Vortex lines in a ferromagnetic spin-triplet superconductor*

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(Received 2 September 2011; revised manuscript received 28 December 2011)

Based on Duan's topological current theory, we show that in a ferromagnetic spin-triplet superconductor there is a topological defect of string structures which can be interpreted as vortex lines. Such defects are different from the Abrikosov vortices in one-component condensate systems. We investigate the inner topological structure of the vortex lines. The topological charge density, velocity, and topological current of the vortex lines can all be expressed in terms of δ function, which indicates that the vortices can only arise from the zero points of an order parameter field. The topological charges of vortex lines are quantized in terms of the Hopf indices and Brouwer degrees of ϕ-mapping. The divergence of the self-induced magnetic field can be rigorously determined by the corresponding order parameter fields and its expression also takes the form of a δ-like function. Finally, based on the implicit function theorem and the Taylor expansion, we conduct detailed studies on the bifurcation of vortex topological current and find different directions of the bifurcation.

Keywords: vortex lines, spin-triplet superconductor, topological current, bifurcation

PACS: 74.20.–z, 74.25.Ha, 02.40.Pc **DOI:** 10.1088/1674-1056/21/5/057401

1. Introduction

Topological defects formed in symmetry-breaking phase transitions play an increasingly important role in both condensed matter systems and high-energy physics.[1−4] Many features and properties of ordered media, such as crystalline solids, magnets, and liquid crystals, are controlled by their defects: the points or lines of discontinuity where the order changes abruptly. Recently, superconductors with triplet pairing have also allowed for a rich variety of topological defects and phase transitions,[5−10] such as the half-quantum vortex (the Alice string),[5,6] and the split phase transitions in triplet systems.[6,8,9] In a superconducting film, there are gapless superconductivity and spin-dependent density of states both in and out the energy gap. This indicates the appearance of inverse-proximity-effect-induced ferromagnetism and the coexistence of ferromagnetism and superconductivity near the interface.[11] There have also been particularly interesting studies of split transitions in a neutral p-wave superfluid in connection with thin films of liquid ^{3}He.[12,13] The questions of various partial symmetry breakdowns are also relevant and have been studied on spin-1 Bose condensates in optical traps[14] and bilayer quantum Hall systems.[15]

In the simplest one-component condensate system, the Abrikosov vortices with quantum flux are the only topological defects, while in the multi-component case other types of topological defects arise. The recent advent of spinor Bose-Einstein condensates (BECs) and triplet superconductors (e.g., UPt_3[16] and Sr_2RuO_4[17]), which have the multi-component order parameter, have opened a new opportunity for us to study much more complex (especially structurally complicated) topological defects in condensed matter physics. These defects include coreless magnetic skymions,[18] vector vortices[19] in the three-component order parameter model triplet superconductors, and the vortices[20] in the two-component order parameter model triplet superconductors. In

*Project supported by the National Natural Science Foundation of China (Grant Nos. 10905026 and 10905027), the Specialized Research Fund for the Doctoral Program of Higher Education (Grant No. 20090211120030), and the Lanzhou Development of Science and Technology Program, China (Grant No. 2010-1-129).

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Ref. [21], Stoof and Khawaja have shown that ferromagnetic condensates have long-lived skyrmion excitations, which are nonsingular but topologically nontrivial points like spin textures. Coreless vortices have been demonstrated to be thermodynamically stable in ferromagnetic spinor condensates with the spin $F = 1$ under rotation.[22] Linear defects have been studied by Leonhardt and Volovik,[23] who have pointed out the existence of Alice strings in the condensate of ^{23}Na. In Ref. [24], Babaev has presented a Ginzburg–Landau functional for a single-condensate spin-triplet superconductor by means of an exact duality mapping, which also leads to another version of the Faddeev O(3) σ-model in the ferromagnetic superconductor. This model describes topological excitations in the form of stable and finite-length knotted closed vortices (knotted solitons), and the resulting effective model for the ferromagnetic spin-triplet superconductor is surprisingly similar to that for two-band superconductors discussed in Ref. [26]. Most of the studies on the topological defects are based on the original identification of the order parameter space proposed by Ho.[27]

In condensed matter physics, the theory of a homotopy group, which uses the fundamental group of the order parameter space of a medium, provides a natural setting for description and classification of topological defects.[28] However, unlike previous studies based on the homotopy group theory,[29,30] we concentrate mainly on investigating the inner topological structure of topological defects in ferromagnetic spin-triplet superconductor systems. For example, the problem of defining quantities such as the topological current of vortices and an associated vortex velocity field is explored. The presence of nontrivial self-induced magnetic field in ferromagnetic spin-triplet superconductors allows the formation of unstable vortex-line structures. We investigate the inner topological structure of vortex lines in spin-triplet superconductor systems and particularly obtain the non-trivial divergence of the self-induced magnetic field. These results are obtained by virtue of Duan's ϕ-mapping current theory,[31] which has been extensively used in studying topological invariant and topological structure of various physical systems.

Our paper is organized as follows: in Section 2, we give a brief review of the Ginzburg–Landau free energy density of ferromagnetic spin-1 superconductors with knotted soliton structures; in Section 3, we investigate the inner topological current of vortex lines and classify these topological defects according to the Brouwer degrees and the Hopf indices of ϕ-mapping. The quantities such as position, density, and the velocity of vortex lines are expressed in terms of the two-dimensional order parameter fields; in Section 4, the vortex lines in a two-gap superconductor are found to encounter, split or merge at bifurcation points; our discussion and conclusion are presented in Section 5.

2. A review of knotted soliton in spin-triplet superconductors

Firstly, let us consider the important features of a charged spin-1 Bose condensate in the following. The order parameter of the spin-1 Bose condensate can be written as $\Psi_a(\boldsymbol{x},t) = \sqrt{n(\boldsymbol{x},t)}\zeta_a(\boldsymbol{x},t)$ $(a = 1,2,3)$, where n is the total density and ζ is a normalized spinor $(\zeta^\dagger\cdot\zeta = 1)$. Then the Ginzburg–Landau free energy density of the spin-1 superconductor in the "ferromagnetic" state reads[24]

$$F = \frac{\hbar^2}{2M}(\boldsymbol{\nabla}\sqrt{n})^2 + \frac{\hbar^2 n}{2M}\left|\left(\boldsymbol{\nabla} + \mathrm{i}\frac{2e}{\hbar c}\boldsymbol{A}'\right)\zeta\right|^2 - \mu n + \frac{n^2}{2}\left[c_0 + c_2\langle\boldsymbol{F}\rangle^2\right] + \frac{\boldsymbol{B}'^2}{8\pi}, \quad (1)$$

where $\langle\boldsymbol{F}\rangle = \zeta_a^* F_{ab}\zeta_b$ is the spin, μ is the chemical potential, M is the mass of the atom, $\boldsymbol{A}'$ is the external magnetic vector potential, and $\boldsymbol{B}'$ is the external magnetic field. All degenerate spinors are related to each other by gauge transformation $\mathrm{e}^{\mathrm{i}\theta}$ and spin rotation $\mathcal{U}(\alpha,\beta,\tau) = \mathrm{e}^{-\mathrm{i}F_x\alpha}\,\mathrm{e}^{-\mathrm{i}F_y\beta}\,\mathrm{e}^{-\mathrm{i}F_z\tau}$, where α, β, and τ are the Euler angles. The matrices $\boldsymbol{F}_x$, $\boldsymbol{F}_y$, and $\boldsymbol{F}_z$ are respectively expressed as

$$\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{-\mathrm{i}}{\sqrt{2}} & 0 \\ \frac{\mathrm{i}}{\sqrt{2}} & 0 & \frac{-\mathrm{i}}{\sqrt{2}} \\ 0 & \frac{\mathrm{i}}{\sqrt{2}} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0\,0 & 0 \\ 0\,0 & 0 \\ 0\,0 & -1 \end{pmatrix},$$

which are the generators of SO(3) group satisfying $[F^i, F^j] = \mathrm{i}\epsilon^{ijk}F_k$ $(i,j,k = 1,2,3)$. By minimizing the energy with a fixed particle number, the ground state structure of $\Psi_a(\boldsymbol{r})$ can be found.[27] In ferromagnetic state, when $c_2 < 0$, the energy is minimized by setting $\langle\boldsymbol{F}\rangle^2 = 1$. The ground state spinor and density are[26]

$$\zeta = \mathrm{e}^{\mathrm{i}(\theta-\tau)}\begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha}\cos^2\frac{\beta}{2} \\ \sqrt{2}\cos\frac{\beta}{2}\sin\frac{\beta}{2} \\ \mathrm{e}^{\mathrm{i}\alpha}\sin^2\frac{\beta}{2} \end{pmatrix}, \quad n^{\mathrm{o}}(\boldsymbol{r}) = \frac{1}{c_0+c_2}\mu. \quad (2)$$

Since the distinct configurations of ζ in Eq. (2) are given in the full range of the Euler angles, the symmetry group of the ferromagnetic state is SO(3).[26] Introducing new variables

$$\boldsymbol{s} = (s^1, s^2, s^3) = (\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta), \quad (3)$$

and

$$\boldsymbol{C} = \frac{M}{en}\boldsymbol{J}, \quad (4)$$

where

$$\boldsymbol{J} = \frac{\mathrm{i}\hbar en}{M}\left(\zeta^\dagger\boldsymbol{\nabla}\zeta - \boldsymbol{\nabla}\zeta^\dagger\zeta\right) - \frac{4e^2n}{Mc}\boldsymbol{A}' \quad (5)$$

is the supercurrent, we can write the free energy density in terms of gauge-invariant variables as[24]

$$\begin{aligned} F = & \frac{\hbar^2}{2M}(\boldsymbol{\nabla}\sqrt{n})^2 + \frac{\hbar^2 n}{4M}(\boldsymbol{\nabla}\boldsymbol{s})^2 + \frac{n}{8M}\boldsymbol{C}^2 \\ & + \frac{\hbar^2c^2}{128\pi e^2}\left(\epsilon_{abc}s^a\boldsymbol{\nabla}s^b\times\boldsymbol{\nabla}s^c - \frac{1}{\hbar}\boldsymbol{\nabla}\times\boldsymbol{C}\right)^2 \\ & - \mu n + \frac{n^2}{2}(c_0 + c_2)\,. \end{aligned} \quad (6)$$

Hereafter, summation over the repeated indices is assumed. The above expression is a version of the Faddeev nonlinear O(3) σ-model introduced in Ref. [25]. It is concluded that this system allows stable knotted solitons characterized by a nontrivial Hopf invariant and the stability is ensured by the fourth-order derivative term (the Faddeev term) for the vector field $\boldsymbol{s}$,[24] which is related to the nontrivial helical configuration part of magnetic field. This helical part has the physical meaning of the self-induced magnetic field in the presence of a nontrivial spin texture (in this paper we discuss a system with no external field).

As shown by Eq. (6), the magnetic field of the system can be divided into two parts: (i) the contribution from the supercurrent $\boldsymbol{J}$, which is introduced by the external magnetic vector potential $\boldsymbol{A}'$ and can provide us with the topological defects named Abrikosov vortices, resembling that in a single spin-0 condensate system; (ii) the contribution of the term $\epsilon_{abc}s^a\boldsymbol{\nabla}s^b\times\boldsymbol{\nabla}s^c$ to the magnetic field is a fundamentally important property of a spin-triplet system which has no counterpart in a single spin-0 condensate system. Actually, because of the presence of this term, the spin-triplet system has properties which are quite different from those of the single spin-0 condensate system. Note that this term has the physical meaning of the self-induced magnetic field in the presence of a nontrivial spin texture, which originates from the nontrivial electromagnetic interaction between the components ζ. In what follows, it is therefore necessary for us to investigate this self-induced magnetic field in more detail by virtue of Duan's topological current theory.

3. The topological current of vortex lines

In this section, by means of Duan's topological current theory, it is shown that there is a two-dimensional topological current which can be derived from the nontrivial topological term $\epsilon_{abc}s^a\boldsymbol{\nabla}s^b\times\boldsymbol{\nabla}s^c$, and the vortex line structure inheres in this topological current. The self-induced magnetic field B^i is expressed as

$$B^i = \frac{\hbar c}{8e}\epsilon^{ijk}\epsilon_{abc}s^a\partial_j s^b\partial_k s^c, \quad (7)$$

where ϵ^{ijk} and ϵ_{abc} represent the fully antisymmetric tensors. In Ref. [24], Babaev has only considered the superconductor in a simply-connected space, which means that the defects in the superconductor do not reflect the zero points of the order parameter field. Conventionally, the unit vector $\boldsymbol{s}$ can be expressed by Eq. (3). With this normal expression, one can immediately draw a conclusion that the divergence of the self-induced magnetic field is equal to zero, as shown by Maxwell's equation. In fact, an important fact that the angles of vector field, β and α, are undefined at the vortex cores (i.e., the zero points of order parameter) is ignored in Eq. (3). Therefore, we cannot express the wave function in the form of Eq. (3) at the zero points of the order parameter field. In order to investigate the unusual property of B^i at these special points, we rewrite the unit vector s^a as

$$\begin{aligned} & s^a = \frac{\varphi^a}{||\varphi||}, \quad ||\varphi|| = \sqrt{\varphi^a\varphi^a}, \\ & s^a s^a = 1, \quad a = 1, 2, 3, \end{aligned} \quad (8)$$

instead of the well-known expression (Eq. (3)). Based on Eq. (8), it is clearly shown that the field $\boldsymbol{s}$ takes an uncertain value at the zero points of $\varphi^a(x)$, which are called the singular points of s^a. Moreover, the singularity leads to an unusual behaviour of the self-induced magnetic field B^i at these points. Obviously, the three-component vector order parameter field $\varphi(\boldsymbol{r}, t)$ determines the defect properties of the system, and there is a smooth mapping between the $(3+1)$-dimensional spacetime X (with the local coordinate x) and the 3-dimensional Euclidean space R^3 as $\varphi : x \mapsto \boldsymbol{\varphi}(x) \in \mathrm{R}^3$. From Eq. (8), it can seen

that the vector field $\boldsymbol{s}$ is regarded as the section of 2-dimensional unit sphere $S^2(X)$.

By virtue of Duan's topological current theory,[31,32] when the φ^a field possesses several zero points, the divergence of the self-induced magnetic field is no longer zero but takes the form of the δ-like function. To discuss the self-induced magnetic field in a spin-triplet superconductor in $(3+1)$-dimensional spacetime, it is necessary to reexpress the magnetic field as

$$B^\mu = \frac{\hbar c}{8e}\epsilon^{\mu\lambda\nu}\epsilon_{abc}s^a\partial_\lambda s^b\partial_\nu s^c, \quad \mu,\nu,\lambda = 0,1,2,3. \tag{9}$$

If we introduce the field tensor $k_{\mu\nu}$ as

$$k_{\mu\nu} = \epsilon_{abc}s^a\partial_\mu s^b\partial_\nu s^c, \tag{10}$$

which is a topological term describing the non-uniform distribution of the unit vector $\boldsymbol{s}$ at large distances in space, one can obtain

$$B^\mu = \frac{\hbar c}{8e}\epsilon^{\mu\lambda\nu}k_{\lambda\nu}. \tag{11}$$

According to Duan–Ge's U(1) gauge potential decomposition theory, the tensor $k_{\mu\nu}$ can be changed into a U(1) field tensor form[33–35]

$$k_{\mu\nu} = \epsilon_{abc}s^a\partial_\mu s^b\partial_\nu s^c = \partial_\mu W_\nu - \partial_\nu W_\mu. \tag{12}$$

Here, W_μ is the Wu–Yang potential obtained from the two-dimensional vector field $\boldsymbol{e}$. It can be expressed as

$$W_\mu = \boldsymbol{e}_1\cdot\partial_\mu\boldsymbol{e}_2, \tag{13}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the unit vectors normal to $\boldsymbol{s}$ (i.e., $\boldsymbol{e}_1\perp\boldsymbol{s}$, $\boldsymbol{e}_2\perp\boldsymbol{s}$). $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{s}$ satisfy the normalizing condition and form an orthogonal frame, i.e.,

$$\boldsymbol{s} = \boldsymbol{e}_1\times\boldsymbol{e}_2,\ \boldsymbol{e}_1\cdot\boldsymbol{e}_2 = 0, \tag{14}$$

$$\boldsymbol{e}_1\cdot\boldsymbol{e}_1 = \boldsymbol{e}_2\cdot\boldsymbol{e}_2 = 1. \tag{15}$$

In order to explain the topological structure, we introduce a new 2-dimensional vector order parameter $\boldsymbol{\phi} = (\phi^1,\phi^2)$ based on $\boldsymbol{e}$, which resides in the plane formed by the unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ and satisfies

$$e_1^a = \frac{\phi^a}{\|\boldsymbol{\phi}\|}, \quad e_2^a = \epsilon_{ab}\frac{\phi^b}{\|\boldsymbol{\phi}\|},$$
$$\|\boldsymbol{\phi}\|^2 = \phi^a\phi^a,\ a,b = 1,2, \tag{16}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the sections of the sphere bundle $S^1(X)$ and regarded as a projection of X onto a 1-dimensional unit sphere S^1. Obviously, the zero points of $\boldsymbol{\phi}$ are exactly the 2-dimensional singular points of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$. Based on the $\boldsymbol{\phi}$ field, the Wu–Yang potential W_μ and the field tensor $k_{\mu\nu}$ can be respectively expressed as

$$W_\mu = \epsilon_{ab}\frac{\phi^a}{\|\boldsymbol{\phi}\|}\partial_\mu\frac{\phi^b}{\|\boldsymbol{\phi}\|}, \tag{17}$$

$$k_{\mu\nu} = 2\epsilon_{ab}\partial_\mu\frac{\phi^a}{\|\boldsymbol{\phi}\|}\partial_\nu\frac{\phi^b}{\|\boldsymbol{\phi}\|}. \tag{18}$$

Similarly, the divergence Q of the self-induced magnetic field can be rigorously proved

$$Q = \partial_\mu\boldsymbol{B}^\mu = -\frac{\hbar c}{8e}\epsilon^{\mu\nu\lambda}\epsilon_{abc}\partial_\mu\phi^d\partial_\nu\phi^b\partial_\lambda\phi^c \times\frac{\partial}{\partial\phi^d}\frac{\partial}{\partial\phi^a}\left(\frac{1}{||\phi||}\right). \tag{19}$$

Using the equation that $\partial_\mu(\phi^a/\|\boldsymbol{\phi}\|) = \partial_\mu\phi^a/\|\boldsymbol{\phi}\| + \phi^a\partial_\mu(1/\|\boldsymbol{\phi}\|)$ and the Green-function relation in ϕ-space $\partial_a\partial_a\ln\|\phi\| = 2\pi\delta^2(\vec{\phi})$ ($\partial_a = \partial/\partial\phi^a$), we indeed obtain the δ-function structure of the divergence Q and the tensor $k_{\mu\nu}$ as expected

$$k_{\mu\nu} = 4\pi\epsilon_{\mu\nu\lambda\rho}\delta^2(\boldsymbol{\phi})D^{\lambda\rho}(\phi/x), \tag{20}$$

$$Q = \partial_\mu B^\mu = \frac{\hbar\pi c}{2e}\delta(\boldsymbol{\phi})D^0(\phi/x), \tag{21}$$

where $D^{\lambda\rho}(\phi/x) = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\epsilon_{ab}\partial_\mu\phi^a\partial_\nu\phi^b$. The δ-like function shows that the divergence Q is not equal to zero when $\boldsymbol{\phi}\neq 0$, which indicates that in the spin-triplet superconductor system there are some kinds of topological defects located at the zero points of field $\boldsymbol{\phi}$. From the above discussion, we see that the zero points of $\boldsymbol{\phi}$ play an important role in the topological current theory, therefore we discuss them more deeply. According to Duan's topological current theory,[36] the 2-dimensional topological tensor current is defined as

$$K^{\mu\nu} = \frac{1}{2\pi}\frac{1}{2}\epsilon^{\mu\nu\lambda\rho}\epsilon_{ab}\partial_\lambda s^a\partial_\rho s^b = \delta^2(\boldsymbol{\phi})D^{\mu\nu}(\phi/x). \tag{22}$$

It is therefore revealed that there is a 2-dimensional topological tensor current in the field tensor $k_{\mu\nu}$

$$k_{\mu\nu} = 4\pi\epsilon_{\mu\nu\lambda\rho}K^{\lambda\rho}. \tag{23}$$

By defining a topological current j^i as the spatial component of $K^{\mu\nu}$, one can prove that the topological current also takes the form of a δ-function

$$j^i = K^{0i} = \frac{1}{8\pi}\epsilon^{ijk}k_{jk} = \delta^2(\boldsymbol{\phi})D^i(\phi/x), \tag{24}$$

where $D^i(\phi/x) = \epsilon^{ijk}\epsilon_{ab}\partial_j\phi^a\partial_k\phi^b/2$ is the Jacobian vector. Equation (24) implies an important conclusion

$$j^{\mu\nu}\begin{cases} = 0, \text{ if and only if } \boldsymbol{\phi}\neq 0, \\ \neq 0, \text{ if and only if } \boldsymbol{\phi} = 0. \end{cases} \tag{25}$$

Therefore, it is necessary to study the zero points of $\boldsymbol{\phi}$ to determine the non-zero solutions of j^i. The implicit

function theorem[37] says that when these zero points are regular points of the ϕ-mapping and the Jacobian vector

$$\boldsymbol{D}\left(\frac{\phi}{x}\right)=\left[D^1\left(\frac{\phi}{x}\right),D^2\left(\frac{\phi}{x}\right),D^3\left(\frac{\phi}{x}\right)\right]\neq 0, \quad (26)$$

the general solutions of

$$\begin{cases}\phi^1(x^0=t,x^1,x^2,x^3)=0,\\ \phi^2(x^0=t,x^1,x^2,x^3)=0,\end{cases} \quad (27)$$

can be expressed as

$$x^1=x_j^1(s,t),\quad x^2=x_j^2(s,t),\quad x^3=x_j^3(s,t),$$
$$j=1,2,\ldots,N. \quad (28)$$

The above solutions represent the evolution surfaces of N isolated strings L_j in $(3+1)$-dimensional spacetime manifold X with s being the string parameter. The topological string structures are exactly the vortex lines in the ferromagnetic spin-triplet superconductor. The time-like evolution parameter t and space-like string parameter s span a 2-dimensional sub-manifold which is called the i-th singular submanifold M_i $(i=1,2,\ldots,N)$ in X. Therefore, besides the closed knotted solitons mentioned above,[24] the nontrivial topological term can lead to the vortex line structures in spin-triplet superconductors by making use of the ϕ-mapping theory. These kinds of topological defects are qualitatively different from the Abrikosov vortices in one-component condensate systems for these systems are governed by the self-induced magnetic field.

In what follows, we will answer the questions: what are the topological charges of the topological defects and how to obtain the inner topological structure of line defects. To further understand the inner topological and physical meaning of j^μ, we need a more explicit expression for $\delta(\boldsymbol{\phi})$. For this purpose, we study the spatial structure of these vortex lines by fixing the time coordinate t. In δ-function theory,[38] it can be expanded as follows:

$$\delta^2(\boldsymbol{\phi})=\sum_{j=1}^{N}\beta_j\int_{\mathrm{L}_j}\frac{\delta^3(\boldsymbol{x}-\boldsymbol{x}_j(s))}{|D(\phi/u)|_{\Sigma_j}}\mathrm{d}s, \quad (29)$$

where

$$\left|D\left(\frac{\phi}{u}\right)\right|_{\Sigma_j}=\left(\frac{1}{2}\epsilon^{ik}\epsilon_{mn}\frac{\partial\phi^m}{\partial u^i}\frac{\partial\phi^n}{\partial u^k}\right),$$

and Σ_j is the j-th planar element transversal to L_j with local coordinates $(u^1,\ u^2)$. The positive integer β_j is the Hopf index of ϕ-mapping. From the definition of the Jacobian vector, we can obtain the direction vector of the vortex lines

$$\left.\frac{\mathrm{d}x^i}{\mathrm{d}s}\right|_{\boldsymbol{x}_j}=\left.\frac{D^i(\phi/x)}{D(\phi/u)_{\Sigma_j}}\right|_{\boldsymbol{x}_j}, \quad (30)$$

which leads to

$$\left.\frac{\mathrm{d}x^i}{\mathrm{d}x^k}\right|_{\boldsymbol{x}_j}=\left.\frac{D^i(\phi/x)}{D^k(\phi/x)}\right|_{\boldsymbol{x}_j},\quad i,k=1,2,3. \quad (31)$$

Therefore, from Eqs. (24), (29), and (30), we obtain

$$j^i=\sum_{j=1}^{N}\beta_j\eta_j\int_{\mathrm{L}_j}\frac{\mathrm{d}x^i}{\mathrm{d}s}\delta^3(\boldsymbol{x}-\boldsymbol{x}_j(s))\mathrm{d}s, \quad (32)$$

where $\eta_j=\mathrm{sign}[D(\phi/u)|_{\boldsymbol{x}_j}]=\pm1$ is the Brouwer degree of ϕ-mapping. Let Σ be an arbitrary surface, and suppose that there are M line defects passing through it. According to the above equations, one can prove that

$$Q=\int_\Sigma j^i\mathrm{d}\sigma_i=\sum_{j=1}^{M}W_j, \quad (33)$$

where $W_j=\beta_j\eta_j$ is the winding number of $\boldsymbol{\phi}$ around L_j. Equations (32) and (33) show that they are exactly the topological current and charge of a system of N vortices moving in the (3+1)-dimensional spacetime which are quantized by the winding number W_j. Here, the Hopf indices β_j and Brouwer degrees η_j can be

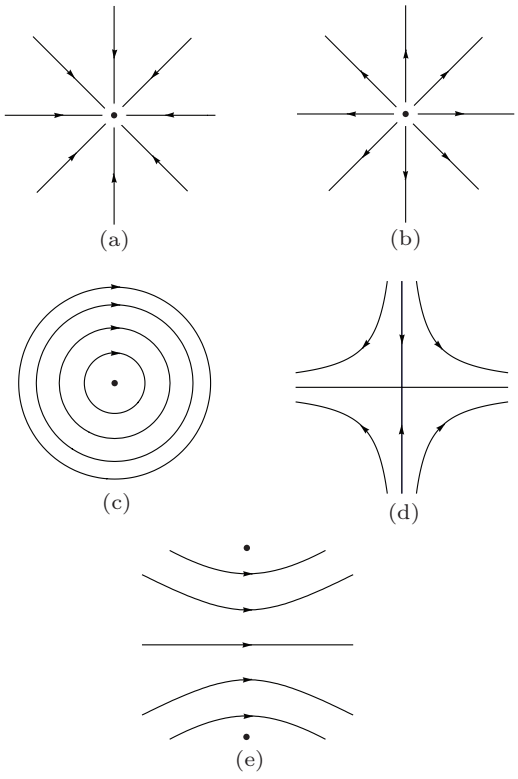


Fig. 1. Distributions of the order parameter field around the vortex lines with different Hopf indices and Brouwer degrees, i.e., different winding numbers. The values of the winding number for Figs. 1(a)–1(e) are (a) $W_j=+1$, (b) $W_j=+1$, (c) $W_j=+1$, (d) $W_j=-1$, and (e) $W_j=+2$, respectively.

used to classify the strings. The Hopf indices β_j characterize the absolute values of the topological quantization. The Brouwer degrees $\eta_j = +1$ correspond to strings, while $\eta_j = -1$ correspond to antistrings. In topology, the winding number means that when the point $\boldsymbol{x}$ covers the neighborhood of the zero point $\boldsymbol{x}_j$ once, the vector field $\boldsymbol{\phi}$ covers the corresponding region in ϕ-space for W_j times, which is a topological invariant and is also called the degree of Gauss map. It should be pointed out that there are different kinds of $\boldsymbol{\phi}$ field distribution around the strings which may lead to different winding numbers (see Fig. 1).

4. The bifurcation of vortex lines in ferromagnetic spin-triplet superconductor

From the above discussion, we know that the results in the above section are obtained straightly from the topological point of view under the condition that $D^i(\phi/x) \neq 0$. When this condition fails, i.e., $D^i(\phi/x) = 0$, the above results (Eq. (28)) will change in some way. It is interesting to discuss what will happen, and what the correspondence in physics will be when $D^i(\phi/x) = 0 \quad (i = 1, 2, 3)$.

For simplicity, let us take an arbitrary cross section normal to the z axis, i.e., $(2 + 1)$-dimensional spacetime with coordinates $x_1 = x$, $x_2 = y$, and $x_0 = t$.

The intersection line $x(t)$ between the evolution surface N_i and the cross section S is exactly the motion curve of the line defect on the cross section (see Fig. 2). The generalized velocity at the intersection point between L_j and the cross section is given by

$$v_j^i = \left.\frac{\mathrm{d}x^i}{\mathrm{d}t}\right|_{\boldsymbol{x}_j} = \left.\frac{D^i(\phi/x)}{D^0(\phi/x)}\right|_{\boldsymbol{x}_j}, \quad i = 1, 2, \tag{34}$$

where $D^0(\phi/x) = \epsilon_{ab}\partial_1\phi^a\partial_2\phi^b$, $D^1(\phi/x) = \epsilon_{ab}\partial_2\phi^a\partial_0\phi^b$, and $D^2(\phi/x) = \epsilon_{ab}\partial_0\phi^a\partial_1\phi^b$. Equation (34) determines the motion of the j-th vortex line. It is obvious that in the case of

$$D^0\left(\frac{\phi}{x}\right) = 0, \quad D^i\left(\frac{\phi}{x}\right) = 0, \quad i = 1, 2, \tag{35}$$

at some points (marked as (z_j^{*i}, t^*)) along L_j, the functional relationship between t and x or t and y is not unique in the neighborhood of (z_j^{*i}, t^*). This fact can be easily seen from Eq. (34), which directly shows that under the restraint (35) the tangential direction of the integral curve is indefinite, i.e., the velocity field of vortices is indefinite at the point (z_j^{*i}, t^*). Hence, Eq. (34) does not satisfy the the existence and uniqueness theorem of the solutions of a differential equation, and the very point (z_j^{*i}, t^*) is called the bifurcation point of the 2-dimensional vector order parameter $\boldsymbol{\phi}(x, y, t)$ in the (2+1)-dimensional spacetime.

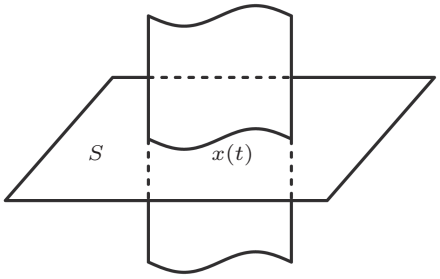


Fig. 2. Intersection line ($x(t)$) between the evolution surface of a line defect and the cross section (S), i.e., the movement curve of the line defect on the cross section. S is normal to the z axis.

Next we will search for different directions of all branch curves or the velocity field of line defects at the bifurcation point. It is known that,[26] at the bifurcation point (z_j^{*i}, t^*), the rank of the Jacobian matrix $[\partial\phi/\partial x]$ is smaller than 2 (for $D^i(\phi/x) = 0$ $(i = 0, 1, 2)$). Here, we suppose that the rank of the Jacobian matrix is 1. According to Duan's topological current theory, the Taylor expansion of the solution of $\boldsymbol{\phi} = 0$ in the neighborhood of (z_j^{*i}, t^*) can be generally expressed as[39]

$$\begin{aligned} &A(x^1 - z_j^{*^1})^2 + 2B(x^1 - z_j^{*1})(t - t^*) \\ &+C(t - t^{*2})^2 + \cdots = 0, \end{aligned} \tag{36}$$

which leads to

$$A\left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2 + 2B\frac{\mathrm{d}x^1}{\mathrm{d}t} + C = 0, \tag{37}$$

or

$$C\left(\frac{\mathrm{d}t}{\mathrm{d}x^1}\right)^2 + 2B\frac{\mathrm{d}t}{\mathrm{d}x^1} + A = 0, \tag{38}$$

where A, B, and C are three constants determined by the two-component vector order parameter $\boldsymbol{\phi}$. The direction field of vortex lines at the bifurcation point is

$$\boldsymbol{m} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, 1\right), \tag{39}$$

where the second component $\mathrm{d}y/\mathrm{d}t$ of the direction vector $\boldsymbol{m}$ can be given by:[26] $\mathrm{d}y/\mathrm{d}t = f_x^y\mathrm{d}x/\mathrm{d}t + f_t^y$, where the partial derivative coefficients f_x^y and f_t^y are also determined by the order parameter field at the bifurcation point. The solutions of Eq. (36) give different directions of the branch curves at the bifurcation

point. There are four possible cases that will reveal the physical meanings of the bifurcation points:[39]

Case 1 If $(A \neq 0)$ and $\Delta = 4(B^2 - AC) > 0$, from Eq. (37) we obtain two different motion directions of the zero point on the cross section: $(\mathrm{d}x^1/\mathrm{d}t)|_{1,2} = (-B \pm \sqrt{B^2 - AC})/A$, where two world-lines of two zero points intersect with different directions at the bifurcation point (see Fig. 3). This shows that two line defects meet and then separate at the bifurcation point, which is called the intersection of two vortex lines of different directions.

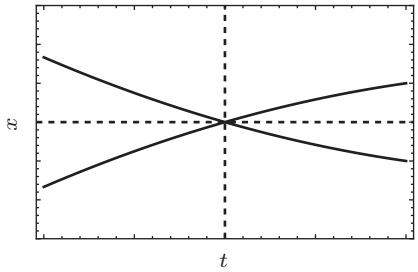


Fig. 3. Two line defects colliding with different directions of motion at the bifurcation point in the $(2+1)$-dimensional spacetime.

Case 2 If $(A \neq 0)$ and $\Delta = 4(B^2 - AC) = 0$, we get only one direction at the bifurcation point: $(\mathrm{d}x^1/\mathrm{d}t)|_{1,2} = -B/A$. This includes three sub-cases: (i) two vortex lines tangentially contact (intersect) (see Fig. 4); (ii) two vortex lines merge into one line (see Fig. 5); (iii) one vortex line splits into two lines (see Fig. 6). It is clear that a pair of defects can be transformed into a single defect with a total net winding number, without requiring surgery to extend beyond the interior of any contour surrounding the pair, as considered by Mermin.[28]

Case 3 If $(A = 0, C \neq 0)$ and $\Delta = 4(B^2 - AC) > 0$, from Eq. (37) we obtain $(\mathrm{d}t/\mathrm{d}x^1)|_{1,2} = (-B \pm \sqrt{B^2 - AC})/C = 0, -2B/C$. This includes two sub-cases: (i) three vortex lines merge into one line (see Fig. 7); (ii) one vortex line splits into three lines (see Fig. 8).

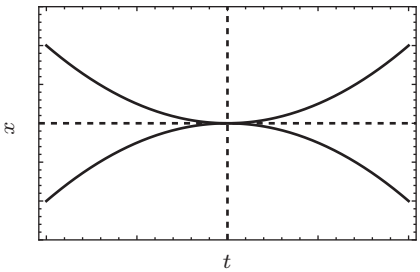


Fig. 4. Two line defects colliding at the bifurcation point.

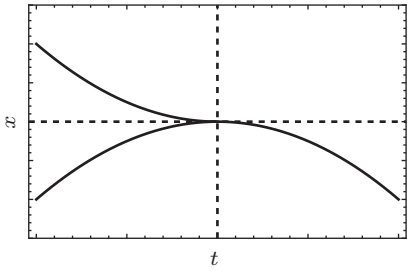


Fig. 5. Two line defects merging into one line defect at the bifurcation point.

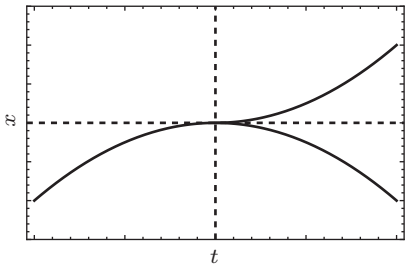


Fig. 6. One line defect splitting into two line defects at the bifurcation point.

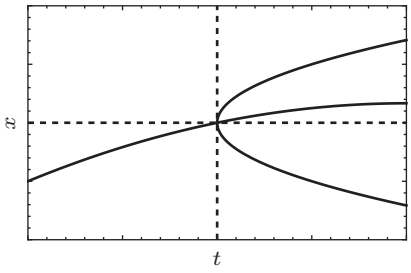


Fig. 7. Three line defects merging into one line defect at the bifurcation point.

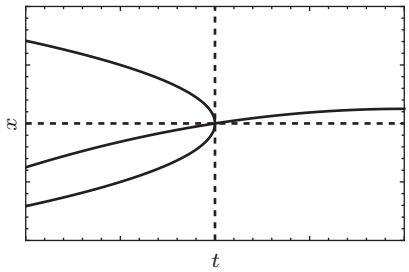


Fig. 8. One line defect splitting into three line defects at the bifurcation point.

Case 4 If $(A = C = 0)$, equation (37) leads to $\mathrm{d}x^1/\mathrm{d}t = 0$, $\mathrm{d}t/\mathrm{d}x^1 = 0$. This case shows that two curves normally intersect at the bifurcation point, which is similar to the Case 3).

In the above discussion, we consider only the case that two vortex lines pass through the bifurcation point. The configurations where more vortex lines pass through the bifurcation point are high-energy and unstable states, and will fall apart as quickly as they form. However, it should be pointed out that when

more than two vortex lines cross, split or merge at a certain time t^*, the branch conditions are still valid ($\boldsymbol{D}(\phi/x) \neq 0$), although these configurations are unstable. Moreover, the continuity equation for the system shows that the sum of the topological charge of final vortex line (s) should be equal to that of the initial line (s) at the bifurcation point for a fixed index j: $\sum_{\rm f} \beta_{j\rm f}\eta_{j\rm f} = \sum_{\rm i} \beta_{j\rm i}\eta_{j\rm i}$, where "i" stands for "initial" and "f" for "final".

5. Discussion and conclusion

In summary, by making use of the ϕ-mapping theory, we draw a conclusion that ferromagnetic spin-triplet superconductors allow the formation of vortex line structures and the vortex lines are located at the zero points of the two-dimensional vector field $\boldsymbol{\phi}$, namely, at the singular points of the field $\boldsymbol{e}$ defined by Eq. (16). We mainly investigate the topological properties rather than the dynamic properties of the vortex lines. Indeed, in contrast to the Abrikosov vortices in one-component condensate systems, these defects cannot be created by simply applying an external magnetic field due to the contribution of the term $\epsilon^{abc} s_a \boldsymbol{\nabla} s_b \times \boldsymbol{\nabla} s_c$. Their topological charges can be expressed in terms of the Hopf indices and the Brouwer degrees of the ϕ-mapping. We stress that quantities such as the topological charge density and velocity of the vortex lines can be rigorously determined by the corresponding order parameter fields of ϕ-mapping. The exact expression of the divergence of the self-induced magnetic field is given, which is no longer always zero but takes the form of δ-like function. The kernel of ϕ-mapping is the singularity of the topological current j^i in the (3+1)-dimensional spacetime, and the inner structure of topological current is labelled by the winding number of ϕ-mapping. In particular, the velocity of vortex lines is revealed to be indefinite during the branch processes. This is similar to the case of the vortex excitation in an anti-ferromagnetic BEC considered in Ref. [40].

All the results in this paper are obtained from the viewpoint of topology, without using any particular models or hypotheses. The inherent conservation structure of the topological current are obtained, and the interaction of vortex lines on the basis of this structure are studied at the bifurcation point. One can apply the branch conditions to the branch processes (crossing, splitting, and merging) of several vortex lines in superconductors in the neighborhood of a branch point. At the bifurcation point of the vector order parameter, a line defect with a topological charge W may split into several line defects whose total topological charge is W. Conversely, several line defects whose total topological charge is W can merge into one line defect with a topological charge W. Moreover, when two world-lines intersect with different directions, two line defects meet and then separate at the bifurcation point. These results indicate that the line defects are unstable at these branch points of the vector order parameter. We should point out that such branch processes are stochastic processes, and the topological charges of line defects must be preserved during the branch processes. Finally, it should be pointed out that because of different natures from Abrikosov vortices, these kinds of topological defects (i.e., vortex lines) may exhibit some unconventional phenomena and thus may be interesting objects for experimental studies.

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