The stability of a non-extensive relativistic Fermi system*

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Based on the statistical theory of non-extensive relativity, and using theoretical analysis and numerical simulation, the non-extensive mechanical stability of ultra-relativistic free Fermi gas is investigated. The expressions of the stability conditions under high and low temperatures are given, and the mechanisms of the influences of temperature, ultra-relativistic effect, and non-extensive parameter q on stability are analysed. Our results show that at high temperature and under the condition of $q < 1$, the stability of a non-extensive system is weaker than that of an extensive system, and the relativistic effect reduces system stability as compared with a non-relativistic system. However, under the condition of $q > 1$, the stability of the non-extensive system is stronger than that of the extensive system, and the relativistic effect strengthens the system stability as compared with the non-relativistic system. In addition, under the condition of low temperature, the variation of the stability of the non-extensive system with temperature has a turning point.

Keywords: Fermi gas, relativity, stability

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1. Introduction

Since the advent of the concept of non-extensive entropy, the non-extensive statistical thermodynamic theory has been developed quickly and used widely in areas such as physics, astrophysics, and medicine. Non-extensive statistical mechanics in particular has great potential applications in long-range interactions, memory effects, and non-complete systems. In recent years, theoretical studies relevant to non-extensive statistics were investigated intensely.[1−10] In the meantime, the theoretical and the experimental studies of the extensive system stability are increasing gradually. According to Ref. [12], the degenerated Fermi gas can collapse. In Ref. [13] the instability of Fermi gases in constricted weak interaction was investigated and its density condition was analysed. In Ref. [14] the weakly interacting Fermi gases were studied and the criterion of instable system was given based on the particle number density. In addition, the mechanical stability of weakly interacting Fermi gases was studied and the condition of system instability was also analysed in Ref. [15].

Based on the non-extensive statistical theory of the extreme relativism, using the theoretical analysis and the numerical simulation, we analyse the mechanisms of the influences of temperature, extreme relativistic effect and non-extensive parameter q on system stability.

2. Analysis of stability conditions

2.1. Stability condition of the non-extensive extreme relativistic system

Supposing the energy spectrum of a single particle $\varepsilon = cp$, where c is the light velocity and p is momentum, the stability condition expressions are obtained by using the appropriate method.

2.1.1. The case of $q \leq 1$ and $\mu < 0$

According to the document,[11] the total energy can be written as

$$E = 3N\frac{\alpha_4}{\alpha_5}\left[\frac{n(ch)^3}{8g\pi}\frac{\alpha_4}{\alpha_1}\right]^{(1-q)/(4-3q)} \times \left(\frac{1-q}{KT}\right)^{1/(3q-4)}, \quad (1)$$

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where N means the particle number, n is the particle number density, and K the Boltzmann constant. So the pressure can be expressed as

$$P = n\frac{\alpha_4}{\alpha_5}\left[\frac{n(ch)^3}{8g\pi}\frac{\alpha_4}{\alpha_1}\right]^{(1-q)/(4-3q)} \times \left(\frac{1-q}{KT}\right)^{1/(3q-4)}, \quad (2)$$

and the stability condition is

$$\frac{\partial P}{\partial V} = -\frac{5-4q}{4-3q}n^{(5-4q)/(4-3q)}\frac{1}{V}\frac{\alpha_4}{\alpha_5} \times \left[\frac{(ch)^3}{8g\pi}\frac{\alpha_4}{\alpha_1}\right]^{(1-q)/(4-3q)}\left(\frac{1-q}{KT}\right)^{1/(3q-4)} < 0, \quad (3)$$

where

$$\alpha_i = \Gamma\left(\frac{1}{1-q} + \mathrm{i}\right)$$

and V is the volume of this system. According to expression (3), $\partial P/\partial V < 0$ will hold true as long as $q \leq 1$.

2.1.2. The case of $q \leq 1$ and $\mu > 0$

According to Ref. [11], the expression of energy can be expressed as

$$E = \frac{3}{4}N\mu\left[1 + \frac{1}{(1-q)\beta\mu}\right]\frac{(2-q)(3-2q)x^2 - 4(3-2q)x - 12(1-q)(2-q)}{(2-q)(3-2q)x^2 - 3(3-2q)x - 6(1-q)(2-q)}. \quad (4)$$

Considering $x = 1 + (1-q)\beta\mu \gg 1$ and $\beta = 1/KT$, the pressure can be shown as

$$P = \frac{1}{4}n\mu\left[1 + \frac{1}{(1-q)\beta\mu}\right]\frac{(2-q)(3-2q)x^2 - 4(3-2q)x - 12(1-q)(2-q)}{(2-q)(3-2q)x^2 - 3(3-2q)x - 6(1-q)(2-q)} \approx \frac{1}{4}n\mu\left[1 + \frac{1}{(1-q)\beta\mu}\right], \quad (5)$$

and the stability condition is

$$\frac{\partial P}{\partial V} = -\frac{1}{4}\frac{n\mu}{V}\left[1 + \frac{1}{(1-q)\beta\mu}\right] + \frac{1}{4}n\frac{\partial\mu}{\partial V} < 0. \quad (6)$$

From Ref. [11], we can have the following expressions:

$$\frac{3n}{8g\pi}(hc)^3 \approx \mu^3\left[1 + \frac{3}{2-q}\frac{KT}{\mu}\right], \quad (7)$$

$$\mu \approx \mu_0\left[1 - \frac{1}{2-q}\frac{KT}{\mu_0}\right], \quad (8)$$

$$\mu_0 = hc\left(\frac{3n}{8g\pi}\right)^{1/3}. \quad (9)$$

In such a case, the stability condition is

$$\frac{\partial P}{\partial V} = -\frac{n}{12V}\left[4\mu_0 + 3KT\left(\frac{1}{1-q} - \frac{1}{2-q}\right)\right] < 0. \quad (10)$$

Considering $q \leq 1$, we can have $1/(1-q) - 1/(2-q) > 0$, thus $\partial P/\partial V < 0$ can be established when q is approximately equal to 1 and lower than 1.

2.1.3. The case of $q \geq 1$ and $\mu < 0$

According to Ref. [11], the energy can be written as

$$E = 3NKT\frac{\alpha_{-4}}{\alpha_{-3}}\frac{1}{q-1} \times \left[\frac{n(q-1)^3\alpha_0}{8g\pi\alpha_{-3}}\left(\frac{hc}{KT}\right)^3\right]^{(1-q)/(4-3q)}, \quad (11)$$

thus the pressure of the system is

$$P = n^{(5-4q)/(4-3q)}KT\frac{\alpha_{-4}}{\alpha_{-3}}\frac{1}{q-1} \times \left[\frac{(q-1)^3\alpha_0}{8g\pi\alpha_{-3}}\left(\frac{hc}{KT}\right)^3\right]^{(1-q)/(4-3q)}. \quad (12)$$

The stability condition is

$$\frac{\partial P}{\partial V} = -\frac{5-4q}{4-3q}n^{(5-4q)/(4-3q)}\frac{KT}{V}\frac{\alpha_{-4}}{\alpha_{-3}}\frac{1}{q-1} \times \left[\frac{(q-1)^3\alpha_0}{8g\pi\alpha_{-3}}\left(\frac{hc}{KT}\right)^3\right]^{(1-q)/(4-3q)} < 0, \quad (13)$$

where $\alpha_{-i} = \Gamma\left(1/(q-1) - \mathrm{i}\right)$. From expression (13), we can obtain $\partial P/\partial V > 0$ under the condition of $5/4 < q < 4/3$, namely, this system is instable.

2.1.4. The case of $1 < q < 5/4$ and $0 < \mu < 1/\beta(q-1)$

According to Ref. [11], we can have the following expression:

$$f_{q,D} \approx \frac{Z_q^{(1-q)D}}{(q-1)^D}\frac{(Z_q^{q-1}-1)^D}{D} + \frac{Z_q^{1+(1-q)D}}{(q-1)^D}$$

$$\times \frac{\left[1-\dfrac{1-D(q-1)}{q-1}\dfrac{1}{1-Z_q^{q-1}}\dfrac{1}{-D+q(D-1)}\right]}{\dfrac{1}{q-1}-D}. \tag{14}$$

Considering $(q-1)\beta\mu \sim 1$, we can have

$$\frac{f_{q,4}}{f_{q,3}} \approx \frac{x}{q-1}\frac{1-3(q-1)}{1-4(q-1)} = \left[\frac{1}{q-1}-\beta\mu\right]\frac{4-3q}{5-4q}. \tag{15}$$

Then the energy of the system can be written as

$$E = \frac{4-3q}{5-4q}N\left[\frac{KT}{q-1}-\mu\right], \tag{16}$$

the pressure of the system is

$$P = \frac{4-3q}{3(5-4q)}n\left[\frac{KT}{q-1}-\mu\right], \tag{17}$$

and the stability condition is

$$\frac{\partial P}{\partial V} = -\frac{4-3q}{3(5-4q)}n\frac{1}{V}\left(\frac{KT}{q-1}-\mu\right) \times \left(1+\frac{1-q}{4-3q}\right). \tag{18}$$

For the chemical potential of the system, we can have

$$\mu \approx KT \times \frac{1-\left[\dfrac{nh^3}{8g\pi}\left(\dfrac{c}{KT}\right)^3(q-1)^2(4-3q)\right]^{(1-q)/(4-3q)}}{q-1}, \tag{19}$$

$$\frac{\partial \mu}{\partial V} = -\frac{1-q}{(4-3q)\,V}\left(\mu-\frac{kT}{1-q}\right). \tag{20}$$

Thus the stability condition of the system can be expressed as

$$\frac{\partial P}{\partial V} = -\frac{4-3q}{3(5-4q)}\frac{n}{V}\left(\frac{KT}{q-1}-\mu\right) \times \left(1+\frac{1-q}{4-3q}\right) < 0. \tag{21}$$

In such a case, from the condition of $\mu < 1/\beta(q-1)$, we can have $(KT/(q-1)-\mu) > 0$. Thus from expression (21), the system can be stable, if the limit $1 < q < 5/4$ holds.

2.2. Stability condition of the extensive extreme relativity system

Under the high temperature, the pressure of an extensive extreme relativity system, according to Ref. [16], can be written as

$$P \approx nKT\left[1+\frac{z_0}{16}\right], \tag{22}$$

where

$$z_0 = \frac{nh^3}{64g\pi}\left(\frac{c}{KT}\right)^3.$$

So the stability condition of the system can be expressed as

$$\frac{\partial P}{\partial V} = -\frac{n}{V}KT\left[1+\frac{nh^3}{8\times 64g\pi}\left(\frac{ch}{KT}\right)^3\right] < 0. \tag{23}$$

Under low temperature, the pressure of the extensive extreme relativistic system, also according to Ref. [16] can be obtained as

$$P = \frac{ch}{4\sqrt{\pi}}\left(\frac{3}{4g\pi}\right)^{1/3}n^{4/3} \times \left[1+\frac{2\pi^2}{3}\left(\frac{4g\pi}{3n}\right)^{2/3}\left(\frac{KT}{ch}\right)^2\right], \tag{24}$$

and the stability condition of the system can be expressed as

$$\frac{\partial P}{\partial V} = -\frac{ch}{3\sqrt{\pi}}\left(\frac{3}{4g\pi}\right)^{1/3}\frac{n^{4/3}}{V} \times \left[1+\frac{\pi^2}{3}\left(\frac{4g\pi}{3n}\right)^{2/3}\left(\frac{KT}{ch}\right)^2\right] < 0. \tag{25}$$

2.3. Stability condition of the non-extensive non-relativistic system at high temperature

For the non-extensive system of non-relativity at high temperature, considering the conditions of $q < 1$, $\mu < 0$, and $(q-1)\beta\mu < 1$, from Ref. [16], we can have

$$N = \frac{gV}{h^3}\pi^{3/2}(2mKT)^{3/2}f_{3/2}(Z_q), \tag{26}$$

$$E = \frac{3}{2}NKTf_{5/2}(Z_q)/f_{3/2}(Z_q), \tag{27}$$

$$\mu = \frac{\left[\frac{n}{g}\frac{h^3}{(2\pi mkT)^{3/2}}\frac{\alpha_0}{\alpha_{-3/2}}(q-1)^{3/2}\right]^{2(1-q)/(5-3q)} - 1}{1-q} KT, \tag{28}$$

where

$$\alpha_{-\mathrm{i}} = \Gamma\left(\frac{1}{1-q} - \mathrm{i}\right).$$

Supposing

$$f_{q,D}(Z_q) \approx \frac{Z_q^{1+D(1-q)}}{(1-q)^D}\frac{\Gamma\left(\frac{1}{1-q}+1\right)}{\Gamma\left(\frac{1}{1-q}+D+1\right)},$$

the pressure of the system is

$$P = \frac{2}{3}\frac{E}{V} = nKT\left(\frac{1}{1-q} + \beta\mu\right)\frac{\alpha_{5/2}}{\alpha_{7/2}}, \tag{29}$$

where

$$\alpha_i = \Gamma\left(\frac{1}{1-q} + \mathrm{i}\right),$$

and the stability condition is

$$\frac{\partial P}{\partial V} = -\frac{n}{V}KT\frac{\alpha_{5/2}}{\alpha_{7/2}}\frac{7-5q}{5-3q}\left(\frac{1}{1-q} + \frac{\mu}{KT}\right) < 0. \tag{30}$$

Considering the condition $(q-1)\beta\mu < 1$, we can know

$$\frac{1}{1-q} + \frac{\mu}{KT} > 0.$$

Thus, under the conditions discussed here, $\partial P/\partial V < 0$ holds true.

Considering $q > 1$ and $\mu < 0$, according to Ref. [16], the particle number and the energy of the system are the same as expressions (26) and (27), respectively. However, the variance is

$$f_{q,D}(Z_q) \approx \frac{Z_q^{1-D(q-1)}}{(q-1)^D}\frac{\Gamma\left(\frac{1}{q-1} - D\right)}{\Gamma\left(\frac{1}{q-1}\right)}.$$

In such a case, the pressure of the system can be expressed as

$$P = \frac{2}{3}\frac{E}{V} = -nKT\left(\frac{1}{1-q} + \beta\mu\right)\frac{\alpha_{-5/2}}{\alpha_{-3/2}}, \tag{31}$$

and the condition of the system stability can be written as

$$\frac{\partial P}{\partial V} = \frac{n}{V}KT\frac{\alpha_{-5/2}}{\alpha_{-3/2}}\frac{7-5q}{5-3q}\left(\frac{1}{1-q} + \frac{\mu}{KT}\right) < 0. \tag{32}$$

With the condition of $q > 1$ and $\mu < 0$, we can find

$$\frac{1}{1-q} + \frac{\mu}{KT} < 0.$$

Assume that $7/5 < q < 5/3$, then we will obtain $\partial P/\partial V > 0$, namely, this system is instable.

3. Influences of temperature, parameter q, and relativistic effects

Supposing that the particle number density $n = 10^{15}$ and $V = 1$, from expressions (3), (13), and (23), we can obtain the results as shown in Fig. 1.

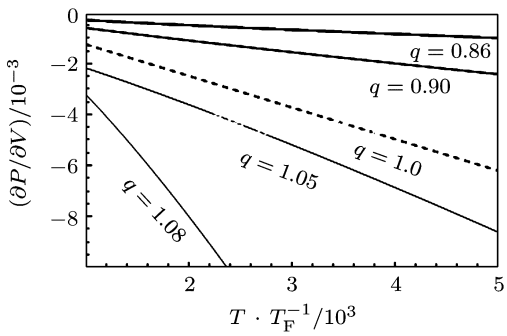


Fig. 1. Stabilities of non-extensive system varying with temperature (in the case of high temperature).

And from expressions (6), (21), and (25), we can have the results as exhibited in Fig. 2.

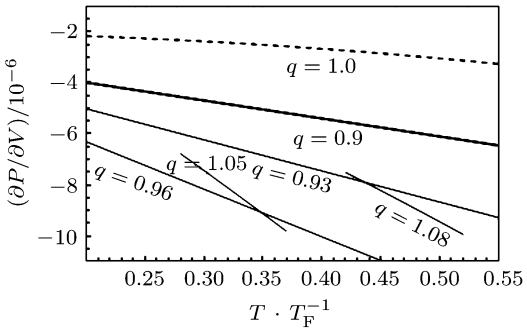


Fig. 2. Stabilities of non-extensive system varying with temperature (in the case of low temperature).

Under high temperature, to compare the stability of the relativistic system with that of the non-relativistic system, according to expressions (3), (13), (30), and (32), we can have the results as shown in Figs. 3 and 4, in which the "non" refers to the non-relativistic system and the "re" denotes the relativistic system.

According to Fig. 1, under high temperature, different values of parameter q have obvious influences on the system stability. When $q < 1$, the stability of the non-extensive system is weaker than that of the extensive system. On the contrary, when $q > 1$, the stability of the non-extensive system is stronger than that of the extensive one. This figure also shows that the greater the parameter q, the stronger the system stability will be and that the stability changes more significantly with the increase of temperature. In addition, the variances of system stability caused by different values of parameter q can be amplified with the increase of temperature.

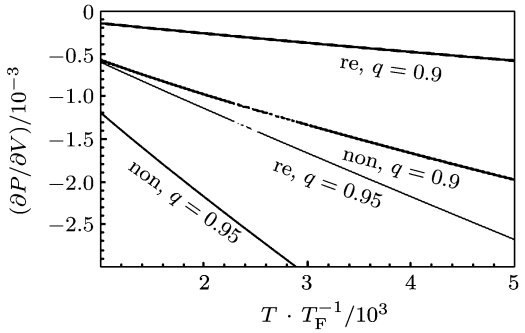


Fig. 3. Stabilities varying with temperature in the case of $q < 1$.

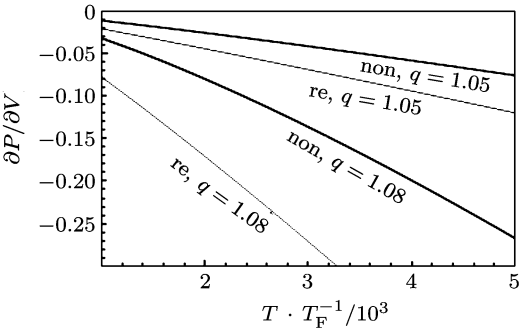


Fig. 4. Stabilities varying with temperature in the case of $q > 1$.

According to Fig. 2, under low temperature, different values of parameter q lead to much more complicated influence on the system stability than under high temperature. When $q < 1$, the stability of the non-extensive system is better than that of the extensive system. With temperature increasing, the difference between the influences caused by different values of q increases, which is consistent with the result at high temperature. In addition, the stabilities of $q > 1$ and $q < 1$ versus the temperature have turning points. When the temperature of the system is lower than the turning temperature, the system stability of $q < 1$ is greater than that of $q > 1$; however, when the temperature is higher than the turning temperature, the stability of the system with $q > 1$ is stronger than the other one. What is more, for the non-extensive system of $q > 1$, the turning temperature will rise with the value of q increasing.

Comparisons of stability between the relativistic system and the non-relativistic system in Figs. 3 and 4 reveal that in the case of high temperature, the relativistic effect can affect system stability, which implies that, the influence caused by the extreme relativistic effect can be controlled by the value of parameter q, particularly when $q < 1$, the extreme relativistic effect reduces the system stability. However, when $q > 1$, the extreme relativistic effect strengthens the system stability. In addition, with the increase of temperature, the influence of the extreme relativistic effect can be strengthened.

4. Conclusion

In this paper, the stability condition expressions under high temperature and low temperature by the theoretical analysis and the numerical simulation according to the statistic theory of the non-extensive extreme relativity are presented, and the mechanisms of the influences of temperature, ultra-relativistic effect, and non-extensive parameter q on system stability are also analysed. It is shown that at high temperature, when $q < 1$, the non-extensive system stability is inferior to that of the extensive system and the extreme relativistic effect will reduce the system stability compared with the non-relativistic situation; when $q > 1$, the stability of non-extensive system is stronger than that of the extensive system and the extreme relativistic effect will strengthen the system stability. However, under the condition of $5/4 < q < 4/3$, the extreme relativistic system is instable. In the case of low temperature, the stabilities of $q > 1$ and $q < 1$ have turning points. When the temperature of the system is lower than the turning temperature, the stability of the system with $q < 1$ is greater than that with $q > 1$. However, when the temperature is higher than the turning temperature, the stability of the system with $q > 1$ is greater than that with $q < 1$. In addition, for the non-extensive system of $q > 1$, the turning temperature rises with the value of q increasing.

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