Fractional backward Kolmogorov equations*

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This paper derives the fractional backward Kolmogorov equations in fractal space–time based on the construction of a model for dynamic trajectories. It shows that for the type of fractional backward Kolmogorov equation in the fractal time whose coefficient functions are independent of time, its solution is equal to the transfer probability density function of the subordinated process $X(S_\alpha(t))$, the subordinator $S_\alpha(t)$ is termed as the inverse-time α-stable subordinator and the process $X(\tau)$ satisfies the corresponding time homogeneous Itô stochastic differential equation.

Keywords: anomalous diffusive, fractional backward Kolmogorov equations, subordinated process

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1. Introduction

Over the past decades the study of anomalous diffusion has attracted growing attention in various fields of physics and related sciences.[1−3] There are two popular and widely used stochastic models to describe the conventional behaviour of anomalous diffusion. The first of them is the fractional Brownian motion (FBM), introduced by Kolmogorov in 1940, which is a Gaussian process with stationary increments and with self-similarity properties.[4,5] Long memory FBM is neither a semimartingale (except when the Hurst parameter $H = 1/2$) nor a Markov process. However, it is closely connected with fractional calculus and has been widely used in many fields.

The second popular model of anomalous diffusion, the continuous time random walk (CTRW), considers random walks in space and time by means of independent and identically distributed (iid) couples of space and time random steps, which was introduced by Montroll and Weiss in 1965 and developed in a number of succeeding publications.[6] The stochastic behavior of the CTRWs is just a semimartingale. Many commonly used fractional kinetic equations of the diffusion, diffusion–advection, and Fokker–Planck type are derived asymptotically from basic random walk models, and from a generalized master equation as a complementary tool in the description of transport dynamics in complex systems which are governed by anomalous diffusion and non-exponential relaxation patterns. It should be noted that these fractional equations are based on the construction of random wandering process rather than on the construction of a model for dynamic trajectories.

From CTRWs, the three types of fractional anomalous subdiffusive dynamics models in the presence of different force fields; time-independent force field $F(x)$, space-independent force field $F(t)$, and time- and space-dependent force field $F(x,t)$, have been derived using a generalized master equation.[7] These fractional anomalous subdiffusive dynamics models are studied to describe the evolution of the probability density function $p(x,t)$ for subdiffusion.[8]

As we know, the temporal and spatial random steps of CTRW are discrete. By passing from discrete steps to a continuous limit, the CTRW description will lead to the subordination of one random process by another. In 2007, Magdziarz *et al.*[9,10] proved that the solution $p(x,t)$ of the fractional anomalous subdiffusive dynamics models in the presence of time-independent force field $F(x)$, is equal to the probability density function (PDF) of the subordinated process

$$Z(t) = X(S_\alpha(t)), \tag{1}$$

using the Laplace transform method, where the process $X(\tau)$ is the solution of the following Itô stochastic differential equation

$$\mathrm{d}X(\tau) = F[X(\tau)]\eta_\alpha^{-1}\mathrm{d}\tau + (2\kappa_\alpha)^{1/2}\mathrm{d}B(\tau) \tag{2}$$

driven by the standard Brownian motion $B(\tau)$, which is assumed to be independent of the process $S_\alpha(t)$.

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The subordinator $S_\alpha(t)$ is termed as the inverse-time α-stable subordinator. It is defined as

$$S_\alpha(t) = \inf\{\tau : U(\tau) > t\}, \tag{3}$$

where $U(\tau)$ denotes a strictly increasing α-stable Lévy motion,[11,12] i.e., an α-stable process with Laplace transform $\langle \mathrm{e}^{-kU(\tau)}\rangle = \mathrm{e}^{-\tau k^\alpha}$.

Two years later, Magdziarz constructed a stochastic process:[9,13]

$$Z(t) = \int_0^t F(u)\mathrm{d}(S_\alpha(u)) + B(S_\alpha(t)), \quad t > 0, \tag{4}$$

whose probability density function is the solution of the fractional Fokker–Planck equation in the presence of space-independent force field $F(t)$ by showing that their moments coincide.

In 2010, Henry *et al.*[7,14] derived a fractional Fokker–Planck equation for subdiffusion in a general space- and time-dependent force field from power-law waiting time continuous-time random walks biased by Boltzmann weights. The governing equation is derived from a generalized master equation and is shown to be equivalent to the corresponding stochastic Langevin equation for the subordinated process $Z(t) = X(S_\alpha(t))$.

There are other fractional kinetics introduced by Zaslavsky to describe random wandering in the fractal space–time based on the construction of a model for dynamic trajectories:[15,16]

$$\begin{aligned}\frac{\partial^\alpha p(x,t)}{\partial t^\alpha} &= \frac{\partial^\beta}{\partial(-x)^\beta}(\mathscr{A}(x)p(x,t)) \\ &\quad + \frac{\partial^{\beta_1}}{\partial(-x)^{\beta_1}}(\mathscr{B}(x)p(x,t)),\end{aligned} \tag{5}$$

where $\mathscr{A}(x), \mathscr{B}(x)$ are the coefficient functions, the orders $\alpha \in (0,1]$, β, $\beta_1 \in (0,2]$, $\beta < \beta_1$, and the operators $\partial^\alpha/\partial t^\alpha$, $\partial^\alpha/\partial(-t)^\alpha$ for arbitrary α are the Riemann–Liouville integrals and derivatives:

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t \frac{f(t')\mathrm{d}t'}{(t-t')^{\alpha-n+1}}, \tag{6}$$

$$\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}(-t)^\alpha} = \frac{-1}{\Gamma(n-\alpha)}\frac{\mathrm{d}^n}{\mathrm{d}(-t)^n}\int_0^t \frac{f(t')\mathrm{d}t'}{(t'-t)^{\alpha-n+1}}, \tag{7}$$

here n is an integer part of $\alpha > 0$. The fractional kinetics were successfully used to describe anomalous transport in systems with Hamiltonian chaos.

It is known to all that the backward Kolmogorov equations are widely used in many fields, such as the first-passage problem, the stochastic optimal control of dynamical systems, and economics, which are commonly given on the terminal hour conditions.

In this article, we shall study the fractional backward Kolmogorov equations which are derived on the base of the construction of a model for dynamic trajectories in the fractal space–time. In particular, we shall show that the type of the fractional backward Kolmogorov equations in the fractal time:

$$\begin{aligned}&\frac{\partial p(x,t;x_0,t_0)}{\partial t_0} \\ &= {}_0D_{t_0}^{1-\alpha}\left[-a(x_0)\frac{\partial}{\partial x_0} - \frac{1}{2}b(x_0)\frac{\partial^2}{\partial x_0^2}\right]p(x,t;x_0,t_0),\end{aligned} \tag{8}$$

whose coefficient functions are independent of time t_0, where

$${}_0D_{t_0}^{1-\alpha} f(t_0) = \frac{\mathrm{d}}{\mathrm{d}t_0} I_{t_0}^\alpha f(t_0)$$

and

$${}_0I_{t_0}^\alpha f(t_0) = \frac{1}{\Gamma(\alpha)}\frac{\mathrm{d}}{\mathrm{d}t_0}\int_0^{t_0}\frac{f(t')\mathrm{d}t'}{(t_0-t')^{1-\alpha}} \tag{9}$$

is also the Riemann–Liouville fractional integral of order $\alpha \in (0,1]$, to be equivalent to the time homogeneous subordinated Itô stochastic differential equation for the subordinated process $Z(t) = X(S_\alpha(t))$:

$$\mathrm{d}X(\tau) = m(X(\tau))\mathrm{d}\tau + \sigma(X(\tau))\mathrm{d}B(\tau) \tag{10}$$

driven by the standard Brownian motion $B(\tau)$, which means the solution $p(x,t;x_0,t_0)$ of Eq. (8) is equal to the transfer probability density function of the subordinated process $Z(t) = X(S_\alpha(t))$.

2. Fractional backward Kolmogorov equations

Let $p(x,t;x',t')$ be the probability density of having a particle at the position x at the time t if at the time $t' \le t$ the particle was at the point x'. The Markov process equation can be written in the form

$$\begin{aligned}&p(x_3,t_3;x_1,t_1) \\ &= \int P(x_3,t_3;x_2,t_2)P(x_2,t_2;x_1,t_1)\mathrm{d}x_2.\end{aligned} \tag{11}$$

In the derivation of FPK equations, the probability density $p(x,t;x_0,t_0)$ can be viewed as the function of variables x,t and parameters x_0, t_0. To the contrary, in this paper we can treat the probability density $p(x,t;x_0,t_0)$ as the function of variables x_0,t_0 and parameters x,t, which satisfies the (fractional) backward Kolmogorov equation.

Now we consider the evolution of $p(x,t;x_0,t_0)$ during an infinitesimal time Δt_0 and use the expansion

$$\begin{aligned} &p(x,t;x_0,t_0-\Delta t_0)\\ &= p(x,t;x_0,t_0)-\frac{\partial p}{\partial t_0}\Delta t_0+O((\Delta t_0)^2). \end{aligned} \tag{12}$$

Equation (12) is valid providing the limit:

$$\begin{aligned} &\lim_{\Delta t_0\to 0}\frac{1}{\Delta t_0}[p(x,t;x_0,t_0)\\ &-p(x,t;x_0,t_0-\Delta t_0)]=\frac{\partial p}{\partial t_0} \end{aligned} \tag{13}$$

has a sense.

The distribution $p(x,t;x_0,t_0-\Delta t_0)$ is called the transfer probability and it satisfies the normalization condition:

$$\int p(y,t_0;x_0,t_0-\Delta t_0)\,\mathrm{d}y=1, \tag{14}$$

and the chain equation of the Markov-type process:

$$\begin{aligned} &p(x,t;x_0,t_0-\Delta t_0)\\ &=\int p(x,t;y,t_0)p(y,t_0;x_0,t_0-\Delta t_0)\,\mathrm{d}y. \end{aligned} \tag{15}$$

Therefore equation (13) can be written as

$$\begin{aligned} \frac{\partial p}{\partial t_0} &= \lim_{\Delta t_0\to 0}\frac{1}{\Delta t_0}\int[p(x,t;x_0,t_0)-p(x,t;y,t_0)]\\ &\times p(y,t_0;x_0,t_0-\Delta t_0)\,\mathrm{d}y. \end{aligned} \tag{16}$$

The probability density $p(x,t;y,t_0)$ viewed as the function of variables of y can be the Taylor series expansion:

$$\begin{aligned} p(x,t;y,t_0) &= p(x,t;x_0,t_0)+(y-x_0)\frac{\partial p(x,t;x_0,t_0)}{\partial x_0}\\ &+\frac{1}{2}(y-x_0)^2\frac{\partial^2 p(x,t;x_0,t_0)}{\partial x_0^2}\\ &+O((y-x_0)^3). \end{aligned} \tag{17}$$

Using the Kolmogorov conditions, there exist the limits

$$a(x_0,t_0)\equiv\lim_{\Delta t_0\to 0}\frac{1}{\Delta t_0}\int(y-x_0)p(y,t_0;x_0,t_0-\Delta t_0)\,\mathrm{d}y, \tag{18}$$

$$b(x_0,t_0)\equiv\lim_{\Delta t_0\to 0}\frac{1}{\Delta t_0}\int(y-x_0)^2p(y,t_0;x_0,t_0-\Delta t_0)\,\mathrm{d}y, \tag{19}$$

$$\lim_{\Delta t_0\to 0}\frac{1}{\Delta t_0}\int(y-x_0)^m p(y,t_0;x_0,t_0-\Delta t_0)\,\mathrm{d}y=0\quad(m\geq 3), \tag{20}$$

and substituting Eq. (17) into Eq. (16) gives

$$\begin{aligned} \frac{\partial p(x,t;x_0,t_0)}{\partial t_0} &= -a(x_0,t_0)\frac{\partial p(x,t;x_0,t_0)}{\partial x_0}\\ &-\frac{1}{2}b(x_0,t_0)\frac{\partial^2 p(x,t;x_0,t_0)}{\partial x_0^2}, \end{aligned} \tag{21}$$

which is called the backward Kolmogorov equations.

To introduce a process of wandering along a fractal space–time set $\{x_0,t_0\}$, we consider a new expansion of the probability density $p(x,t;x_0,t_0)$ instead of Eq. (12)

$$\begin{aligned} &p(x,t;x_0,t_0-\Delta t_0)-p(x,t;x_0,t_0)\\ &=-\frac{\partial^\alpha p}{\partial t_0^\alpha}(\Delta t_0)^\alpha+O((\Delta t_0)^{\alpha_1}),\quad \alpha_1>\alpha, \end{aligned} \tag{22}$$

where the derivative of order α $(0<\alpha\leq 1)$ is the Riemann–Liouville fractional calculus.

The main feature of Eq. (22) is that in the limit $\Delta t_0\to 0$ the right-hand side is proportional to $(\Delta t_0)^\alpha$ and it shows a singular behavior, thus there exits the limit:

$$\begin{aligned} &\lim_{\Delta t_0\to 0}\frac{1}{(\Delta t_0)^\alpha}[p(x,t;x_0,t_0)\\ &-p(x,t;x_0,t_0-\Delta t_0)]=\frac{\partial^\alpha p}{\partial t_0^\alpha}. \end{aligned} \tag{23}$$

With the help of Eqs. (14) and (15), we can transform Eq. (23) into

$$\begin{aligned} \frac{\partial^\alpha p}{\partial t_0^\alpha} &= \lim_{\Delta t_0\to 0}\frac{1}{\Delta t_0^\alpha}\int[p(x,t;x_0,t_0)-p(x,t;y,t_0)]\\ &\times p(y,t_0;x_0,t_0-\Delta t_0)\,\mathrm{d}y. \end{aligned} \tag{24}$$

In analogy to expansion (17) the probability density $p(x,t;y,t_0)$ for an infinitesimal $|y-x_0|$ fractional generalization can be considered

$$\begin{aligned} &p(x,t;y,t_0)\\ &=p(x,t;x_0,t_0)+\frac{1}{\Gamma(1+\beta)}\frac{\partial p^\beta(x,t;x_0,t_0)}{\partial x_0^\beta}(y-x_0)^\beta\\ &+\frac{1}{\Gamma(1+2\beta)}\frac{\partial^{(2\beta)}p(x,t;x_0,t_0)}{\partial x_0^{(2\beta)}}(y-x_0)^{(2\beta)}\\ &+O((y-x_0)^{\beta_1}) \end{aligned} \tag{25}$$

with an appropriate fractal space dimension characteristic $0<\beta\leq 1$, and $\beta_1>2\beta$.

Considering the generalized Kolmogorov conditions, there exist the limits:

$$\mathscr{A}(x_0,t_0) \equiv \frac{1}{\Gamma(1+\beta)} \lim_{\Delta t_0 \to 0} \frac{1}{\Delta t_0} \int (y-x_0)^{\beta} p(y,t_0;x_0,t_0-\Delta t_0)\,\mathrm{d}y, \tag{26}$$

$$\mathscr{B}(x_0,t_0) \equiv \frac{1}{\Gamma(1+2\beta)} \lim_{\Delta t_0 \to 0} \frac{1}{\Delta t_0} \int (y-x_0)^{(2\beta)} p(y,t_0;x_0,t_0-\Delta t_0)\,\mathrm{d}y, \tag{27}$$

$$\lim_{\Delta t_0 \to 0} \frac{1}{\Delta t_0} \int (y-x_0)^{\beta_1} p(y,t_0;x_0,t_0-\Delta t_0)\,\mathrm{d}y = 0 \quad (\beta_1 \geq 2\beta), \tag{28}$$

the fractional backward Kolmogorov equations (FBKE) can be derived from Eq. (24) with expansion (25)

$$\frac{\partial^{\alpha} p(x,t;x_0,t_0)}{\partial t_0^{\alpha}} = -\mathscr{A}(x_0,t_0)\frac{\partial p^{\beta}(x,t;x_0,t_0)}{\partial x_0^{\beta}} - \mathscr{B}(x_0,t_0)\frac{\partial^{(2\beta)} p(x,t;x_0,t_0)}{\partial x_0^{(2\beta)}}. \tag{29}$$

For $\alpha = \beta = 1$, equation (29) reduces to the regular backward Kolmogorov equations.

Equation (29) can also be simplified in the case that the coefficient functions are independent of time t_0 and $\beta = 1$,

$$\frac{\partial^{\alpha} p(x,t;x_0,t_0)}{\partial t_0^{\alpha}} = -a(x_0)\frac{\partial p(x,t;x_0,t_0)}{\partial x_0} - b(x_0)\frac{\partial p(x,t;x_0,t_0)}{\partial x_0}, \tag{30}$$

or

$$\frac{\partial p(x,t;x_0,t_0)}{\partial t_0} = {}_0D_{t_0}^{1-\alpha}\left[-a(x_0)\frac{\partial}{\partial x_0} - \frac{1}{2}b(x_0)\frac{\partial^2}{\partial x_0^2}\right] \times p(x,t;x_0,t_0). \tag{31}$$

3. Subordinated stochastic process and the corresponding FBKE

Now we consider the subordinated process,

$$Z(t) = X(S_{\alpha}(t)).$$

Here, the process $X(\tau)$ is given by the time homogeneous Itô stochastic differential equation (SDE)

$$\mathrm{d}X(\tau) = m(X(\tau))\mathrm{d}\tau + \sigma(X(\tau))\mathrm{d}B(\tau), \tag{32}$$

driven by the standard Brownian motion $B(\tau)$ with mean 0 and $\langle B^2(\tau)\rangle = \tau$. Let us stress that the role of $S_{\alpha}(t)$ is analogous to the role of the fractional operator ${}_0D_{t_0}^{1-\alpha}$ in Eq. (31).

Using the total probability formula and the independence of $X(\tau)$ and $S_{\alpha}(t)$, we obtain that the transfer probability $p(x,t;x_0,t_0)$ of $X(S_{\alpha}(t))$ is given by

$$p(x,t;x_0,t_0) = \int_0^{\infty}\int_0^{\infty} g(\tau_1,t) f(x,\tau_1;x_0,\tau_2) \times g(\tau_2,t_0)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2. \tag{33}$$

Here, by $f(x,\tau_1;x_0,\tau_2)$ and $g(\tau,t)$ we denote the transfer probability of $X(\tau)$ and the PDFs of $S_{\alpha}(t)$, respectively. Equivalently, in the Laplace space with respect to t_0, the above formula yields

$$\hat{p}(x,t;x_0,k) = \int_0^{\infty}\int_0^{\infty} g(\tau_1,t) f(x,\tau_1;x_0,\tau_2) \times \hat{g}(\tau_2,k)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2. \tag{34}$$

Next, we find the expression for $\hat{g}(\tau,k)$. It is easy to verify[17] that the relation

$$g(\tau,t) = \frac{t}{\alpha\tau}u(t,\tau) \tag{35}$$

holds, where by $u(t,\tau)$ we denote the PDF of the process $U(\tau)$ from definition (3). Consequently, we can calculate the Laplace transform

$$\hat{g}(\tau,k) = \int_0^{\infty} \mathrm{e}^{-kt}\frac{t}{\alpha\tau}u(t,\tau)\,\mathrm{d}t = k^{\alpha-1}\,\mathrm{e}^{-\tau k^{\alpha}}. \tag{36}$$

Using the above result in combination with Eq. (34), we obtain

$$\hat{p}(x,t;x_0,k) = \int_0^{\infty}\int_0^{\infty} g(\tau_1,t) f(x,\tau_1;x_0,\tau_2) k^{\alpha-1} \times \mathrm{e}^{-\tau_2 k^{\alpha}}\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 = k^{\alpha-1}\int_0^{\infty} \hat{f}(x,\tau;x_0,k^{\alpha}) g(\tau,t)\,\mathrm{d}\tau. \tag{37}$$

Since the process $X(\tau)$ is given by Eq. (32), its transfer probability $f(x,\tau_1;x_0,\tau_2)$ obeys the equation

$$\frac{\partial f(x,\tau_1;x_0,\tau_2)}{\partial \tau_2} = -a(x_0)\frac{\partial f(x,\tau_1;x_0,\tau_2)}{\partial x_0} - \frac{1}{2}b(x_0)\frac{\partial^2 f(x,\tau_1;x_0,\tau_2)}{\partial x_0^2}. \tag{38}$$

Equivalently, in the Laplace space with respect to τ_2, we have

$$k\hat{f}(x,\tau_1;x_0,k) - f(x,\tau_1;x_0,0) = -a(x_0)\frac{\partial \hat{f}(x,\tau_1;x_0,k)}{\partial x_0}$$

$$-\frac{1}{2}b(x_0)\frac{\partial^2\hat{f}(x,\tau_1;x_0,k)}{\partial x_0^2}. \tag{39}$$

The above formula after the change of variables $k \to k^\alpha$ gives

$$\begin{aligned} & k^\alpha\hat{f}(x,\tau_1;x_0,k^\alpha) - f(x,\tau_1;x_0,0) \\ =\ & -a(x_0)\frac{\partial\hat{f}(x,\tau_1;x_0,k^\alpha)}{\partial x_0} \\ & -\frac{1}{2}b(x_0)\frac{\partial^2\hat{f}(x,\tau_1;x_0,k^\alpha)}{\partial x_0^2}. \end{aligned} \tag{40}$$

Now by integrating Eq. (40) multiplied by $g(\tau_1,t)$ over τ_1, in combination with Eq. (37) we can obtain

$$\begin{aligned} & k\hat{p}(x,t;x_0,k) - p(x,t;x_0,0) \\ =\ & k^{1-\alpha}\Big[-a(x_0)\frac{\partial\hat{p}(x,t;x_0,k)}{\partial x_0} \\ & -\frac{1}{2}b(x_0)\frac{\partial^2\hat{p}(x,t;x_0,k)}{\partial x_0^2}\Big]. \end{aligned} \tag{41}$$

Inverting the Laplace transform, equation (41) gives the fractional backward Kolmogorov equations in the fractal time as

$$\begin{aligned} & \frac{\partial p(x,t;x_0,t_0)}{\partial t_0} \\ =\ & {}_0D_{t_0}^{1-\alpha}\left[-a(x_0)\frac{\partial}{\partial x_0} - \frac{1}{2}b(x_0)\frac{\partial^2}{\partial x_0^2}\right] \\ & \times p(x,t;x_0,t_0). \end{aligned} \tag{42}$$

Thus, we have proved that the transfer probability $p(x,t;x_0,t_0)$ of the process $X(S(t))$ is a solution of the fractional backward Kolmogorov equations (42).

4. Conclusion

The fractional backward Kolmogorov equations in the fractal space–time are derived based on the construction of a model for dynamic trajectories. The type of fractional backward Kolmogorov equation in the fractal time, whose coefficient functions are independent of the time t_0, are proved to be equivalent to the time homogeneous subordinated Itô stochastic differential equation for the subordinated process $Z(t) = X(S_\alpha(t))$: $\mathrm{d}X(\tau) = m(X(\tau))\mathrm{d}\tau + \sigma(X(\tau))\mathrm{d}B(\tau)$, driven by the standard Brownian motion $B(\tau)$. There are the problems that one does not know if the fractional backward Kolmogorov equations in the fractal time, whose coefficient functions are modified to be dependent on time t_0 or on the space–time (x_0,t_0).

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