Interactions between bright solitons in different species of Bose–Einstein condensates*

Jia-Ren Yan(颜家壬)† and Jie Zhou(周 杰)

Department of Physics, Hunan Normal University, Changsha 410081, *China*

(Received 7 November 2011; revised manuscript received 10 February 2012)

The dynamics of a bright–bright vector soliton in a cigar-shaped Bose–Einstein condensate trapping in a harmonic potential is studied. The interaction between bright solitons in different species with small separation is derived. Unlike the interaction between solitons of the same species, it is independent of the phase difference between solitons. It may be of attraction or repulsion. In the former case, each soliton will oscillate about and pass through each other around the mass-center of the system, which will also oscillate harmonically due to the harmonic trapping potential.

Keywords: interactions between bright solitons, Bose–Einstein condensates, dynamics of solitons

PACS: 03.75.Lm, 05.45.Yv, 42.81.Dp **DOI:** 10.1088/1674-1056/21/6/060304

1. Introduction

Bose–Einstein condensates (BECs) in ultra-cold gases provide an ideal test ground for nonlinear science. BECs of weakly interacting atoms offer fascinating possibilities for the study of nonlinear phenomena, as they are very pure samples of ultra-cold gases building up an effective macroscopic wave function of up to mm size. Among many features of nonlinear phenomena, the formation of solitons is perhaps one of the most interesting. In many physical fields, the interactions between solitons have great potential in a wide range of applications.[1] At temperatures close to absolute zero, the condensate can be accurately described by the Gross–Pitaevskii (GP) equation,[2] which has the form of a nonlinear Schrödinger (NLS) equation, where the nonlinear term arises from interactions between atoms. It is well known that NLS equation also appears in nonlinear optics and it supports soliton solutions.[3] Both bright and dark solitons have been observed in experiments of BECs as well as optical fibers. Bright solitons characterized as non-spreading matter-wave packets, have been observed in BECs with attractive interaction,[4−6] where they represented the ground state of the system. In a repulsively interacting condensate confined in a periodic potential, bright gap solitons have been realized[7,8] by modelling a suitable anomalous dispersion. Repulsive condensates in harmonic traps, however, can only support dark solitons, which correspond to propagating one-dimensional localized density minima. Dark solitons have been generated experimentally,[9−12] and their temperature properties have been studied theoretically in three-dimensional geometries, where they exhibit dynamical instabilities as those in one dimension.

Multi-component solitons also appear in BECs. Theoretical[13−15] and experimental[16,17] studies have shown that interspecies interaction plays crucial roles in these systems. Recently, two-species BECs with turnable interaction[18,19] motivates further exploration of two-species BECs. The mean-field dynamics of a two-species BEC in the limit of strong transverse confinement is governed by the following dimensionless coupled equation[20,21]

$$\begin{aligned} &\mathrm{i}\partial_t\psi_1 + \frac{1}{2}\partial_x^2\psi_1 - g_{11}|\psi_1|^2\psi_1 - g_{12}|\psi_2|^2\psi_1 = 0,\\ &\mathrm{i}\partial_t\psi_2 + \frac{\alpha}{2}\partial_x^2\psi_2 - g_{21}|\psi_1|^2\psi_2 - g_{22}|\psi_2|^2\psi_2 = 0, \end{aligned} \quad (1)$$

where x is the dimensionless longitudinal spatial variable measured in units of $a_{1\perp} = \sqrt{\hbar/m_1\omega_{1\perp}}$, with $\omega_{j\perp}$ being the transverse confining frequency for species j, t is the time measured in terms of $1/\omega_{1\perp}$, g_{ij} are constants proportional to a_{ij}:

$$g_{11} = \frac{2N_1}{a_{1\perp}}a_{11}\ , \quad g_{12} = g_{21} = \frac{2\gamma N_1}{a_{1\perp}}\frac{1+\alpha}{\gamma+\alpha}a_{12},$$

*Project supported by the National Natural Science Foundation of China (Grant No. 10775049) and the Natural Science Foundation of Hunan Province of China (Grant No. 09JJ6008).

†Corresponding author. E-mail: jiarenyan@hunnu.edu.cn

© 2012 Chinese Physical Society and IOP Publishing Ltd

$$g_{22} = \frac{2\gamma N_1}{a_{1\perp}} a_{22} \tag{2}$$

with $\alpha = m_1/m_2$ and $\gamma = \omega_{2\perp}/\omega_{1\perp}$, ψ_j is normalized such that $\int |\psi_1|^2 \mathrm{d}x = 1$ and $\int |\psi_2|^2 \mathrm{d}x = N_2/N_1$, with N_j being the particle number of species j. Since a_{12} and a_{jj} $(j = 1, 2)$ is the interspecies and intraspecies s-wave scattering length, respectively, we hereafter refer to g_{ij} also as the appropriate (s-wave) scattering length (measured in suitable units).

It was pointed out by Pérez-Garcia *et al.*[20] that multi-component BECs even support solitons which do not exist in single species. These solitons were referred to in Ref. [20] as "symbiotic solitons". For instance, robust bright–bright (BB) vector solitons exist in a two-component BECs constituting of two species of repulsively interacting condensates ($g_{jj} > 0$, $j = 1, 2$), provided the interspecies attraction is strong enough to overcome the self-repulsion of each atomic cloud ($g_{12} < 0$, $g_{12}^2 > g_{11}g_{22}$). In Ref. [20], collisions between different vector solitons were studied in terms of simulations. Starting from Eq. (1), Liu *et al.*[21] studied vector solutions in two-species BECs with a tunable interaction. It was pointed out that by turning the interspecies interaction via the Feshbach resonance technique,[18] different types (bright–bright, dark–dark, and bright–dark) of vector solitons (VS) can be realized and even transformed to each other.

In this paper, we shall study the existence and properties of interaction between components of a bright–bright (BB) vector soliton, namely, interaction between bright solitons in different species, with the help of the perturbation theories for NLS equation.[22,23] At first, we shall study the interaction between solitons with large separation in Sections 2. In this case the coupling terms are small and can be taken as perturbations. In Section 3, we study interaction between solitons with small separation. Since here the coupling is strong, the key is of course to find the correct perturbations. In the derivation of interactions, we have employed some integral formulas (Eqs. (15)–(17)) which were proved in Appendix A.

2. Interaction between solitons with large separation

We start our study from Eq. (1). Since the coupling between solitons with large separation is weak, in order to support BB solitons, we assume $g_{jj} < 0$ $(j = 1, 2)$. Letting $\psi_1 = \phi_1/\sqrt{|g_{11}|}$ and $\psi_2 = \sqrt{\kappa}\phi_2/\sqrt{|g_{22}|}$, equation (1) becomes

$$\begin{aligned} &\mathrm{i}\partial_t\phi_1 + \frac{1}{2}\partial_x^2\phi_1 + |\phi_1|^2\phi_1 = \mathrm{i}f_1[\phi_j], \\ &\mathrm{i}\partial_t\phi_2 + \frac{\alpha}{2}\partial_x^2\phi_2 + \alpha|\phi_2|^2\phi_2 = \mathrm{i}f_2[\phi_j], \end{aligned} \tag{3}$$

where

$$\mathrm{i}f_1[\phi_j] = \alpha\frac{g_{12}}{|g_{22}|}|\phi_2|^2\phi_1, \quad \mathrm{i}f_2[\phi_j] = \frac{g_{12}}{|g_{11}|}|\phi_1|^2\phi_2. \tag{4}$$

It is easy to check directly that if $f_1[\phi_j] = f_2[\phi_j] = 0$, equation (3) possesses the following BB vector soliton solution

$$\phi_{1\mathrm{B}} = \mathrm{e}^{\mathrm{i}\varphi_1} b \operatorname{sech}\theta_1, \quad \phi_{2\mathrm{B}} = \mathrm{e}^{\mathrm{i}\varphi_2} b \operatorname{sech}\theta_2, \tag{5}$$

where $\varphi_1 = c_1\theta_1/b + \delta_1$, $\varphi_2 = c_2\theta_2/\kappa b + \delta_2$, and $\theta_j = b(x - \xi_j)$. b, c_j, ξ_j, and δ_j are parameters of soliton, among which b and c_j are constants, and the time-dependence of ξ_j and δ_j are expressed by

$$\begin{aligned} &\partial_t\xi_1 = c_1, \quad \partial_t\delta_1 = \frac{1}{2}(b^2 + c_1^2), \\ &\partial_t\xi_2 = c_2, \quad \partial_t\delta_2 = \frac{1}{2}\left(\alpha b^2 + \frac{c_2^2}{\alpha}\right). \end{aligned} \tag{6}$$

Note that $\phi_{j\,\mathrm{B}}$ is strongly localized such that it differs clearly from zero only in the neighbors of $\theta_j = 0$ $(x = \xi_j)$. If the separation between solitons is large $(b|\xi_1 - \xi_2| \gg 1)$, $|\phi_{1\mathrm{B}}\ \phi_{2\mathrm{B}}|$ is small, and then the right-hand side of Eq. (3) $f_j[\phi_{i\,\mathrm{B}}]$ can be taken as perturbations. The standard perturbed NLS equation (the first equation in Eq. (3)) has been well studied in Refs. [22] and [23]. The perturbation affects the bright soliton in two respects. First, the soliton parameters b and c_1 are now slowly varying functions of time t instead of constants. Their time-dependence are given by Refs. [22] and [23]

$$\begin{aligned} \partial_t b &= \mathrm{Re}\int_{-\infty}^{\infty} \mathrm{d}\theta_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\mathrm{i}\varphi_1} f_1[\phi_{j\,\mathrm{B}}], \\ \partial_t c_1 &= \mathrm{Im}\int_{-\infty}^{\infty} \mathrm{d}\theta_1 \frac{\sinh\theta_1}{\cosh^2\theta_1}\, \mathrm{e}^{-\mathrm{i}\varphi_1} f_1[\phi_{j\,\mathrm{B}}], \end{aligned} \tag{7}$$

while the time-dependence of parameters ξ_1 and δ_1 are modified as

$$\begin{aligned} \partial_t\xi_1 &= c_1 + \frac{1}{b^2}\,\mathrm{Re}\int_{-\infty}^{\infty} \mathrm{d}\theta_1\theta_1 \operatorname{sech}\theta_1\, \mathrm{e}^{-\mathrm{i}\varphi_1} f_1[\phi_{j\,\mathrm{B}}], \\ \partial_t\delta_1 &= \frac{1}{2}(b^2 + c_1^2) + \frac{1}{b}\,\mathrm{Im}\int_{-\infty}^{\infty} \mathrm{d}\theta_1 \\ &\quad \times \operatorname{sech}\theta_1(1 - \theta_1\tanh\theta_1)\, \mathrm{e}^{-\mathrm{i}\varphi_1} f_1[\phi_{j\,\mathrm{B}}], \end{aligned} \tag{8}$$

instead of Eq. (6). Besides, perturbation will also give rise to a small correction on the form of soliton. But it is ignored here as it does not affect the interaction

essentially. As to the extended perturbed NLS equation (the second equation in Eq. (3)), a similar study leads to

$$\partial_t b = \mathrm{Re}\int_{-\infty}^{\infty} \mathrm{d}\theta_2 \,\mathrm{sech}\,\theta_2\, \mathrm{e}^{-\mathrm{i}\varphi_2} f_2[\phi_{j\,\mathrm{B}}],$$
$$\partial_t c_2 = \alpha\,\mathrm{Im}\int_{-\infty}^{\infty} \mathrm{d}\theta_2 \frac{\sinh\theta_2}{\cosh^2\theta_2}\,\mathrm{e}^{-\mathrm{i}\varphi_2} f_2[\phi_{j\,\mathrm{B}}], \quad (9)$$

and

$$\begin{aligned}\partial_t \xi_2 &= c_2 + \frac{1}{b^2}\,\mathrm{Re}\int_{-\infty}^{\infty} \mathrm{d}\theta_2\theta_2\,\mathrm{sech}\,\theta_2\,\mathrm{e}^{-\mathrm{i}\varphi_2} f_2[\phi_{j\,\mathrm{B}}],\\ \partial_t \delta_2 &= \frac{1}{2}\left(\alpha b^2 + \frac{c_2^2}{\alpha}\right) + \frac{1}{b}\,\mathrm{Im}\int_{-\infty}^{\infty}\mathrm{d}\theta_2\\ &\quad\times \mathrm{sech}\,\theta_2(1-\theta_2\tanh\theta_2)\,\mathrm{e}^{-\mathrm{i}\varphi_2} f_2[\phi_{j\,\mathrm{B}}]. \quad (10)\end{aligned}$$

Inserting the perturbation terms f_j given in Eq. (4) into Eqs. (7)–(10), after performing all the integrals, we obtain the time-dependence of all the soliton parameters below

$$\begin{aligned}\partial_t b &= 0, \quad (11)\\ \partial_t c_1 &= -\alpha b^3\frac{g_{12}}{|g_{22}|}\left(6\frac{\cosh\Delta}{\sinh^3\Delta}\right.\\ &\quad\left.-4\frac{\Delta}{\sinh^2\Delta}-6\frac{\Delta}{\sinh^4\Delta}\right),\\ \partial_t c_2 &= \alpha b^3\frac{g_{12}}{|g_{11}|}\left(6\frac{\cosh\Delta}{\sinh^3\Delta}\right.\\ &\quad\left.-4\frac{\Delta}{\sinh^2\Delta}-6\frac{\Delta}{\sinh^4\Delta}\right), \quad (12)\end{aligned}$$

where $\Delta = b(\xi_1-\xi_2)$ is proportional to the separation between solitons

$$\begin{aligned}\partial_t\xi_1 &= c_1, \quad \partial_t\xi_2 = c_2, \quad (13)\\ \partial_t\delta_1 &= \frac{1}{2}(b^2+c_1^2) - \alpha b^2\frac{g_{12}}{|g_{22}|}\left(6\frac{\Delta\cosh\Delta}{\sinh^3\Delta}\right.\\ &\quad\left.-\frac{3}{\sinh^2\Delta}-2\frac{\Delta^2}{\sinh^2\Delta}-3\frac{\Delta^2}{\sinh^4\Delta}\right),\\ \partial_t\delta_2 &= \frac{1}{2}(\alpha b^2+\frac{c_2^2}{\alpha}) - \alpha b^2\frac{g_{12}}{|g_{11}|}\left(6\frac{\Delta\cosh\Delta}{\sinh^3\Delta}\right.\\ &\quad\left.-\frac{3}{\sinh^2\Delta}-2\frac{\Delta^2}{\sinh^2\Delta}-3\frac{\Delta^2}{\sinh^4\Delta}\right). \quad (14)\end{aligned}$$

In the derivation of Eqs. (12) and (14), we have employed the following integral formulas which will be proved in Appendix A

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{sech}^2(x-x_1)\,\mathrm{sech}^2(x-x_2)\\ &= 4\frac{(x_1-x_2)\cosh(x_1-x_2)}{\sinh^3(x_1-x_2)} - \frac{4}{\sinh^2(x_1-x_2)}, \quad (15)\\ &\int_{-\infty}^{\infty}\mathrm{d}x\frac{\sinh(x-x_1)}{\cosh^3(x-x_1)}\,\mathrm{sech}^2(x-x_2)\\ &= 6\frac{\cosh(x_1-x_2)}{\sinh^3(x_1-x_2)} - 4\frac{x_1-x_2}{\sinh^2(x_1-x_2)}\\ &\quad -6\frac{x_1-x_2}{\sinh^4(x_1-x_2)}, \quad (16)\\ &\int_{-\infty}^{\infty}\mathrm{d}x\frac{(x-x_1)\sinh(x-x_1)}{\cosh^3(x-x_1)}\,\mathrm{sech}^2(x-x_2)\\ &= -2\frac{(x_1-x_2)\cosh(x_1-x_2)}{\sinh^3(x_1-x_2)}\\ &\quad -\frac{1}{\sinh^2(x_1-x_2)} + 2\frac{(x_1-x_2)^2}{\sinh^2(x_1-x_2)}\\ &\quad +3\frac{(x_1-x_2)^2}{\sinh^4(x_1-x_2)}. \quad (17)\end{aligned}$$

Combining Eqs. (12) and (13) leads to the Newton's dynamical equations of solitons

$$\begin{aligned}M_1\ddot{\xi}_1 &= -M_2\ddot{\xi}_2 = F(\Delta)\\ &= -b^3 g_{12}\left(6\frac{\cosh\Delta}{\sinh^3\Delta} - 4\frac{\Delta}{\sinh^2\Delta}\right.\\ &\quad\left.-6\frac{\Delta}{\sinh^4\Delta}\right), \quad (18)\end{aligned}$$

where an overdot stands for derivative respect to time t, the effective masses of solitons are

$$M_1 = |g_{22}|/\alpha\,, \quad M_2 = |g_{11}|/\alpha. \quad (19)$$

$F(\Delta)$ is the force acting on the first soliton by the other one. Equations (18) and (19) indicate that in the case of weak coupling ($\Delta \gg 1$), the intraspecies scattering length g_{jj} determines the effective masses M_j, while interaction $F(\Delta)$ is determined by the interspecies scattering length g_{12} alone. Introducing the interaction potential $V(\Delta)$, we can rewrite Eq. (18) as

$$\begin{aligned}M_j\ddot{\xi}_j &= -\partial_{\xi_j}V(\Delta),\\ V(\Delta) &= 2b^2 g_{12}\left(\frac{\Delta\cosh\Delta}{\sinh^3\Delta} - \frac{1}{\sinh^2\Delta}\right). \quad (20)\end{aligned}$$

It must be emphasized that the results obtained in this section hold only for the case of $|\Delta| \gg 1$. Under this condition, $V(\Delta)$ can be approximately expressed as

$$V(\Delta) \sim 8b^2 b_{12}|\Delta|\,\mathrm{e}^{-2|\Delta|}, \quad \text{for } |\Delta| \gg 1. \quad (21)$$

Equations (20) and (21) reveal that interaction between solitons in different species is over a short range. It attenuates exponentially with the separation between them. Obviously, when $g_{12} < 0$, solitons attract each other, and otherwise ($g_{12} > 0$) they repel. Moreover, the interaction $V(\Delta)$ is independent of the phase difference of soliton $\Delta\varphi = \varphi_1 - \varphi_2$. This is the most distinction from the interaction between solitons in the same species.

3. Interaction between solitons with small separation

We still start our study from Eq. (1). Following Ref. [21], we define two constants:

$$D_1 = \frac{g_{22} - \alpha g_{12}}{g_{12}^2 - g_{11}g_{22}}, \quad D_2 = \frac{\alpha g_{11} - g_{12}}{g_{12}^2 - g_{11}g_{22}}, \tag{22}$$

which are restricted by the following relations

$$g_{11}D_1 + g_{12}D_2 = -1, \quad g_{21}D_1 + g_{22}D_2 = -\alpha. \tag{23}$$

As was pointed out in Ref. [21] that equation (1) with arbitrary g_{ij} possesses different vector soliton solutions whose types are determined by D_j uniquely. For example, when $D_j > 0$ $(j = 1, 2)$, it possesses BB soliton solution. Without loss of generality, we assume that $\alpha g_{11} < g_{22}/\alpha$. Under this condition, turning the value (magnitude and sign) of g_{12} via the Feshbach resonance, we find that $D_j > 0$ holds in any one of the following cases.

Case 1 $-\sqrt{g_{11}g_{22}} < g_{12} < 0 < \alpha g_{11} < g_{22}/\alpha$.

Case 2 $g_{12} < \alpha g_{11} < 0 < g_{22}/\alpha$. (The BB solitons in Cases 1 and 2 have been studied in Refs. [20] and [21] respectively).

Case 3 If $\alpha g_{11} < g_{22}/\alpha < 0$, a careful analysis indicates that $D_j > 0$ holds, provided $g_{12} < \alpha g_{11}$ or $g_{22}/\alpha < g_{12} < \sqrt{g_{11}g_{22}}$.

Now we return to Eq. (1). If $D_j > 0$, letting $\psi_j = \sqrt{D_j}\phi_j$, and employing Eq. (23), we can rewrite Eq. (1) in the form

$$\begin{aligned} &\mathrm{i}\partial_t\phi_1 + \frac{1}{2}\partial_x^2\phi_1 + |\phi_1|^2\phi_1 \\ &= g_{12}D_2(|\phi_2|^2 - |\phi_1|^2)\phi_1\,, \\ &\mathrm{i}\partial_t\phi_2 + \frac{\alpha}{2}\partial_x^2\phi_2 + \alpha|\phi_2|^2\phi_2 \\ &= g_{12}D_1(|\phi_1|^2 - |\phi_2|^2)\phi_2. \end{aligned} \tag{24}$$

It is easy to check directly that equation (24) possesses the following BB soliton solution[21] (note that here the right-hand side of Eq. (24) vanishes)

$$\begin{aligned} \phi_{1\mathrm{B}}^0(x,t) &= b\,\mathrm{sech}[b(x-ct)]\exp\left[\mathrm{i}cx + \frac{\mathrm{i}}{2}(b^2 - c^2)t\right], \\ \phi_{2\mathrm{B}}^0(x,t) &= b\,\mathrm{sech}[b(x-ct)] \\ &\quad \times \exp\left[\mathrm{i}\frac{c}{\alpha}x + \frac{\mathrm{i}}{2}\left(\alpha b^2 - \frac{c^2}{\alpha}\right)t\right]. \end{aligned} \tag{25}$$

The BB soliton given in Eq. (25) is a special one in that the center of the two components are always overlapping, and there is no interaction between them. This can be understood as a certain equilibrium state. Therefore, to show the existence of interaction, we should slightly separate the two solitons from each other, namely, to give solitons different positions and velocities, so we still choose $\phi_{j\,\mathrm{B}}(x,t)$ (given by Eq. (5)) as the BB soliton solution instead of $\phi_{j\,\mathrm{B}}^0(x,t)$. Since $b(\xi_1 - \xi_2) \ll 1$, $|\phi_1|^2 - |\phi_2|^2$ is small, and then equations (24) are two independent perturbed NLS equations whose perturbation terms are

$$\begin{aligned} \mathrm{i}f_1'[\phi_j] &= g_{12}D_2(|\phi_2|^2 - |\phi_1|^2)\phi_1, \\ \mathrm{i}f_2'[\phi_j] &= g_{12}D_1(|\phi_1|^2 - |\phi_2|^2)\phi_2. \end{aligned} \tag{26}$$

Repeating the steps from Eq. (7) to Eq. (10) in the last section, we obtain the time-dependence of soliton parameters

$$\begin{aligned} \partial_t b &= \mathrm{Re}\int_{-\infty}^{\infty}\mathrm{d}\theta_1\ \mathrm{sech}\,\theta_1\,\mathrm{e}^{-\mathrm{i}\varphi_1}f_1'[\phi_{j\,\mathrm{B}}] = 0, \\ \partial_t c_1 &= \mathrm{Im}\int_{-\infty}^{\infty}\mathrm{d}\theta_1\frac{\sinh\theta_1}{\cosh^2\theta_1}\,\mathrm{e}^{-\mathrm{i}\varphi_1}f_1'[\phi_{j\,\mathrm{B}}] \\ &= -g_{12}D_2b^3\left(6\frac{\cosh\Delta}{\sinh^3\Delta} - 4\frac{\Delta}{\sinh^2\Delta}\right. \\ &\quad \left. - 6\frac{\Delta}{\sinh^4\Delta}\right); \end{aligned} \tag{27}$$

$$\begin{aligned} \partial_t\xi_1 &= c_1 + \frac{1}{b^2}\,\mathrm{Re}\int_{-\infty}^{\infty}\mathrm{d}\theta_1\theta_1\ \mathrm{sech}\,\theta_1\,\mathrm{e}^{-\mathrm{i}\varphi_1}f_1'[\phi_{j\,\mathrm{B}}] = c_1, \\ \partial_t\delta_1 &= \frac{1}{2}(b^2 + c_1^2) + \frac{1}{b}\,\mathrm{Im}\int_{-\infty}^{\infty}\mathrm{d}\theta_1 \\ &\quad \times \mathrm{sech}\,\theta_1(1 - \theta_1\tanh\theta_1)\,\mathrm{e}^{-\mathrm{i}\varphi_1}f_1'[\phi_{j\,\mathrm{B}}] \\ &= \frac{1}{2}(b^2 + c_1^2) - g_{12}D_2b^2\left(6\frac{\Delta\cosh\Delta}{\sinh^3\Delta} - \frac{3}{\sinh^2\Delta}\right. \\ &\quad \left. - 2\frac{\Delta^2}{\sinh^2\Delta} - 3\frac{\Delta^2}{\sinh^4\Delta}\right); \end{aligned} \tag{28}$$

$$\begin{aligned} \partial_t b &= \mathrm{Re}\int_{-\infty}^{\infty}\mathrm{d}\theta_2\ \mathrm{sech}\,\theta_2\,\mathrm{e}^{-\mathrm{i}\varphi_2}f_2'[\phi_{j\,\mathrm{B}}] = 0, \\ \partial_t c_2 &= \alpha\,\mathrm{Im}\int_{-\infty}^{\infty}\mathrm{d}\theta_2\frac{\sinh\theta_2}{\cosh^2\theta_2}\,\mathrm{e}^{-\mathrm{i}\varphi_2}f_2'[\phi_{j\,\mathrm{B}}] \\ &= \alpha g_{12}D_1b^3\left(6\frac{\cosh\Delta}{\sinh^3\Delta} - 4\frac{\Delta}{\sinh^2\Delta}\right. \\ &\quad \left. - 6\frac{\Delta}{\sinh^4\Delta}\right); \end{aligned} \tag{29}$$

and

$$\begin{aligned} \partial_t\xi_2 &= c_2 + \frac{1}{b^2}\,\mathrm{Re}\int_{-\infty}^{\infty}\mathrm{d}\theta_2\theta_2 \\ &\quad \times \mathrm{sech}\,\theta_2\,\mathrm{e}^{-\mathrm{i}\varphi_2}f_2'[\phi_{j\,\mathrm{B}}] = c_2, \\ \partial_t\delta_2 &= \frac{1}{2}\left(\alpha b^2 + \frac{c_2^2}{\alpha}\right) + \frac{1}{b}\,\mathrm{Im}\int_{-\infty}^{\infty}\mathrm{d}\theta_2 \\ &\quad \times \mathrm{sech}\,\theta_2(1 - \theta_2\tanh\theta_2)\,\mathrm{e}^{-\mathrm{i}\varphi_2}f_2'[\phi_{j\,\mathrm{B}}] \\ &= \frac{1}{2}\left(\alpha b^2 + \frac{c_2^2}{\alpha}\right) - g_{12}D_1b^2\left(6\frac{\Delta\cosh\Delta}{\sinh^3\Delta}\right. \end{aligned}$$

$$-\frac{3}{\sinh^2\Delta}-2\frac{\Delta^2}{\sinh^2\Delta}-3\frac{\Delta^2}{\sinh^4\Delta}\Big). \quad (30)$$

From Eqs. (27)–(30), we obtain the following dynamical equation

$$M_j'\ddot{\xi}_j=-\partial_{\xi_j}V(\Delta),$$
$$V(\Delta)=2b^2g_{12}\Big(\frac{\Delta\cosh\Delta}{\sinh^3\Delta}-\frac{1}{\sinh^2\Delta}\Big), \quad (31)$$

where M_1' and M_1' are effective masses of solitons:

$$M_1'=\frac{1}{D_2}=\frac{g_{12}^2-g_{11}g_{22}}{\alpha g_{11}-g_{12}},$$
$$M_2'=\frac{1}{\alpha D_1}=\frac{g_{12}^2-g_{11}g_{22}}{\alpha g_{22}-\alpha^2 g_{12}}. \quad (32)$$

Note that the interaction potential $V(\Delta)$ given in Eq. (31) is just the same of that in Eq. (20), but the effective masses of solitons are different ($M_j\neq M_j'$). We have seen now that, in the case of strong coupling ($\Delta\ll 1$), g_{12} not only determines the interaction $F(\Delta)$, but also affects the effective masses M_j'. Of course, equation (31) holds only when $|\Delta|\ll 1$. Under this condition, the interaction is approximately a quasi-elastic one as follows:

$$V(\Delta)\approx b^2g_{12}\Big(\frac{2}{3}-\frac{4}{15}\Delta^2\Big) \quad \text{for } |\Delta|\ll 1. \quad (33)$$

Inserting Eq. (33) into the first equation of Eq. (31), we obtain the Newton's dynamical equation governing the relative motion of solitons:

$$M'\ddot{\Delta}=\frac{8}{15}b^4g_{12}\Delta, \quad \text{for } |\Delta|\ll 1, \quad (34)$$

where M' is the reduced effective mass: $M'=M_1'M_2'/(M_1'+M_2')=(g_{12}^2-g_{11}g_{22})/[\alpha(g_{11}+g_{22})-(1+\alpha^2)g_{12}]$. Obviously, when $g_{12}<0$, the solitons will attract each other, and then they will oscillate slightly about and pass through each other around the mass center with frequency $\omega=\sqrt{8b^4|g_{12}|/15M'}$.

Up to now, we have studied interaction between solitons in two extreme cases: $|\Delta|\gg 1$ and $|\Delta|\ll 1$. The only difference between these different cases is the inequality of the effective mass $M_j'\neq M_j$. In order to study interaction applying to any Δ, we must assume $|g_{12}|\ll|g_{jj}|$ as well as $g_{jj}<0$. Under these assumptions, we can start our study from Eq. (3) or Eq. (24), whose right-hand side can be taken as perturbations simultaneously. Therefore, all the equations in the last and this sections still hold regardless of Δ. Neglecting g_{12} in Eq. (32), yields the expected results $M_1'=-g_{22}/\alpha=M_1$ and $M_2'=-g_{11}/\alpha=M_2$. In summary, under conditions: $g_{jj}<0$ and $g_{12}^2\ll g_{11}g_{22}$, equations (18)–(20) hold for any value of Δ.

4. Conclusion

In this paper, we have derived the interactions between bright solitons in different species with large separation ($|\Delta|\gg 1$) and small separation ($|\Delta|\ll 1$) by means of the perturbation theory for the NLS equation. In these cases, the expression of interaction potential is the same: $V(\Delta)=2b^2g_{12}\big(\Delta\cosh\Delta/\sinh^3\Delta-1/\sinh^2\Delta\big)$, but the effective mass of each soliton is different: $M_j\neq M_j'$.

It follows from the expression of $V(\Delta)$ that (i) the interaction is characterized by the interspecies s-wave scattering length g_{12}: solitons attract/repel each other as $g_{12}<0/g_{12}>0$. (ii) In the case of large separation ($|\Delta|\gg 1$), $V(\Delta)$ attenuates exponentially with Δ, and it is of short range. In the case of $|\Delta|\ll 1$, $V(\Delta)$ reduces to elastic potential. (iii) It must be emphasized that unlike the interaction between solitons in the same species, our $V(\Delta)$ is independent of the phase-difference of solitons $\Delta\varphi=\varphi_1-\varphi_2$.

In the special case of $g_{12}^2\ll g_{11}g_{22}$, the above expression for $V(\Delta)$ holds for any value of Δ. In this case, we have $M_j=M_j'$.

Appendix A: Derivation of some integral formulas

To derive Eqs. (15) and (16), we start from the calculation of the integral

$$I_1(x_1,x_2)=\int_{-\infty}^{\infty}\mathrm{d}x\tanh(x-x_1)\,\mathrm{sech}^2(x-x_2). \quad (A1)$$

Following the lines of Ref. [23], we consider a complex integral along a closed path c in plane $z=x+\mathrm{i}y$ as follows:

$$\oint_c f_1(z)\mathrm{d}z=\oint_c z\tanh(z-x_1)\,\mathrm{sech}^2(z-x_2)\mathrm{d}z. \quad (A2)$$

Since the factor $\tanh(z-x_1)\,\mathrm{sech}^2(z-x_2)$ is a periodic function with an imaginary period $\mathrm{i}\pi$, we choose c to be the boundary of a rectangular region infinitely long: $-\infty<x<\infty$, $0\leq y\leq\pi$. In this region the integrand is analytic except for a single pole $z_1=x_1+\mathrm{i}\pi/2$ and a pole of the second order $z_2=x_2+\mathrm{i}\pi/2$. We note that the integrand attenuates exponentially for large $|x|$. Then the integral along the two straight line segments: $x=\pm\infty$, $0\leq y\leq\pi$ should be zero, and it follows that

$$\oint_c f_1(z)\mathrm{d}z=-\mathrm{i}\pi I_1(x_1,x_2). \quad (A3)$$

On the other hand, according to the residue theorem

$$\oint_c f_1(z)\,\mathrm{d}z = 2\pi[\,a\mathrm{Res}\,f_1(z_1) + \mathrm{Res}\,f_1(z_2)], \quad \text{(A4)}$$

where $\mathrm{Res}\,f_1(z_j)$ stands for the residue of $f_1(z)$ at singularity z_j, $j = 1, 2$, which can be easily obtained by the standard method

$$\begin{aligned}\mathrm{Res}\,f_1(z_1) &= \lim_{z\to z_1}[(z-z_1)f_1(z)] \\ &= -\frac{x_1 + \mathrm{i}\pi/2}{\sinh^2(x_1-x_2)}, \\ \mathrm{Res}\,f_1(z_2) &= \lim_{z\to z_2}\frac{\mathrm{d}}{\mathrm{d}z}[z-z_2)^2 f_1(z)] \\ &= \frac{x_2 + \mathrm{i}\pi/2}{\sinh^2(x_1-x_2)} + \frac{\cosh(x_1-x_2)}{\sinh(x_1-x_2)}. \quad \text{(A5)}\end{aligned}$$

Substituting Eqs. (A4) and (A5) into Eq. (A3) leads to

$$\begin{aligned}I_1 &= \int_{-\infty}^{\infty}\mathrm{d}x\tanh(x-x_1)\,\mathrm{sech}^2(x-x_2) \\ &= 2\frac{x_1-x_2}{\sinh^2(x_1-x_2)} - 2\frac{\cosh(x_1-x_2)}{\sinh(x_1-x_2)}. \quad \text{(A6)}\end{aligned}$$

Differentiating Eq. (A6) with respect to x_1 gives Eq. (15)

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{sech}^2(x-x_1)\,\mathrm{sech}^2(x-x_2) \\ &= 4\frac{(x_1-x_2)\cosh(x_1-x_2)}{\sinh^3(x_1-x_2)} - \frac{4}{\sinh^2(x_1-x_2)}. \quad \text{(A7)}\end{aligned}$$

Differentiating Eq. (A7) with respect to x_1, we obtain Eq. (16) immediately

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{\sinh(x-x_1)}{\cosh^3(x-x_1)}\,\mathrm{sech}^2(x-x_2) \\ &= 6\frac{\cosh(x_1-x_2)}{\sinh^3(x_1-x_2)} - 4\frac{x_1-x_2}{\sinh^2(x_1-x_2)} \\ &\quad - 6\frac{x_1-x_2}{\sinh^4(x_1-x_2)}. \quad \text{(A8)}\end{aligned}$$

To prove Eq. (17), we start from the calculation of

$$I_2(x_1,x_2) = \int_{-\infty}^{\infty}\mathrm{d}x\,x\tanh(x-x_1)\,\mathrm{sech}^2(x-x_2). \quad \text{(A9)}$$

Similarly, let us consider the following complex integral

$$\oint_c f_2(z)\,\mathrm{d}z = \oint_c z^2\tanh(z-x_1)\,\mathrm{sech}^2(z-x_2)\,\mathrm{d}z, \quad \text{(A10)}$$

which can be reduced to $\oint_c f_2(z)\,\mathrm{d}z = -2\pi\mathrm{i}I_2 + \pi^2 I_1$. It follows that

$$I_2(x_1,x_2) = -\mathrm{i}\frac{\pi}{2}I_1 - [\,\mathrm{Res}\,f_2(z_1) + \mathrm{Res}\,f_2(z_2)], \quad \text{(A11)}$$

where

$$\begin{aligned}\mathrm{Res}\,f_2(z_1) &= \lim_{z\to z_1}[(z-z_1)f_2(z)] = -\frac{(x_1+\mathrm{i}\pi/2)^2}{\sinh^2(x_1-x_2)}, \\ \mathrm{Res}\,f_2(z_2) &= \lim_{z\to z_2}\frac{\mathrm{d}}{\mathrm{d}z}[z-z_2)^2 f_2(z)] \\ &= \frac{(x_2+\mathrm{i}\pi/2)^2}{\sinh^2(x_1-x_2)} \\ &\quad + 2(x_2+\mathrm{i}\pi/2)\frac{\cosh(x_1-x_2)}{\sinh(x_1-x_2)}. \quad \text{(A12)}\end{aligned}$$

Inserting Eqs. (A6) and (A12) into Eq. (A11) yields

$$\begin{aligned}I_2 &= \int_{-\infty}^{\infty}\mathrm{d}x\,x\tanh(x-x_1)\,\mathrm{sech}^2(x-x_2) \\ &= \frac{x_1^2-x_2^2}{\sinh^2(x_1-x_2)} - 2x_2\frac{\cosh(x_1-x_2)}{\sinh(x_1-x_2)}. \quad \text{(A13)}\end{aligned}$$

Differentiating Eq. (A13) with respect to x_1 twice leads to

$$\begin{aligned}-\frac{1}{2}\frac{\partial^2}{\partial x_1^2}I_2 &= \int_{-\infty}^{\infty}\mathrm{d}x\frac{x\,\sinh(x-x_1)}{\cosh^3(x-x_1)}\,\mathrm{sech}^2(x-x_2) \\ &= (4x_1+2x_2)\frac{\cosh(x_1-x_2)}{\sinh^3(x_1-x_2)} \\ &\quad - \frac{2x_1^2-2x_2^2+1}{\sinh^2(x_1-x_2)} - \frac{3(x_1^2-x_2^2)}{\sinh^4(x_1-x_2)}. \quad \text{(A14)}\end{aligned}$$

Finally, combining Eqs. (A8) and (A14), we obtain Eq. (17)

$$\begin{aligned}&\int_{-\infty}^{\infty}\mathrm{d}x\,\frac{(x-x_1)\sinh(x-x_1)}{\cosh^3(x-x_1)}\,\mathrm{sech}^2(x-x_2) \\ &= -2\frac{(x_1-x_2)\cosh(x_1-x_2)}{\sinh^3(x_1-x_2)} \\ &\quad - \frac{1}{\sinh^2(x_1-x_2)} + 2\frac{(x_1-x_2)^2}{\sinh^2(x_1-x_2)} \\ &\quad + 3\frac{(x_1-x_2)^2}{\sinh^4(x_1-x_2)}. \quad \text{(A15)}\end{aligned}$$

References

[1] Stegeman G I and Segev M 1999 *Science* **286** 1518 and references therein

[2] Pitaevskii L P and Stringari SW 2003 *Bose–Einstein Condensation* (Oxford: Clarendon Press)

[3] Kivshar Y S and Luther-Davies B 1998 *Phys. Rep.* **298** 81

[4] Strecker K E, Partridge G B, Truscott A G and Hulet R G 2002 *Nature* **417** 150

[5] Khaykovich L, Schreck1 F, Ferrari1 G, Bourdel1 T, Cubizolles1 J, Carr1 L D, Castin1 Y and Salomon1 C 2002 *Science* **296** 1290

[6] Cornish S W L, Thompson S T and Wieman C E 2006 *Phys. Rev. Lett.* **96** 170401
[7] Eiermann B, Treutlein P, Anker T, Albiez M, Taglieber M, Marzlin K P and Oberthaler M K 2003 *Phys. Rev. Lett.* **91** 060402
[8] Eiermann B, Anker T, Albiez M, Taglieber M, Treutlein P, Marzlin K P and Oberthaler M K 2004 *Phys. Rev. Lett.* **92** 230401
[9] Burger S, Bongs K, Dettmer S, Ertmer W, Sengstock K, Sanpera1 A, Shlyapnikov G V and Lewenstein M *Phys. Rev. Lett.* **83** 5198
[10] Denschlag J, Simsarian J E, Feder D L, Clark C W, Collins L A, Cubizolles J, Deng L, Hagley E W, Helmerson K, Reinhardt W P, Rolston S L, Schneider B I and Phillips W D 2001 *Science* **287** 97
[11] Dutton Z, Budde M, Slowe C and Hau L V 2001 *Science* **293** 663
[12] Anderson B P, Haljan P C, Regal C A, Feder D L, Collins L A, Clark C W and Cornell E A 2001 *Phys. Rev. Lett.* **86** 2926
[13] Ho T L and Shenoy V B 1996 *Phys. Rev. Lett.* **77** 3276
[14] Esry B D, Green C H, Burke J P and Bohn J L 1997 *Phys. Rev. Lett.* **78** 3594
[15] Pu H and Bigelow N P 1998 *Phys. Rev. Lett.* **80** 1130
[16] Myatt C J, Burt E A, Ghrist R W, Cornell E A and Wieman C E 1997 *Phys. Rev. Lett.* **78** 586
[17] Modugno G, Ferrari G, Roati G, Brecha R J, Simoni A and Inguscio M 2001 *Science* **294**, 1320
[18] Thalhammer G, Barontini G, De Sarlo L, Catani J, Minardi F and Inguscio M 2008 *Phys. Rev. Lett.* **100** 210402
[19] Papp S B, Pino J M and Wieman C E 2008 *Phys. Rev. Lett.* **101** 040402
[20] Pérez-Garcia V M and Beitia J B 2005 *Phys. Rev.* A **72** 033620
[21] Liu X, Pu H, Xiong B, Liu W M and Gong J B 2009 *Phys. Rev.* A **79** 013423
[22] Kaup D J 1990 *Phys. Rev.* A **42** 5689
[23] Yan J R, Tang Y, Zhou G H and Chen Z H 1008 P*hys. Rev.* E **58**1064