive system within an infinite time horizon. Nevertheless, from a practical point of view, it is valuable to achieve synchronization in a finite time. To achieve fast convergence speed in a control system, finite-time control techniques are known to be efficient. In Ref. [32], a sliding mode approach has been proposed to synchronize the noise-perturbed Genesio system in a given finite time. Only the reaching phase of the proposed sliding mode approach has finite-time convergence. Li and Tian[33] have used finite-time control techniques for synchronization of chaotic systems without unknown parameters and input nonlinearities. Recently, Wang *et al.*[34,35] have addressed the finite-time synchronization of uncertain unified chaotic system based on the control Lyapunov function method and the finite-time control method. More recently, Aghababa *et al.*[36] have proposed a sliding mode approach for synchronizing chaotic systems with unknown parameters. To the best of our knowledge, however, for the finite-time synchronization of two different chaotic systems with both unknown parameters and input nonlinearities, there is no result in the literature until now, and it remains open and challenging.

The main contribution of this paper is to propose a nonsingular terminal sliding mode scheme to realize chaos synchronization between two different chaotic systems with fully unknown parameters and input nonlinearities. The proposed technique has finite-time convergence and stability in both reaching phase and sliding mode. Since the traditional terminal sliding surfaces suffer from singularities, they cannot be realized in practical applications. Therefore, in this paper, a nonsingular terminal sliding manifold is introduced to overcome the singularity problem. Suitable adaptation laws are designed to tackle the unknown parameters of the systems. Based on the adaptation laws and finite-time control technique, an adaptive sliding mode controller is introduced to guarantee the existence of the sliding motion in a given finite time. Some numerical simulations are presented to validate the efficiency and applicability of the proposed finite-time control strategy.

2. Problem formulation and preliminary lemmas

In this section, first, the system description and synchronization problem formulation are presented. Then, the precise definition of finite-time synchronization and some necessary lemmas are presented.

Consider the following two different n-dimensional chaotic systems with unknown parameters and input nonlinearities.

Master system:

$$\begin{aligned}
\dot{x}_1(t) &= f_1(x_1, x_2, \ldots, x_n) + F_1(x_1, x_2, \ldots, x_n)\theta,\\
\dot{x}_2(t) &= f_2(x_1, x_2, \ldots, x_n) + F_2(x_1, x_2, \ldots, x_n)\theta,\\
&\cdots,\\
\dot{x}_n(t) &= f_n(x_1, x_2, \ldots, x_n) + F_n(x_1, x_2, \ldots, x_n)\theta. \quad (1)
\end{aligned}$$

Slave system:

$$\begin{aligned}
\dot{y}_1(t) &= g_1(y_1, y_2, \ldots, y_n)\\
&\quad + G_1(y_1, y_2, \ldots, y_n)\psi + \phi_1(u_1),\\
\dot{y}_2(t) &= g_2(y_1, y_2, \ldots, y_n)\\
&\quad + G_2(y_1, y_2, \ldots, y_n)\psi + \phi_2(u_2),\\
&\cdots,\\
\dot{y}_n(t) &= g_n(y_1, y_2, \ldots, y_n)\\
&\quad + G_n(y_1, y_2, \ldots, y_n)\psi + \phi_n(u_n), \quad (2)
\end{aligned}$$

where $x(t) = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$ is the state vector of the master system, $f_i(x)$, $i = 1, 2, \ldots, n$ is a continuous smooth nonlinear function, $F_i(x)$, $i = 1, 2, \ldots, n$ is the i-th row of an $n \times m$ matrix, where its elements are continuous nonlinear functions, $\theta = [\theta_1, \theta_2, \ldots, \theta_m]^{\mathrm{T}}$ is an $m \times 1$ unknown vector parameter of the master system, $y(t) = [y_1, y_2, \ldots, y_n]^{\mathrm{T}}$ is the state vector of the slave system, $g_i(y)$, $i = 1, 2, \ldots, n$, is a continuous smooth nonlinear function, $G_i(y)$, $i = 1, 2, \ldots, n$ is the i-th row of an $n \times m$ matrix, where its elements are continuous nonlinear functions, $\psi = [\psi_1, \psi_2, \ldots, \psi_m]^{\mathrm{T}}$ is an $m \times 1$ unknown vector parameter of the slave system, and $\phi_1(u_1)$, $\phi_2(u_2)$, $\ldots$, $\phi_n(u_n)$ with $\phi_i(0) = 0$, $i = 1, 2, \ldots, n$, are continuous nonlinear functions inside the sector $[\rho_i, \upsilon_i]$, $i = 1, 2, \ldots, n$, $\rho_i > 0$, i.e.

$$\rho_i u_i^2 \leq u_i \phi_i(u_i) \leq \upsilon_i u_i^2, \qquad i = 1, 2, \ldots, n. \quad (3)$$

Remark 1 Most of the well-known chaotic systems as paradigms in the research of chaos can be expressed by Eq. (1), including the Rössler system, Liu

system, Lorenz system, Lü system, Chen system, unified system, Genesio system, Duffing oscillator, Van der Pol oscillator, Chua's circuits.

Assumption 1 It is assumed that the unknown vector parameters θ and ψ are norm bounded, i.e.,

$$\|\theta\| \leq D_\theta, \qquad \|\psi\| \leq D_\psi, \tag{4}$$

where D_θ and D_ψ are two positive known constants.

To solve the finite-time synchronization problem, the error between the master and slave systems can be defined as $e(t) = x(t) - y(t)$. Therefore, with subtracting Eq. (2) from Eq. (1) the error dynamics is obtained as follows:

$$\begin{aligned}
&\dot{e}_1(t) = f_1(x) + F_1(x)\theta - g_1(y) - G_1(y)\psi - \phi_1(u_1),\\
&\dot{e}_2(t) = f_2(x) + F_2(x)\theta - g_2(y) - G_2(y)\psi - \phi_2(u_2),\\
&\cdots,\\
&\dot{e}_n(t) = f_n(x) + F_n(x)\theta - g_n(y)\\
&\qquad - G_n(y)\psi - \phi_n(u_n).
\end{aligned} \tag{5}$$

Finite-time chaos synchronization means that the trajectories of the slave system approach to the trajectories of the master system within a finite time. The precise definition of finite-time chaos synchronization is given below.

Definition 1 Consider the master and slave chaotic systems described by Eqs. (1) and (2), respectively. If there exists a constant $T = T(e(0)) > 0$, such that

$$\mathrm{Lim}_{t\to T}\, \|e(t)\| = 0 \tag{6}$$

and $\|e(t)\| \equiv 0$, if $t \geq T$, then the chaos synchronization between systems (1) and (2) is achieved in a finite time.

It is clear that the finite-time synchronization problem can be transformed to the equivalent problem of finite-time stabilization of the error system (5). The main objective of this paper is that for any given master chaotic system (1) and slave chaotic system (2) with unknown parameters and nonlinear inputs, a proper control law $u(t)$ is designed such that the finite-time stability of the resulting error system (5) can be achieved in the sense of Definition 1.

Lemma 1[36] Assume a, b, and $0 < r < 1$ are real numbers, then the following inequality holds

$$(|a| + |b|)^r \leq |a|^r + |b|^r. \tag{7}$$

Lemma 2[37] Consider the system

$$\dot{x} = f(x) \quad \text{with } f(0) = 0,\ x \in R^n, \tag{8}$$

where $f : D \to R^n$ is continuous on an open neighborhood $D \subset R^n$.

Suppose there exists a continuous differential positive-definite function $V(x)$, real numbers $c > 0$, $0 < \xi < 1$, such that

$$\dot{V}(x) + cV^\xi(x) \leq 0, \quad \forall x \in D. \tag{9}$$

Then, the origin of system (8) is a locally finite-time stable equilibrium, and the settling time, depending on the initial state $x(0) = x_0$, satisfies

$$T(x_0) \leq \frac{V^{1-\xi}(x_0)}{c(1-\xi)}. \tag{10}$$

In addition, if $D = R^n$ and $V(x)$ is also radially unbounded (i.e., $V(x) \to +\infty$ as $\|x\| \to +\infty$), then the origin is a globally finite-time stable equlibrium of system (8).

Lemma 3 Suppose $a_1, a_2, \ldots, a_n$ and $0 < q < 2$ are real numbers, then the following inequality holds

$$\begin{aligned}
&|a_1|^q + |a_2|^q + \cdots + |a_n|^q\\
&\geq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)^{q/2}.
\end{aligned} \tag{11}$$

Proof For $n = 1$, it is obvious that inequality (11) is satisfied. For $n = 2$, using Lemma 1, we have

$$\begin{aligned}
&|a_1|^q + |a_2|^q = \left(|a_1|^2\right)^{q/2} + \left(|a_2|^2\right)^{q/2}\\
&\geq \left(a_1^2 + a_2^2\right)^{q/2}.
\end{aligned} \tag{12}$$

It is assumed that for $n = k$ inequality (11) holds, i.e.,

$$\begin{aligned}
&|a_1|^q + |a_2|^q + \cdots + |a_k|^q\\
&\geq \left(a_1^2 + a_2^2 + \cdots + a_k^2\right)^{q/2}.
\end{aligned} \tag{13}$$

Then for $n = k + 1$, assuming

$$|b| = \left(|a_1|^q + |a_2|^q + \cdots + |a_k|^q\right)^{1/q}$$

and using inequality (12), one has

$$\begin{aligned}
&|a_1|^q + |a_2|^q + \cdots + |a_k|^q + |a_{k+1}|^q\\
&= |b|^q + |a_{k+1}|^q \geq \left(b^2 + a_{k+1}^2\right)^{q/2}.
\end{aligned} \tag{14}$$

From inequality (13), one obtains

$$\begin{aligned}
b^2 &= \left(|a_1|^q + |a_2|^q + \cdots + |a_k|^q\right)^{2/q}\\
&\geq a_1^2 + a_2^2 + \cdots + a_k^2.
\end{aligned} \tag{15}$$

According to the above inequality and inequality (14), one can obtain

$$\left(b^2 + a_{k+1}^2\right)^{q/2} \geq \left(a_1^2 + a_2^2 + \cdots + a_k^2 + a_{k+1}^2\right)^{q/2}. \tag{16}$$

From inequalities (14) and (16), we have

$$|a_1|^q + |a_2|^q + \cdots + |a_k|^q + |a_{k+1}|^q \geq \left(a_1^2 + a_2^2 + \cdots + a_k^2 + a_{k+1}^2\right)^{q/2}. \quad (17)$$

Therefore, by the principle of mathematical induction, one can see that inequality (11) is satisfied for any positive integer n. This completes the proof.

Lemma 4 For $a_1, a_2, \ldots, a_n \in R$, the following inequality holds

$$|a_1| + |a_2| + \cdots + |a_n| \geq \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}. \quad (18)$$

This result is quite straightforward and the proof is omitted here.

Lemma 5 If assume $u_i(t) = \eta_i \operatorname{sgn}(s_i) \operatorname{sgn}(e_i)$, $\eta_i > 0$, then the following inequality holds

$$-s_i \operatorname{sgn}(e_i)\phi_i(u_i) \leq -\rho_i \eta_i |s_i|. \quad (19)$$

Proof Replacing $u_i(t)$ on the left-hand side of Eq. (3) by $\eta_i \operatorname{sgn}(s_i) \operatorname{sgn}(e_i)$, one has

$$\rho_i \eta_i^2 \operatorname{sgn}^2(e_i) \operatorname{sgn}^2(s_i) \leq \eta_i \operatorname{sgn}(e_i) \operatorname{sgn}(s_i)\phi_i(u_i). \quad (20)$$

Since $\operatorname{sgn}^2(e_i) = \operatorname{sgn}^2(s_i) = 1$, we have

$$\rho_i \eta_i \leq \operatorname{sgn}(e_i) \operatorname{sgn}(s_i)\phi_i(u_i). \quad (21)$$

Multiplying $-|s_i|$ to the above inequality, one obtains

$$-s_i \operatorname{sgn}(e_i)\phi_i(u_i) \leq -\rho_i \eta_i |s_i|. \quad (22)$$

Hence the proof is completed.

3. Design of a nonsingular terminal sliding mode controller

In this section, a nonsingular terminal adaptive sliding mode controller is introduced to finite-timely synchronize two different chaotic systems in the presence of both unknown parameters and nonlinear inputs.

Basically, the design procedure of a sliding mode controller involves two main steps. The first step is to establish a stable sliding manifold with the desired properties. The second step is to determine a discontinuous control law to guarantee the occurrence of the sliding motion. In this paper, we propose a nonsingular terminal sliding surface as follows:

$$s_i(t) = |e_i(t)| + \beta_i \int_0^t |e_i(\tau)|^\alpha \, \mathrm{d}\tau, \quad i = 1, 2, \ldots, n, \quad (23)$$

where $\beta_i > 0$ and $0 < \alpha < 1$ are constants.

When the system operates in the sliding mode, it satisfies the following equations[38]

$$s_i(t) = 0 \quad (24)$$

and

$$\dot{s}_i(t) = 0. \quad (25)$$

Thus the following sliding mode dynamics can be obtained

$$\begin{aligned} \dot{s}_i(t) &= \operatorname{sgn}(e_i)\dot{e}_i(t) + \beta_i |e_i(\tau)|^\alpha = 0 \Rightarrow \dot{e}_i(t) \\ &= -\beta_i sgn(e_i) |e_i(\tau)|^\alpha, \quad i = 1, 2, \ldots, n. \end{aligned} \quad (26)$$

Theorem 1 Consider the sliding mode dynamics (26). This system is finite-timely stable and its trajectories converge to the zero equilibrium in the finite time T_1, given by

$$T_1 \leq \frac{\left(0.5 \sum_{i=1}^n e_i^2(0)\right)^{(1-\alpha)/2}}{\mu(1-\alpha)2^{(\alpha-1)/2}}, \quad (27)$$

where $\mu = \min\{\beta_i, i = 1, 2, \ldots, n\}$.

Proof Consider the following positive-definite function as a Lyapunov function candidate

$$V_1(t) = 0.5 \sum_{i=1}^n e_i^2(t). \quad (28)$$

Taking time derivative of the Lyapunov function candidate, one has

$$\dot{V}_1(t) = \sum_{i=1}^n e_i(t)\dot{e}_i(t). \quad (29)$$

Replacing $\dot{e}_i(t)$ from Eq. (26) into Eq. (29), we have

$$\dot{V}_1(t) = \sum_{i=1}^n e_i(t) \left(-\beta_i \operatorname{sgn}(e_i) |e_i(t)|^\alpha\right). \quad (30)$$

It is clear that

$$\dot{V}_1(t) = \sum_{i=1}^n -\beta_i |e_i(t)|^{\alpha+1} \leq -\mu \sum_{i=1}^n |e_i(t)|^{\alpha+1}. \quad (31)$$

From Lemma 3, one can obtain

$$\begin{aligned} \dot{V}_1(t) &\leq -2^{(\alpha-1)/2}\mu \left(0.5 \sum_{i=1}^n e_i^2\right)^{(\alpha+1)/2} \\ &= -2^{(\alpha-1)/2}\mu V_1^{(\alpha+1)/2}. \end{aligned} \quad (32)$$

Therefore, on the basis of Lemma 2, the error $e_i(t)$, $i = 1, 2, \ldots, n$ will converge to the zero equilibrium in the finite time

$$T_1 \leq \frac{\left(0.5 \sum_{i=1}^n e_i^2(0)\right)^{(1-\alpha)/2}}{\mu(1-\alpha)2^{(\alpha-1)/2}}.$$

Thus the proof is achieved completely.

Once an appropriate sliding surface has been selected, it is followed by designing an adaptive sliding mode control law to derive the error system states onto the sliding surface in a finite time and to maintain on it for the subsequent time. Consequently, to guarantee the occurrence of the sliding motion in a finite time, an appropriate control law is proposed as follows:

$$\begin{aligned} u_i(t) = \frac{1}{\rho_i}\Big(& |f_i(x) - g_i(y)| \\ & + \left|F_i(x)\hat{\theta} - G_i(y)\hat{\psi}\right| + \beta_i |e_i|^{\alpha} \\ & + \sigma\left(\left\|\hat{\theta}\right\| + D_\theta + \left\|\hat{\psi}\right\| + D_\psi\right)\frac{|s_i|}{\|S\|^2} + k_i\Big) \\ & \times \operatorname{sgn}(e_i)\operatorname{sgn}(s_i), \end{aligned} \tag{33}$$

where $\hat{\theta}$ and $\hat{\psi}$ are estimations for θ and ψ, respectively, $\sigma = \min\{k_i,\, i = 1,\, 2, \ldots,\, n\} > 0$ is a constant, $S(t) = [s_1,\, s_2, \ldots,\, s_n]^{\mathrm{T}}$, k_i is the switching gain and a positive constant and

$$\frac{s_i}{\|S\|^2} = 0,$$

if $S(t) = 0$.

Now, let the suitable adaptation laws to be the following form

$$\begin{aligned} & \dot{\hat{\theta}}(t) = [F(x)]^{\mathrm{T}}\delta, \qquad \hat{\theta}(0) = \hat{\theta}_0, \\ & \dot{\hat{\psi}}(t) = -[G(y)]^{\mathrm{T}}\delta, \qquad \hat{\psi}(0) = \hat{\psi}_0, \end{aligned} \tag{34}$$

where $\delta = [\operatorname{sgn}(e_1)s_1,\, \operatorname{sgn}(e_2)s_2, \ldots,\, \operatorname{sgn}(e_n)s_n]^{\mathrm{T}}$, and $\hat{\theta}_0$ and $\hat{\psi}_0$ are the initial values of the adaptation parameters $\hat{\theta}$ and $\hat{\psi}$, respectively.

Theorem 2 Consider the error system (5) is controlled by the control signal (33) with adaptation laws (34). The system trajectories approach to the sliding surface $S(t) = 0$ in the finite time T_2, determined by

$$\begin{aligned} T_2 \le \frac{1}{\sigma}\Big(& \sum_{i=1}^{n} s_i^2(0) + \left\|\hat{\theta}(0) - \theta\right\|^2 \\ & + \left\|\hat{\psi}(0) - \psi\right\|^2\Big)^{1/2}. \end{aligned} \tag{35}$$

Proof Choose the candidate Lyapunov function of the system in the form of

$$\begin{aligned} V_2(t) = 0.5\Big(& \sum_{i=1}^{n} s_i^2(t) + \left\|\hat{\theta}(t) - \theta\right\|^2 \\ & + \left\|\hat{\psi}(t) - \psi\right\|^2\Big). \end{aligned} \tag{36}$$

The time derivative of $V_2(t)$ is

$$\begin{aligned} \dot{V}_2(t) = & \sum_{i=1}^{n} s_i(t)\dot{s}_i(t) + (\hat{\theta}(t) - \theta)^{\mathrm{T}}\dot{\hat{\theta}}(t) \\ & + (\hat{\psi}(t) - \psi)^{\mathrm{T}}\dot{\hat{\psi}}(t). \end{aligned} \tag{37}$$

Replacing $\dot{s}_i(t)$ from Eq. (26) into the above equation, one obtains

$$\begin{aligned} \dot{V}_2(t) = & \sum_{i=1}^{n} s_i(t)(\operatorname{sgn}(e_i)\dot{e}_i(t) + \beta_i |e_i(\tau)|^{\alpha}) \\ & + (\hat{\theta}(t) - \theta)^{\mathrm{T}}\dot{\hat{\theta}}(t) + (\hat{\psi}(t) - \psi)^{\mathrm{T}}\dot{\hat{\psi}}(t). \end{aligned} \tag{38}$$

Inserting $\dot{e}_i(t)$ from Eqs. (5) and adaptive laws from Eqs. (34) into the right-hand side of Eq. (38), this yields

$$\begin{aligned} \dot{V}_2(t) = & \sum_{i=1}^{n} s_i(t)(\operatorname{sgn}(e_i)(f_i(x) + F_i(x)\theta - g_1(y) \\ & - G_i(y)\psi - \phi_i(u_i)) + \beta_i |e_i(\tau)|^{\alpha}) \\ & + (\hat{\theta}(t) - \theta)^{\mathrm{T}}[F(x)]^{\mathrm{T}}\delta \\ & - (\hat{\psi}(t) - \psi)^{\mathrm{T}}[G(y)]^{\mathrm{T}}\delta. \end{aligned} \tag{39}$$

Since

$$\sum_{i=1}^{n} s_i \operatorname{sgn}(e_i)F_i(x)\theta = \theta^{\mathrm{T}}[F(x)]^{\mathrm{T}}\delta$$

and

$$\sum_{i=1}^{n} s_i \operatorname{sgn}(e_i)G_i(y)\psi = \psi^{\mathrm{T}}[G(y)]^{\mathrm{T}}\delta,$$

we have

$$\begin{aligned} \dot{V}_2(t) = & \sum_{i=1}^{n} s_i(t)(\operatorname{sgn}(e_i)(f_i(x) - g_i(y) \\ & - \phi_i(u_i)) + \beta_i |e_i(\tau)|^{\alpha}) \\ & + \hat{\theta}^{\mathrm{T}}(t)[F(x)]^{\mathrm{T}}\delta - \hat{\psi}^{\mathrm{T}}(t)[G(y)]^{\mathrm{T}}\delta. \end{aligned} \tag{40}$$

Using Lemma 5 and substituting $u_i(t)$ from Eq. (33) into the above equation, one obtains

$$\begin{aligned} \dot{V}_2(t) \le & \sum_{i=1}^{n} |s_i|\,|f_i(x) - g_i(y)| - |s_i| \\ & \times \Bigg(|f_i(x) - g_i(y)| + |F_i(x)\hat{\theta} \\ & - G_i(y)\hat{\psi}| + \beta_i |e_i|^{\alpha} \\ & + \sigma\left(\left\|\hat{\theta}\right\| + D_\theta + \left\|\hat{\psi}\right\| + D_\psi\right)\left(\frac{|s_i|}{\|S\|^2}\right) + k_i\Bigg) \\ & + s_i(t)\beta_i |e_i(\tau)|^{\alpha} + \hat{\theta}^{\mathrm{T}}(t)[F(x)]^{\mathrm{T}}\delta \\ & - \hat{\psi}^{\mathrm{T}}(t)[G(y)]^{\mathrm{T}}\delta. \end{aligned} \tag{41}$$

It is obvious that

$$\dot{V}_2(t) \le \sum_{i=1}^{n} -|s_i|\left|F_i(x)\hat{\theta} - G_i(y)\hat{\psi}\right| - k_i|s_i|$$

$$- \sigma \left(\left\| \hat{\theta} \right\| + D_\theta + \left\| \hat{\psi} \right\| + D_\psi \right) \frac{|s_i|^2}{\|S\|^2} + \hat{\theta}^{\mathrm{T}}(t)[F(x)]^{\mathrm{T}}\delta - \hat{\psi}^{\mathrm{T}}(t)[G(y)]^{\mathrm{T}}\delta. \quad (42)$$

Using the fact

$$\begin{aligned} & \hat{\theta}^{\mathrm{T}}(t)[F(x)]^{\mathrm{T}}\delta - \hat{\psi}^{\mathrm{T}}(t)[G(y)]^{\mathrm{T}}\delta \\ = & \left(\hat{\theta}^{\mathrm{T}}(t)[F(x)]^{\mathrm{T}} - \hat{\psi}^{\mathrm{T}}(t)[G(y)]^{\mathrm{T}} \right) \delta \\ \leq & \left| \hat{\theta}^{\mathrm{T}}(t)[F(x)]^{\mathrm{T}} - \hat{\psi}^{\mathrm{T}}(t)[G(y)]^{\mathrm{T}} \right| |\delta| \\ = & \left| \hat{\theta}^{\mathrm{T}}(t)[F(x)]^{\mathrm{T}} |\delta| - \hat{\psi}^{\mathrm{T}}(t)[G(y)]^{\mathrm{T}} |\delta| \right|, \end{aligned}$$

where $|\delta| = [|s_1|, |s_2|, \ldots, |s_n|]^{\mathrm{T}}$, one can obtain

$$\begin{aligned} \dot{V}_2(t) \leq & \sum_{i=1}^{n} - |s_i| \left| F_i(x)\hat{\theta} - G_i(y)\hat{\psi} \right| - k_i |s_i| \\ & - \sigma \left(\left\| \hat{\theta} \right\| + D_\theta + \left\| \hat{\psi} \right\| + D_\psi \right) \frac{|s_i|^2}{\|S\|^2} \\ & + \left| \hat{\theta}^{\mathrm{T}}(t)[F(x)]^{\mathrm{T}} |\delta| - \hat{\psi}^{\mathrm{T}}(t)[G(y)]^{\mathrm{T}} |\delta| \right|. \quad (43) \end{aligned}$$

Knowing that

$$\sum_{i=1}^{n} |s_i| F_i(x)\hat{\theta} = \hat{\theta}^{\mathrm{T}}[F(x)]^{\mathrm{T}} |\delta|$$

and

$$\sum_{i=1}^{n} |s_i| G_i(y)\hat{\psi} = \hat{\psi}^{\mathrm{T}}[G(y)]^{\mathrm{T}} |\delta|,$$

one can conclude that

$$\begin{aligned} & \sum_{i=1}^{n} |s_i| \left| F_i(x)\hat{\theta} - G_i(y)\hat{\psi} \right| \\ = & \left| \hat{\theta}^{\mathrm{T}}(t)[F(x)]^{\mathrm{T}} - \hat{\psi}^{\mathrm{T}}(t)[G(y)]^{\mathrm{T}} \right| |\delta|, \end{aligned}$$

this yields

$$\begin{aligned} \dot{V}_2(t) \leq & \sum_{i=1}^{n} -k_i |s_i| \\ & - \sigma \left(\left\| \hat{\theta} \right\| + D_\theta + \left\| \hat{\psi} \right\| + D_\psi \right) \frac{|s_i|^2}{\|S\|^2}. \quad (44) \end{aligned}$$

Using the fact

$$\sum_{i=1}^{n} \frac{|s_i|^2}{\|S\|^2} = 1,$$

we have

$$\dot{V}_2(t) \leq \sum_{i=1}^{n} -k_i |s_i| - \sigma \left(\left\| \hat{\theta} \right\| + D_\theta + \left\| \hat{\psi} \right\| + D_\psi \right). \quad (45)$$

Using Assumption 1 and the triangular inequality, we have

$$\dot{V}_2(t) \leq - \left(\sum_{i=1}^{n} k_i |s_i| + \sigma \left(\left\| \hat{\theta} - \theta \right\| + \left\| \hat{\psi} - \psi \right\| \right) \right). \quad (46)$$

According to Lemma 4, we have

$$\begin{aligned} \dot{V}_2(t) \leq & -\sigma \left(\sum_{i=1}^{n} |s_i| + \left\| \hat{\theta} - \theta \right\| + \left\| \hat{\psi} - \psi \right\| \right) \\ \leq & -\sigma \left(\sum_{i=1}^{n} s_i^2 + \left\| \hat{\theta} - \theta \right\|^2 + \left\| \hat{\psi} - \psi \right\|^2 \right)^{1/2} \\ = & -\sigma V_2^{1/2}(t). \quad (47) \end{aligned}$$

As a result from Lemma 2, the error trajectories $e(t)$ will converge to the sliding surface $S(t) = 0$ in the finite time

$$T_2 \leq \frac{1}{\sigma} \left(\sum_{i=1}^{n} s_i^2(0) + \left\| \hat{\theta}(0) - \theta \right\|^2 + \left\| \hat{\psi}(0) - \psi \right\|^2 \right)^{1/2}.$$

Thus the proof is completed.

Remark 2 The proposed control laws (33) are derived on the basis of the following features.

(i) The control laws should overcome the input nonlinearities. In order to achieve this feature, we propose a switching control law which includes the sign function. This alternative is inspired by the property proven in Lemma 5.

(ii) The control inputs should be robust against the systems' unknown parameters. On the other hand, since the systems' parameters are unknown in engineering practical applications, we use the adaptive control technique to undertake them. Additionally, we have added the parameter error to the Lyapunov function (36) to ensure that the adaptation parameters converge to some constants.

(iii) The control laws should guarantee the finite-time stability of the closed-loop system. Motivated by the finite time control theory (Lemma 2), the control inputs should include some terms to satisfy the condition (9). As a result, we find out that a constant gain with a normalized sliding surface term multiplied by the sum of the bounds of the unknown parameters and adaptive parameters should be employed.

4. Numerical simulations

In this section, two illustrative examples are presented to verify the effectiveness and usefulness of the proposed finite-time control technique in synchronizing two different chaotic systems with unknown parameters and input nonlinearities. Simulation results are obtained using MATLAB software. The ode45 solver is used to solve nonlinear equations. In the following two examples, the vector [2, 2, 2] is selected as

the initial values of the adaptive vector parameters $\hat{\theta}$ and $\hat{\psi}$, and the switching gain k is chosen as [0.1, 0.1, 0.1].

Example 1

To observe the efficient performance of the proposed finite-time control technique, it is assumed that the Lorenz system is the master system and the Chen system is the slave system. The equations of these systems are given below.

Lorenz system:

$$\begin{cases} \dot{x}_1(t) = 10(x_2 - x_1), \\ \dot{x}_2(t) = 28x_1 - x_2 - x_1x_3, \\ \dot{x}_3(t) = -8/3x_3 - x_1x_2. \end{cases} \tag{48}$$

Chen system:

$$\begin{cases} \dot{y}_1(t) = 35(y_2 - y_1), \\ \dot{y}_2(t) = 28y_2 - 7y_1 - y_1y_3, \\ \dot{y}_3(t) = -3y_3 - y_1y_2. \end{cases} \tag{49}$$

One can rewrite Eqs. (48) and (49) in the form of Eqs. (1) and (2) as follows:

$$\dot{x}(t) = \begin{bmatrix} 0 \\ -x_2 - x_1x_3 \\ x_1x_2 \end{bmatrix} + \begin{bmatrix} x_2 - x_1 & 0 & 0 \\ 0 & x_1 & 0 \\ 0 & 0 & -x_3 \end{bmatrix} \begin{bmatrix} 10 \\ 28 \\ 8/3 \end{bmatrix}, \tag{50}$$

and

$$\dot{y}(t) = \begin{bmatrix} 0 \\ y_1y_3 \\ y_1y_2 \end{bmatrix} + \begin{bmatrix} y_2 - y_1 & 0 & 0 \\ -y_1 & y_1 + y_2 & 0 \\ 0 & 0 & -y_3 \end{bmatrix} \begin{bmatrix} 35 \\ 28 \\ 3 \end{bmatrix} + \begin{bmatrix} \phi_1(u_1) \\ \phi_2(u_2) \\ \phi_3(u_3) \end{bmatrix}. \tag{51}$$

According to Eq. (23), three sliding surfaces are defined as

$$s_i(t) = |e_i(t)| + 4\int_0^T |e_i(\tau)|^{1/2}\,\mathrm{d}\tau, \quad i = 1, 2, 3. \tag{52}$$

The input nonlinearities are considered to be $\phi_i(u_i) = [1 + 0.2\sin(t)]u_i$, $i = 1, 2, 3$. Therefore, one can conclude that $\rho_i = 0.8$, $i = 1, 2, 3$. The initial conditions of the Lorenz and Chen systems are randomly selected as: $x_1(0) = 5$, $x_2(0) = 10$, $x_3(0) = 2$, and $y_1(0) = 8$, $y_2(0) = 4$, $y_3(0) = 6$. Also, D_θ and D_ψ are chosen equal to 40 and 60, respectively.

The synchronization errors of the Lorenz and Chen systems are displayed in Fig. 1, where the control input is activated at $t = 3$ s. It is obvious that the synchronization errors converge to the zero very quickly, which implies the synchronization between the Lorenz and Chen systems is realized in a finite time (as illustrated in Fig. 2).

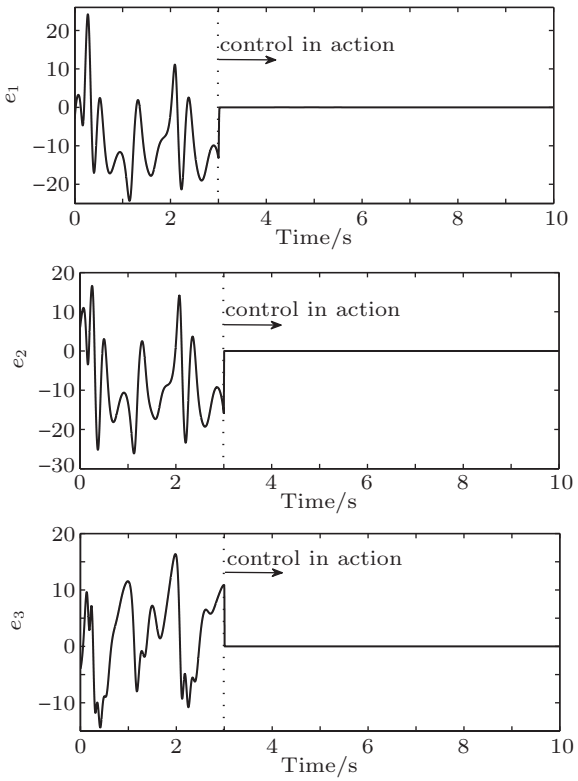


Fig. 1. Synchronization errors between the Lorenz and Chen systems using the proposed sliding mode controller.

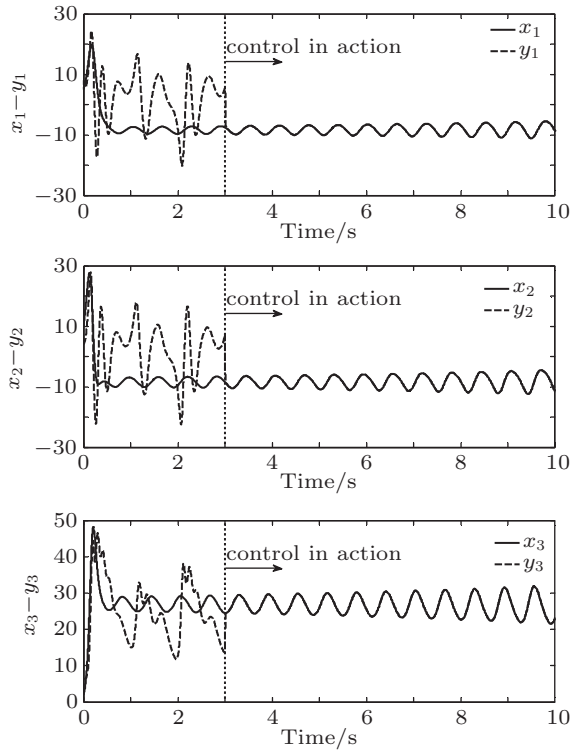


Fig. 2. State trajectories of the Lorenz and Chen systems using the proposed sliding mode controller.

For comparison, the adaptive controller proposed in Ref. [14] is applied to synchronize the Lorenz and Chen chaotic systems with unknown parameters and input nonlinearities. Figure 3 shows the state trajectories of the error system. It is seen that the synchronization errors have permanent oscillations and the synchronization purpose is not completely achieved, as depicted in Fig. 4. Therefore, it can be concluded that the controller in Ref. [14] is not robust against the input nonlinearities. On the other hand, our controller has both robustness against the unknown parameters and input nonlinearities as well as it has the fast convergence properties.

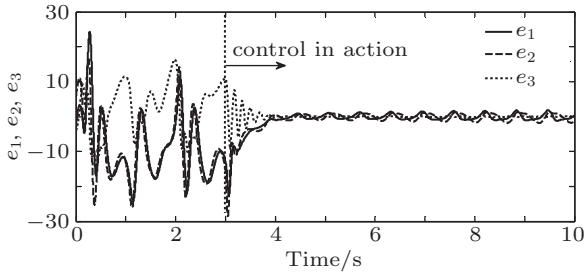


Fig. 3. Synchronization errors between the Lorenz and Chen systems using the adaptive controller in Ref. [14].

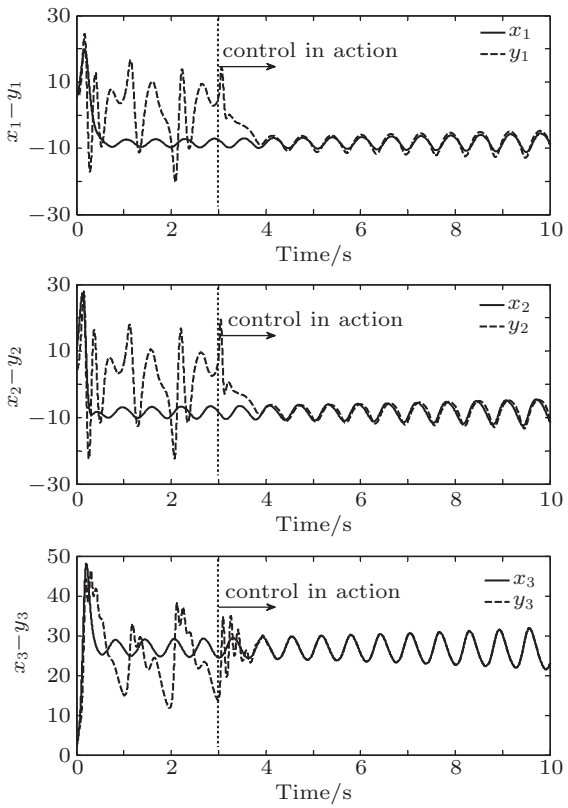


Fig. 4. State trajectories of the Lorenz and Chen systems using the adaptive controller in Ref. [14].

Example 2

This example verifies the effectiveness of the proposed finite-time control approach by synchronizing the Liu and Rössler systems. In this case, the Liu system drives the Rössler system. The nonlinear equations of the Liu and Rössler systems are as follows:

Liu system:

$$\begin{cases} \dot{z}_1(t) = 10(z_2 - z_1), \\ \dot{z}_2(t) = 40z_1 - z_1 z_3, \\ \dot{z}_3(t) = 4z_1^2 - 2.5z_3. \end{cases} \tag{53}$$

Rössler system:

$$\begin{cases} \dot{v}_1(t) = -(v_2 + v_3), \\ \dot{v}_2(t) = 0.2v_2 + v_1, \\ \dot{v}_3(t) = 0.2 + v_1 v_3 - 5.7v_3. \end{cases} \tag{54}$$

The master and slave systems can be rewritten in the form of Eqs. (1) and (2) as follows:

$$\dot{z}(t) = \begin{bmatrix} 0 \\ -z_1 z_3 \\ 4z_1^2 \end{bmatrix} + \begin{bmatrix} z_2 - z_1 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & -z_3 \end{bmatrix} \begin{bmatrix} 10 \\ 40 \\ 2.5 \end{bmatrix}, \tag{55}$$

and

$$\dot{v}(t) = \begin{bmatrix} 0 \\ v_1 \\ 0.2 + v_1 v_3 \end{bmatrix} + \begin{bmatrix} v_2 + v_3 & 0 & 0 \\ -y_1 & v_2 & 0 \\ 0 & 0 & -v_3 \end{bmatrix} \begin{bmatrix} -1 \\ 0.2 \\ 5.7 \end{bmatrix} + \begin{bmatrix} \phi_1(u_1) \\ \phi_2(u_2) \\ \phi_3(u_3) \end{bmatrix}. \tag{56}$$

Subsequently, using Eq. (23), three sliding manifolds are defined as

$$s_i(t) = |e_i(t)| + 4\int_0^T |e_i(\tau)|^{1/2}\,\mathrm{d}\tau, \quad i = 1, 2, 3. \tag{57}$$

The input nonlinearities are considered to be $\phi_i(u_i) = [0.5 + 0.3\sin(t)]u_i$, $i = 1, 2, 3$. Therefore, one can conclude that $\rho_i = 0.2$, $i = 1, 2, 3$. The initial values of the Liu and Rössler systems are randomly chosen as $z_1(0) = 4$, $z_2(0) = 6$, $z_3(0) = 1$, and $v_1(0) = 10$, $v_2(0) = 3$, $v_3(0) = 5$. D_θ and D_ψ are set to be 55 and 15, respectively.

The synchronization errors between the Liu and Rössler systems are shown in Fig. 5, where the control input is activated at $t = 3$ s. It is observed that the synchronization errors approach to the zero very rapidly. This means that the trajectories of the Rössler system are synchronized by the trajectories of

the Liu system as quickly as possible (as illustrated in Fig. 6).

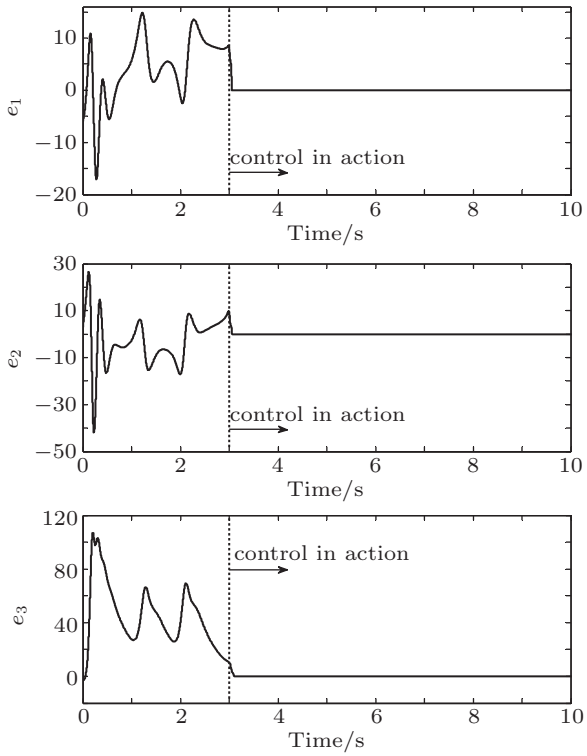


Fig. 5. Synchronization errors between the Liu and Rössler systems using the proposed sliding mode controller.

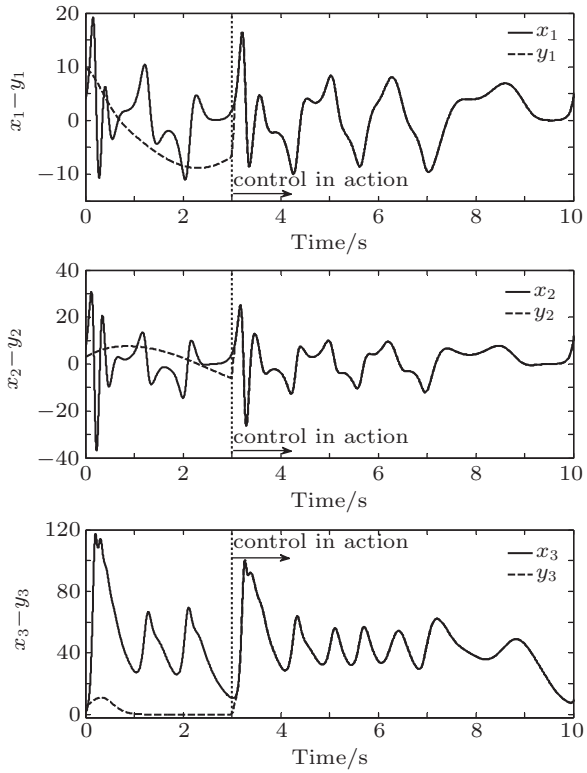


Fig. 6. State trajectories of the Liu and Rössler systems using the proposed sliding mode controller.

To compare the performance of the proposed technique, the adaptive synchronization strategy proposed in Ref. [14] is applied to synchronize the Liu and Rössler systems in the presence of the unknown parameters and input nonlinearities. The system synchronization errors are shown in Fig. 7. One can see there are nonzero steady state oscillations in the error trajectories. This means that the proposed adaptive controller in Ref. [14] cannot synchronize the Liu and Rössler systems, as illustrated in Fig. 8. However, the method proposed in our paper can robustly and quickly synchronize two different chaotic systems in the presence of unknown parameters in both the master and slave systems and nonlinear inputs.

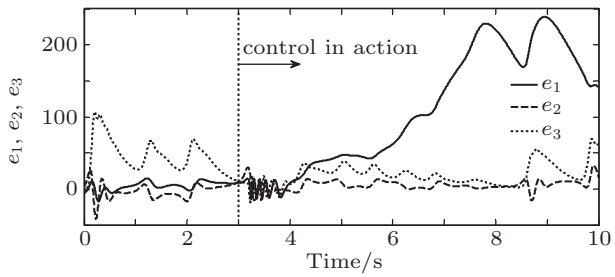


Fig. 7. Synchronization errors between the Liu and Rössler systems using the adaptive controller in Ref. [14].

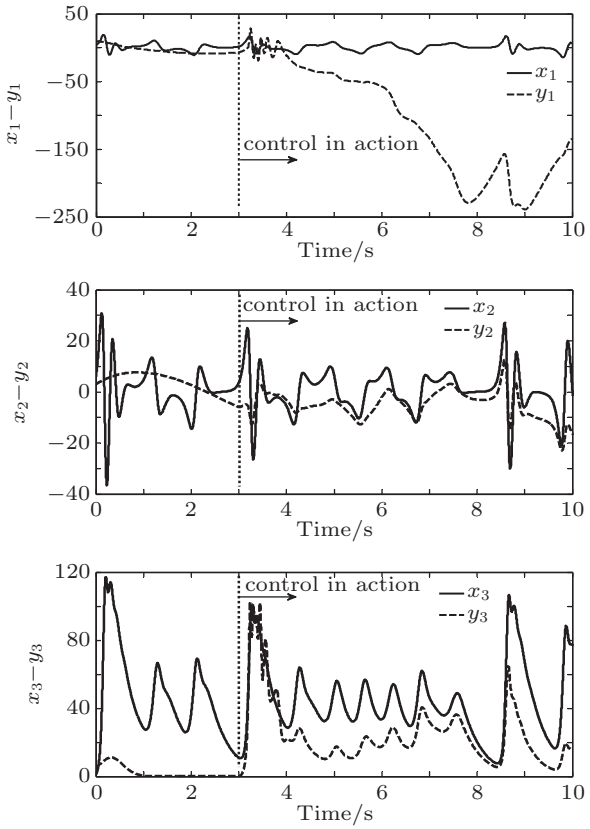


Fig. 8. State trajectories of the Liu and Rössler systems using the adaptive controller in Ref. [14].

5. Conclusion

In this paper, the problem of finite-time chaos synchronization between two different chaotic systems with unknown parameters and input nonlinearities is studied. First, a nonsingular terminal switching manifold is introduced and its finite-time stability and convergence are proved. Then, appropriate adaptation laws are proposed to undertake the unknown parameters. Afterwards, based on the adaptation laws and Lyapunov stability theory, an adaptive finite-time sliding mode controller is designed. Finally, numerical results demonstrate the usefulness and applicability of the proposed finite-time control technique and verify the theoretical results. It should be noted that the introduced nonsingular terminal sliding mode scheme can be applied for a large class of control problems.

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