Electric field distribution and effective nonlinear AC and DC responses of graded cylindrical composites*

Ding Xia(丁 霞)[a], Jia Yan-Xia(贾艳霞)[b], and Wei En-Bo(魏恩泊)[c]†

[a] College of Mathematics and Physics, Qingdao University of Science & Technology, Qingdao 266061, China
[b] Hisense Trans-Tech Company Limited, Qingdao 266071, China
[c] Institute of Oceanology, Chinese Academy of Sciences, Qingdao 266071, China

(Received 8 October 2011; revised manuscript received 25 November 2011)

The perturbation method is used to study the localization of electric field distribution and the effective nonlinear response of graded composites under an external alternating-current (AC) and direct-current (DC) electric field $\boldsymbol{E}_{\rm app} = \boldsymbol{E}_0(1+\sin\omega t)$. The dielectric profile of the cylindrical inclusions is modeled by function $\varepsilon_{\rm i}(r) = C_k r^k$ $(r \le a)$, where r is the radius of the cylindrical inclusion, and C_k, k, a are parameters. In the dilute limit, the local potentials and the effective nonlinear responses at all harmonics are derived. Meanwhile, the general effective nonlinear responses are also derived and compared with the effective nonlinear responses at harmonics under the AC and DC external field. It is found that the effective nonlinear AC and DC responses at harmonics can be calculated by those of the general effective nonlinear of the graded composites under the external DC electric field. Moreover, the obtained local electrical fields show that the electrical field distribution in the cylindrical inclusions is controllable, and the maximum of the electric field inside the cylinder is at its center.

Keywords: graded composite, effective nonlinear alternating-current and direct-current response, local electric field distribution

PACS: 72.80.Tm, 42.79.Ry, 72.20.Ht, 41.20.Cv **DOI:** 10.1088/1674-1056/21/5/057202

1. Introduction

Graded materials, such as liquid crystal droplets and biological cells,[1−3] are abundant in nature, and their material properties can vary spatially.[4] The properties of these materials have received considerable attention because of their advantages over the traditional homogeneous composite materials in various engineering applications. The main advantage is that the graded composites can be tailored for the specific needs in material sciences via changing their compositions or microstructures. For instance, the electric conductivity, the thermal conductivity, and the optical and mechanical properties can be designed to vary along the radial direction of a cylindrical or spherical particle.[4−10] In addition, the effects of external field on the response of nonlinear graded composites should be investigated, because the local field is related to the external electric field. The potentials and the electric field distributions can be controlled in the graded inclusion regions by tuning the grading parameters. For example, the graded metamateriels were designed to control the electromagnetic field to realize cloaking at the microwave frequency experimentally.[11−12] Over the past few years, many methods have been developed to estimate the effective response of the graded composites under external direct-current (DC) fields,[13−19] such as the special-function method, the effective dipole approximation method, and the perturbation method.[20−23] Particularly, many authors have focused on the effective nonlinear alternating-current (AC) response of the nonlinear graded composites when an external AC field is applied.[24,25] Recently, Ding *et al.* have developed a perturbation method to focus on the effective nonlinear AC response of cylindrical graded composites with the presence of the external AC field in the dilute limit.[25,26] When the external AC field is applied to nonlinear graded composites, the distributions of potential and electric field induced in the media region

*Project supported by the National Natural Science Foundation of China (Grant Nos. 40876094 and JQ10974106), the National High Technology Research and Development Program of China (Grant Nos. 2009AA09Z102 and 2008AA09A403), the Excellent Youth Fundation of Shandong Scientific Committee, China (Grant No. JQ201018), and the Natural Science Foundation of Shandong Province, China (Grant No. ZR2009AZ002).

†Corresponding author. E-mail: ebwei@qdio.ac.cn

© 2012 Chinese Physical Society and IOP Publishing Ltd

will, in general, consist of components of all harmonics. Furthermore, the local field at harmonics leads to an important and useful effect on the effective nonlinear response. The effective response is useful to modulate the response of the graded composites from one profile to another.[27−29] For example, the nonlinear susceptibility of electrorheological (ER) fluid (which can be regarded as a kind of nonlinear composites) can be induced at all harmonics under an external AC electric field, and the results are well understood in the spectral representation theory.[21,28,29] However, to the best of our knowledge, there are rarely exact analytical solutions for graded composites with the DC or the AC external field. Therefore, the aim of this work is to investigate the local electrical field distribution of nonlinear cylindrical graded composites and the effects of the applied DC and AC electric field on the effective nonlinear dielectric response.

In this paper, we apply the perturbation method to treat the DC and AC responses of nonlinear graded composites. The external field is the sum of a DC field and a AC field, $\boldsymbol{E}_{\rm app} = \boldsymbol{E}_0(1+\sin\omega t)$. We study the effective dielectric response of a graded cylindrical composite with a power-law dielectric profile $\varepsilon_{\rm i}(r) = C_k r^k$ $(r \le a)$ embedded in a homogeneous host, the inclusions and the host materials are Kerr-like nonlinear in Ω_α

$$\boldsymbol{D}_\alpha = \varepsilon_\alpha \boldsymbol{E} + \chi_\alpha |\boldsymbol{E}|^2 \boldsymbol{E}, \tag{1}$$

where r is the radial variable of the cylindrical particle, subscripts $\alpha = {\rm i}, {\rm h}$ denote the physical quantities in the inclusion and the host regions, respectively, Ω_α $(\alpha = {\rm i}, {\rm h})$ is the region occupied by the α-type material, $\boldsymbol{D}$ and $\boldsymbol{E}$ are the electric displacement field and the electric field, respectively. Both real linear and nonlinear dielectric constants ε and χ are assumed to be independent of the frequency of the applied field, which is a valid assumption for the low-frequency processes. For nonlinear cylindrical composites, the linear dielectric response $\varepsilon_{\rm i}(r)$ is a function of space variable r in the cylindrical region. Dielectric responses $\varepsilon_{\rm h}$, $\chi_{\rm h}$, and $\chi_{\rm i}$ are all constants. The governing equation satisfies Gauss's law without source, $\nabla \cdot \boldsymbol{D} = 0$. For the low frequency, we make an assumption that the quasi-electrostatic field equation satisfies $\nabla \times \boldsymbol{E} = 0$. The boundary conditions are the continuities of potential Φ and the normal components of electric displacement field D on the surfaces of the inclusions.

2. Perturbation method

For external field $\boldsymbol{E}_{\rm app} = \boldsymbol{E}_0(1+\sin\omega t)$, with $\boldsymbol{E}_0 = \boldsymbol{E}_x$ along the x direction and a finite frequency ω, the potential at point x can be expanded as

$$\begin{aligned}\Phi_\alpha(x,t) =& \sum_{k=0} \Phi_{2k}^\alpha(x)\cos(2k\omega t) \\ &+ \sum_{k=0} \Phi_{2k+1}^\alpha(x)\sin[(2k+1)\omega t].\end{aligned} \tag{2}$$

For weakly nonlinear composites, the perturbation method is used to deal with the potentials.[19] We choose the nonlinear dielectric constant of the host medium $\chi_{\rm h}$ as the perturbation parameter, and expand the local potentials Φ_k^α at harmonic $k\omega$ in the following form:

$$\Phi_k^\alpha(x) = \sum_{s=0} \chi_{\rm h}^s \Phi_k^{\alpha,s}(x) \ \text{ in } \ \Omega_\alpha, \ \ \alpha = {\rm i}, {\rm h}. \tag{3}$$

Using electric field equation $\boldsymbol{E}_\alpha(x,\omega,t) = -\nabla\Phi_\alpha(x,\omega,t)$, we can obtain the perturbation expansion of electric displacement $\boldsymbol{D}_\alpha(x,t)$ by substituting Eqs. (2) and (3) into Eq. (1)

$$\begin{aligned}\boldsymbol{D}_\alpha(x,t) =& \sum_{k=0} \boldsymbol{D}_{2k}^\alpha(x)\cos(2k\omega t) \\ &+ \sum_{k=0} \boldsymbol{D}_{2k+1}^\alpha(x)\sin[(2k+1)\omega t], \\ \boldsymbol{D}_k^\alpha(x) =& \sum_{s=0} \chi_{\rm h}^s \boldsymbol{D}_k^{\alpha,s}(x),\end{aligned} \tag{4}$$

where dielectric displacement field $\boldsymbol{D}_k^{\alpha,s}(x)$ at harmonic frequency $k\omega$ is a function of potential $\Phi_k^{\alpha,s}(x)$. From the governing equation $\nabla \cdot \boldsymbol{D}(x,t) = 0$, a set of equations are obtained by comparing coefficients of the same power of the perturbation parameter in Ω_α

$$\nabla \cdot \boldsymbol{D}_k^{\alpha,s}(x) = 0, \quad \alpha = {\rm i}, {\rm h}. \tag{5}$$

The corresponding boundary conditions of each equation for the perturbation potential and the electric displacement field are

$$\begin{aligned}&\Phi_k^{{\rm i},s}(x) = \Phi_k^{{\rm h},s}(x)\,|_{r=a}\ , \\ &\boldsymbol{n}\cdot\boldsymbol{D}_k^{{\rm i},s}(x) = \boldsymbol{n}\cdot\boldsymbol{D}_k^{{\rm h},s}(x)\,|_{r=a}\ ,\end{aligned}$$

on the surface of inclusion region $\Omega_{\rm i}$.

At infinity, the potential must match that of the external applied field. Thus, if $s \ne 0$ and $k \ne 0, 1$, the potentials satisfy

$$\begin{aligned}&\partial\Phi_0^{{\rm h},0}/\partial x = \partial\Phi_1^{{\rm h},0}/\partial x = -E_0, \\ &\partial\Phi_k^{{\rm h},s}/\partial x = 0.\end{aligned}$$

3. Potential distribution

We consider a two-dimensional nonlinear graded composite of cylinder inclusions with dielectric function profile $\varepsilon_{\rm i}(r) = C_k r^k$ $(r \le a)$ randomly embedded in a homogeneous host with linear and nonlinear dielectric constants $\varepsilon_{\rm h}$ and $\chi_{\rm h}$ under external field $\boldsymbol{E}_{\rm app} = \boldsymbol{E}_0(1+\sin\omega t)$. The cylindrical inclusions are aligned in the z direction. In the cylindrical coordinates, the zero-order and the first-order potentials will be derived, and it is enough to calculate the effective cubic nonlinear response of the Kerr composites.

The governing equations of the zero-order potential at the zero frequency in the media regions are Laplace's equations. With the boundary conditions given in Section 2, we obtain the zero-order potential at the zero frequency

$$\Phi_0^{\rm h,0}(r,\varphi) = -(r + b_0 a^2 r^{-1})\cos(\varphi)E_0, \quad r \le a,$$

and

$$\Phi_0^{\rm i,0}(r,\varphi) = -H_0 r^s \cos(\varphi)E_0,$$

where $b_0 = (\varepsilon_{\rm h} - C_k s a^k)/(\varepsilon_{\rm h} + C_k s a^k)$, $s = (\sqrt{k^2+4} - k)/2$, $H_0 = 2\varepsilon_{\rm h} a^{1-s}/(\varepsilon_{\rm h} + sC_k a^k)$.

In addition, it has been pointed out that the governing equations and the boundary conditions of the zero-order potential at basic frequency ω are the same as those of the zero-order potential at the zero frequency. Hence, we have $\Phi_1^{\alpha,0}(r,\varphi) = \Phi_0^{\alpha,0}(r,\varphi)$. For the zero-order potentials of the second, the third, and the higher harmonics, these solutions equal to zero, as their boundary-value problems correspond to Laplace's equations without source.

At the zero frequency, the governing equations of the first-order potentials are

$$\begin{aligned} &\boldsymbol{\nabla}\cdot\varepsilon_\alpha(r)\cdot\boldsymbol{\nabla}\Phi_1^{1\alpha} + \varepsilon_\alpha(r)\boldsymbol{\nabla}^2\Phi_1^{1\alpha} \\ &= -\frac{5}{2}\boldsymbol{\nabla}\cdot(\beta_\alpha\boldsymbol{\nabla}\Phi_1^{0\alpha}\left|\boldsymbol{\nabla}\Phi_1^{0\alpha}\right|^2), \quad \alpha = {\rm i},{\rm h}, \end{aligned} \tag{6}$$

where $\beta_\alpha = \chi_\alpha/\chi_{\rm h}$. The general solutions in the host and the inclusions are

$$\begin{aligned} \Phi_0^{\rm h,1} &= (A_{1,0}^{\rm h,1} r^{-1} + A_{\rm h}^0 r^{-3} + A_{\rm h}^1 r^{-5})\cos(\varphi)E_0^3 \\ &\quad + (A_{3,0}^{\rm h,1} r^{-3} + A_{\rm h}^2 r^{-1})\cos(3\varphi)E_0^3, \\ \Phi_0^{\rm i,1} &= -(H_{1,0}^{\rm i,1} r^s + H_0^{\rm i} r^{3s-k-2})\cos(\varphi)E_0^3 \\ &\quad - (H_{3,0}^{\rm i,1} r^p + G_0^{\rm i} r^{3s-k-2})\cos(3\varphi)E_0^3. \end{aligned}$$

The unknown coefficients of potentials $\Phi_0^{\alpha,1}$ can be determined by the boundary conditions given in Section 2. The useful coefficients, which will be used to calculate the effective response, are

$$\begin{aligned} H_{1,0}^{\rm i,1} &= \Big[\frac{5}{2}\Big(a^2 - 2b_0 - \frac{1}{3}b_0^3 a^{-4}\Big) \\ &\quad - \frac{5}{8}s\beta_{\rm i}H_0^3 a^{3s-1}(3s^2+1) \\ &\quad - \varepsilon_{\rm h}a^{3s-k-1}H_0^{\rm i} - C_k a^{3s-1}(3s-k-2)H_0^{\rm i}\Big] \\ &\quad \times [(\varepsilon_{\rm h} + sC_k a^k)a^{(s+1)}]^{-1}, \\ H_0^{\rm i} &= 5\beta_{\rm i}H_0^3 d_1/2C_k e_1, \\ d_1 &= \Big[\frac{1}{2}s(s-1) - \frac{1}{4}ks\Big](3s^2+1) + \frac{1}{2}(s^2-1), \\ e_1 &= (3s-k-2)(3s-2) - 1. \end{aligned}$$

We deal with the first-order potential at the zero frequency following the above procedure. The first-order potentials of the basic frequency, the second, and the third harmonics are given as follows. (i) The first-order potentials at the basic frequency are

$$\begin{aligned} \Phi_1^{\rm h,1} &= (A_{1,1}^{\rm h,1} r^{-1} + B_{\rm h}^0 r^{-3} + B_{\rm h}^1 r^{-5})\cos(\varphi)E_0^3 \\ &\quad + (A_{3,1}^{\rm h,1} r^{-3} + B_{\rm h}^2 r^{-1})\cos(3\varphi)E_0^3, \\ \Phi_1^{\rm i,1} &= -(H_{1,1}^{\rm i,1} r^s + H_1^{\rm i} r^{3s-k-2})E_0^3\cos(\varphi) \\ &\quad - (H_{3,1}^{\rm i,1} r^p + G_1^{\rm i} r^{3s-k-2})E_0^3\cos(3\varphi). \end{aligned}$$

(ii) The first-order potentials at the second harmonic are

$$\begin{aligned} \Phi_2^{\rm h,1} &= (A_{1,2}^{\rm h,1} r^{-1} + C_{\rm h}^0 r^{-3} + C_{\rm h}^1 r^{-5})\cos(\varphi)E_0^3 \\ &\quad + (A_{3,2}^{\rm h,1} r^{-3} + C_{\rm h}^2 r^{-1})\cos(3\varphi)E_0^3, \\ \Phi_2^{\rm i,1} &= -(H_{1,2}^{\rm i,1} r^s + H_2^{\rm i} r^{3s-k-2})E_0^3\cos(\varphi) \\ &\quad - (H_{3,2}^{\rm i,1} r^p + G_2^{\rm i} r^{3s-k-2})E_0^3\cos(3\varphi). \end{aligned}$$

(iii) The first-order potentials at the third harmonic are

$$\begin{aligned} \Phi_3^{\rm h,1} &= (A_{1,3}^{\rm h,1} r^{-1} + D_{\rm h}^0 r^{-3} + D_{\rm h}^1 r^{-5})\cos(\varphi)E_0^3 \\ &\quad + (A_{3,3}^{\rm h,1} r^{-3} + D_{\rm h}^2 r^{-1})\cos(3\varphi)E_0^3, \\ \Phi_3^{\rm i,1} &= -(H_{1,3}^{\rm i,1} r^s + H_3^{\rm i} r^{3s-k-2})E_0^3\cos(\varphi) \\ &\quad - (H_{3,3}^{\rm i,1} r^p + G_3^{\rm i} r^{3s-k-2})E_0^3\cos(3\varphi). \end{aligned}$$

To derive the effective response, the useful relationships between these equation coefficients are given as

$$\begin{aligned} &H_{1,1}^{\rm i,1} = \frac{3}{2}H_{1,0}^{\rm i,1}, \quad H_{1,2..}^{\rm i,1} = -\frac{3}{5}H_{1,0}^{\rm i,1}, \\ &H_{1,3}^{\rm i,1} = -\frac{1}{10}H_{1,0}^{\rm i,1}. \end{aligned} \tag{7}$$

4. Effective response in the dilute limit

At low concentrations of graded cylindrical inclusions, the Landau method was previously proposed to estimate the effective coupling response at harmonics is applied[13]

$$\begin{aligned}&\frac{1}{V}\int_V [\boldsymbol{D}(x,\omega,t)-(\varepsilon_{\rm h}\boldsymbol{E}(x,\omega,t)\\&-\chi_{\rm h}\left|\boldsymbol{E}(x,\omega,t)\right|^2\boldsymbol{E}(x,\omega,t))]{\rm d}V\\&=\bar{\boldsymbol{D}}(\omega,t)-[\varepsilon_{\rm h}\bar{\boldsymbol{E}}(\omega,t)+\chi_{\rm h}\left|\bar{\boldsymbol{E}}(\omega,t)\right|^2\bar{\boldsymbol{E}}(\omega,t)],\end{aligned}\tag{8}$$

where V describes the volume of composites, i.e., $V=\Omega_{\rm i}+\Omega_{\rm h}$, $\bar{\boldsymbol{D}}$ and $\bar{\boldsymbol{E}}$ denote the volume averages of electric displacement field and electric field, respectively. In the dilute limit, the average electric field $\bar{\boldsymbol{E}}$ can be simply regarded as the external field, since the interaction among particles can be neglected

$$\begin{aligned}&\frac{1}{V}\int_{\Omega_{\rm i}} [\boldsymbol{D}(x,\omega,t)-(\varepsilon_{\rm h}\boldsymbol{E}(x,\omega,t)\\&-\chi_{\rm h}\left|\boldsymbol{E}(x,\omega,t)\right|^2\boldsymbol{E}(x,\omega,t))]{\rm d}V\\&=\bar{\boldsymbol{D}}(\omega,t)-[\varepsilon_{\rm h}\bar{\boldsymbol{E}}(\omega,t)+\chi_{\rm h}\left|\bar{\boldsymbol{E}}(\omega,t)\right|^2\bar{\boldsymbol{E}}(\omega,t)].\end{aligned}\tag{9}$$

Because the external field is the sum of the DC and the AC electric fields, many harmonic components are included in the average electric displacement field. The effective responses of the composites are defined by considering the cylindrical nonlinear composites as an effective homogeneous medium

$$\bar{\boldsymbol{D}}(\omega,t)=\sum\nolimits_{n=0}^{\infty}\bar{\boldsymbol{D}}_n\sin(n\omega t),\tag{10}$$

where $\bar{\boldsymbol{D}}_n=\varepsilon^*_{n\omega}\boldsymbol{E}_\alpha+\chi^*_{n\omega}\left|\boldsymbol{E}_\alpha\right|^2\boldsymbol{E}_\alpha+\cdots$, $n=0,1,2,\ldots$, and $\varepsilon^*_{n\omega}$ and $\chi^*_{n\omega}$ are the linear and the nonlinear effective dielectric responses at the n-th harmonic frequency $n\omega$, respectively.

Substituting the external electric field and Eq. (10) into Eq. (9), we can obtain the effective coupling responses of the nonlinear graded composites at harmonics in the dilute limit

$$\begin{aligned}\varepsilon^*_0&=\varepsilon_{\rm h}+f_{\rm i}H_0\left[\frac{C_k(s+1)a^{s+k-1}}{s+k+1}-\varepsilon_{\rm h}a^{s-1}\right],\\\chi^*_0&=\frac{5}{2}\chi_{\rm h}+\frac{5}{2}H_0^3f_{\rm i}(\chi_{\rm i}-\chi_{\rm h})\frac{3s^3+s^2+s+3}{3s-1}a^{3(s-1)}\\&\quad+\chi_{\rm h}f_{\rm i}H^{\rm i,1}_{1,0}\left[C_ka^{s+k-1}\frac{s+1}{s+k+1}-\varepsilon_{\rm h}a^{s-1}\right]\\&\quad+\chi_{\rm h}f_{\rm i}H^{\rm i}_0\left[C_ka^{3(s-1)}\frac{3s-k-1}{3s-1}-\varepsilon_{\rm h}a^{3s-k-3}\right],\\\varepsilon^*_\omega&=\varepsilon_{\rm h}+f_{\rm i}H_0\left[\frac{C_k(s+1)a^{s+k-1}}{s+k+1}-\varepsilon_{\rm h}a^{s-1}\right],\\\chi^*_\omega&=\frac{15}{4}\chi_{\rm h}+\frac{15}{4}H_0^3f_{\rm i}(\chi_{\rm i}-\chi_{\rm h})\\&\quad\times\frac{3s^3+s^2+s+3}{3s-1}a^{3(s-1)}\\&\quad+\frac{3}{2}\chi_{\rm h}f_{\rm i}H^{\rm i,1}_{1,0}[C_ka^{s+k-1}\frac{s+1}{s+k+1}-\varepsilon_{\rm h}a^{s-1}]\\&\quad+\frac{3}{2}\chi_{\rm h}f_{\rm i}H^{\rm i}_0[C_ka^{3(s-1)}\\&\quad\times\frac{3s-k-1}{3s-1}-\varepsilon_{\rm h}a^{3s-k-3}],\\\varepsilon^*_{2\omega}&=0,\\\chi^*_{2\omega}&=-\frac{3}{2}\chi_{\rm h}-\frac{3}{2}H_0^3f_{\rm i}(\chi_{\rm i}-\chi_{\rm h})\\&\quad\times\frac{3s^3+s^2+s+3}{3s-1}a^{3(s-1)}\\&\quad-\frac{3}{5}\chi_{\rm h}f_{\rm i}H^{\rm i,1}_{1,0}[C_ka^{s+k-1}\frac{s+1}{s+k+1}-\varepsilon_{\rm h}a^{s-1}]\\&\quad-\frac{3}{5}\chi_{\rm h}f_{\rm i}H^{\rm i}_0[C_ka^{3(s-1)}\\&\quad\times\frac{3s-k-1}{3s-1}-\varepsilon_{\rm h}a^{3s-k-3}],\\\varepsilon^*_{3\omega}&=0,\\\chi^*_{3\omega}&=-\frac{1}{4}\chi_{\rm h}-\frac{1}{4}H_0^3f_{\rm i}(\chi_{\rm i}-\chi_{\rm h})\\&\quad\times\frac{3s^3+s^2+s+3}{3s-1}a^{3(s-1)}\\&\quad-\frac{1}{16}\chi_{\rm h}f_{\rm i}H^{\rm i,1}_{1,0}[C_ka^{s+k-1}\frac{s+1}{s+k+1}-\varepsilon_{\rm h}a^{s-1}]\\&\quad-\frac{1}{16}\chi_{\rm h}f_{\rm i}H^{\rm i}_0[C_ka^{3(s-1)}\\&\quad\times\frac{3s-k-1}{3s-1}-\varepsilon_{\rm h}a^{3s-k-3}],\end{aligned}\tag{11}$$

where $f_{\rm i}$ is the volume fraction of the cylindrical inclusion, and the relationships in Eq. (7) have also been taken into account.

5. General nonlinear response

The concept of general nonlinear response in a nonlinear composite is used to estimate the general nonlinear response of graded composites under the AC field. The concept was first proposed by Gu *et al.*[30] to deal with an inhomogeneous nonlinear composite. Here, for the effective homogeneous medium, the general nonlinear response is defined as[19]

$$\begin{aligned}\langle\boldsymbol{D}(x,\omega,t)\rangle&=\varepsilon_e\langle\boldsymbol{E}(x,\omega,t)\rangle\\&\quad+\chi_e\langle\left|\boldsymbol{E}(x,\omega,t)\right|^2\boldsymbol{E}(x,\omega,t)\rangle+\cdots,\end{aligned}\tag{12}$$

where $\varepsilon_{\rm e}$ and $\chi_{\rm e}$ are the linear and the nonlinear effective dielectric responses, respectively. In the dilute

limit, the angled bracket $\langle\cdots\rangle = 1/V\int_V \cdots \mathrm{d}V$ denotes the ensemble average over the whole media region. We extend the Landau and Lifshits formula to calculate the general effective nonlinear response using Eq. (12) and considering the fact that the integration of the left-hand side is zero in the host regions

$$\begin{aligned}&\frac{1}{V}\int_{\Omega_\mathrm{i}}[(\varepsilon_\mathrm{i}(r)-\varepsilon_\mathrm{h})\boldsymbol{E}(x,\omega,t)\\&+(\chi_\mathrm{i}-\chi_\mathrm{h})\left|\boldsymbol{E}(x,\omega,t)\right|^2\boldsymbol{E}(x,\omega,t)\mathrm{d}V\\&=(\varepsilon_\mathrm{e}-\varepsilon_\mathrm{h})\left\langle\boldsymbol{E}(x,\omega,t)\right\rangle+(\chi_\mathrm{e}-\chi_\mathrm{h})\\&\times\left\langle\left|\boldsymbol{E}(x,\omega,t)\right|^2\boldsymbol{E}(x,\omega,t)\right\rangle.\end{aligned}\tag{13}$$

At low inclusion concentrations, the average electric field is approximated by the external field. Substituting the local and the external electric fields into Eq. (13), we obtain the general effective nonlinear response using the relationships in Eq. (7) as

$$\varepsilon_\mathrm{e}=\varepsilon_\mathrm{h}+f_\mathrm{i}H_0\left[\frac{C_k(s+1)a^{s+k-1}}{s+k+1}-\varepsilon_\mathrm{h}a^{s-1}\right],\tag{14}$$

$$\begin{aligned}\chi_\mathrm{e}=&\chi_\mathrm{h}+\frac{1}{4}H_0^3f_\mathrm{i}(\chi_\mathrm{i}-\chi_\mathrm{h})\frac{3s^3+s^2+s+3}{3s-1}a^{3(s-1)}\\&+\frac{2}{5}\chi_\mathrm{h}f_\mathrm{i}H_{1,0}^{\mathrm{i},1}\left[C_ka^{s+k-1}\frac{s+1}{s+k+1}-\varepsilon_\mathrm{h}a^{s-1}\right]\\&+\frac{2}{5}\chi_\mathrm{h}f_\mathrm{i}H_0^\mathrm{i}\left[C_ka^{3(s-1)}\frac{3s-k-1}{3s-1}\right.\\&\left.-\varepsilon_\mathrm{h}a^{3s-k-3}\right].\end{aligned}\tag{15}$$

6. Numerical results of electrical field distribution

With the above solutions of the potentials, the local electric field distribution can be obtained by $\boldsymbol{E}=-\boldsymbol{\nabla}\Phi$. The local electric field at harmonics along the x axis is expressed as

$$E_x=E^\mathrm{dc}+E_\omega\sin\omega t+E_{2\omega}\cos 2\omega t+E_{3\omega}\sin 3\omega t+\cdots.$$

Using Eq. (7), we can obtain the following simple relationship:

$$E_{3\omega}=E_{2\omega}/6.\tag{16}$$

We have calculated the coefficient moduli of the corresponding local electric fields of the DC component and the fundamental-order harmonic, as shown in Figs. 1–4 for different reduced graded composite parameters k and C_k. It can be seen that the harmonics of the local electric field distributions are sensitive to the graded composite parameters k and C_k in the cylindrical inclusions ($0\le r\le 1$). In Figs. 1 and 2, the DC electrical field and the fundamental harmonic field moduli increase toward the center of the cylinder from $r=0$ to $r=0.1$ with the increasing k.

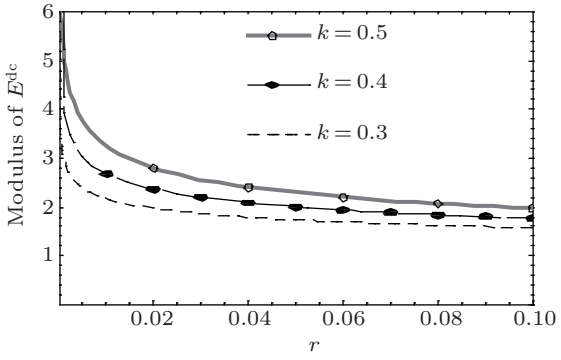


Fig. 1. Distributions of the DC component of electric field modulus $|\boldsymbol{E}|$ along the x axis. The parameters are $C_k=0.9$, $E_0=1$, $\varphi=0$, and $\varepsilon_\mathrm{h}=1$.

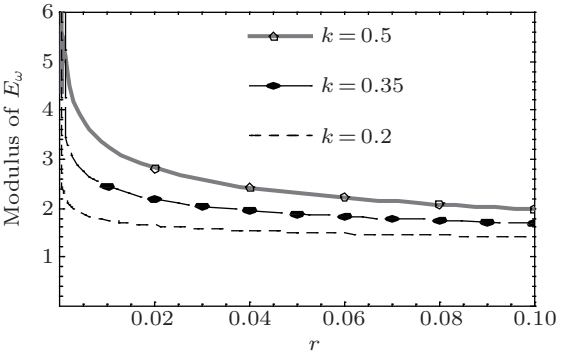


Fig. 2. Distributions of electric field modulus $|\boldsymbol{E}|$ at frequency ω along the x axis, the parameters are the same as those given in Fig. 1.

The same results can be obtained by decreasing grading parameter C_k, as shown in Figs. 3 and 4. It can be clearly seen that the electric field can be strongly concentrated at the center of the cylinder for different k and C_k. Ranging from 0.1 to 1, the electric field is nearly uniform (without any peak). Meanwhile, the local electric field distributions of the second-order and the third-order harmonics varying with parameters k and C_k (figure omitted) are similar to the results shown in Fig. 2.

In summary, the field can be concentrated at the center of the cylinder for different values of k and C_k. The local electric field distributions at harmonics can be exactly controlled by tuning grading parameters k and C_k. The results are similar to those of the localization field properties of graded cylindrical composites under the optical field.[31] Thus, it is also instructive

to generalize the results obtained here to the case of graded cylindrical materials at low frequencies.

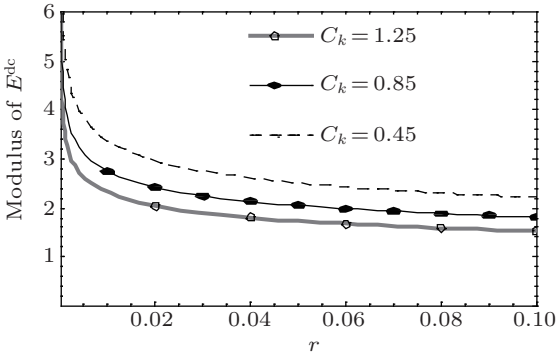


Fig. 3. Distributions of the DC component of electric field modulus $|\boldsymbol{E}|$ along the x axis, where $k = 0.4$, and the other parameters are given in Fig. 1.

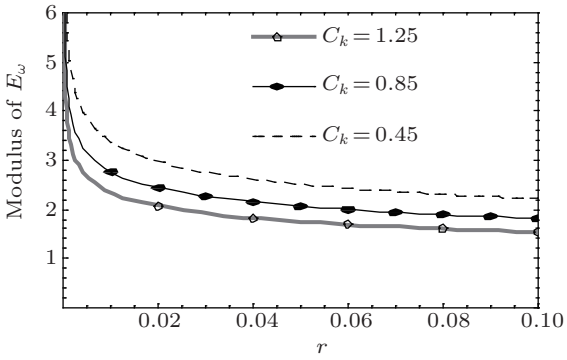


Fig. 4. Distributions of electric field modulus $|\boldsymbol{E}|$ at frequency ω along the x axis, where $k = 0.4$, the other parameters are given in Fig. 2.

7. Discussion

Using Eqs. (11), (14), and (15), we can obtain the following simple relationships:

$$\varepsilon_0^* = \varepsilon_\omega^* = \varepsilon_{\rm e}, \quad \chi_0 = 5\chi_{\rm e}/2, \quad \chi_\omega^* = 15\chi_{\rm e}/4,$$
$$\chi_{2\omega}^* = -3\chi_{\rm e}/2, \quad \chi_{3\omega}^* = -\chi_{\rm e}/4. \tag{17}$$

They imply that the relationships can be used to estimate the effective nonlinear responses at the basic, the second, the third, and the higher harmonics provided that the nonlinear effective response at the zero frequency is known. Furthermore, we can obtain a comprehensive understanding of the nonlinear response in the electrorheological fluid from the above relationships.[21,22]

We will show that the effective coupling responses of nonlinear graded composites at all harmonics can be estimated by the effective response of nonlinear graded composites under the DC external field, which can be used to calculate the effective coupling response of nonlinear graded composites under the AC and DC field. Furthermore, if we substitute the radius of the cylindrical inclusion $a = 1$ into Eqs. (14) and (15), we have the following DC and AC effective responses:

$$\varepsilon_{\rm e} = \varepsilon_{\rm h} + f_{\rm i} H_0 [C_k (s+1)/(s+k+1) - \varepsilon_{\rm h}], \tag{18}$$

$$\chi_{\rm e} = \chi_{\rm h} + \frac{1}{4} H_0^3 f_{\rm i} (\chi_{\rm i} - \chi_{\rm h}) \frac{(3s^3 + s^2 + s + 3)}{3s - 1} + \frac{2}{5} \chi_{\rm h} f_{\rm i} H_{1,0}^{\rm i,1} \left[C_k \frac{s+1}{s+k+1} - \varepsilon_{\rm h} \right] + \frac{2}{5} \chi_{\rm h} f_{\rm i} H_0^{\rm i} \left[C_k \frac{3s-k-1}{3s-1} - \varepsilon_{\rm h} \right], \tag{19}$$

where $f_{\rm i}$ is the volume fraction of the cylindrical inclusions. The relationships in Eq. (7) and coefficients $H_{1,0}^{\rm i,1}$ and $H_0^{\rm i}$ given in Section 3 have been used to derive Eqs. (18) and (19). Comparing our results of Eqs. (18) and (19) with the previous ones under the external DC electric field obtained in Ref. [20], we have $H_{1,0}^{\rm i,1} = 5H_1/2$, $H_0^{\rm i} = 5H/2$. Substituting the relations into Eqs. (18) and (19) , we find that equations (18) and (19) are the effective responses of nonlinear graded composites under the external DC electric field. Then, using the relationships of Eq. (17), the effective nonlinear dielectric DC and AC responses of nonlinear graded composites at harmonics can be calculated by using the general effective response of nonlinear graded composites under the external DC field.

In Fig. 5, with Eq. (18), we plot the effective nonlinear dielectric response of graded cylindrical composites with dielectric profile $\varepsilon_{\rm i}(r) = C_k r^k$ versus parameter k at the volume fraction of $f_{\rm i} = 0.1$. Clearly, the effective nonlinear dielectric response contrast ratio $\chi_{\rm e}/\chi_{\rm h}$ increases (or decreases) as parameter k increases with a large (or small) contrast ratio $\chi_{\rm i}/\chi_{\rm h}$.

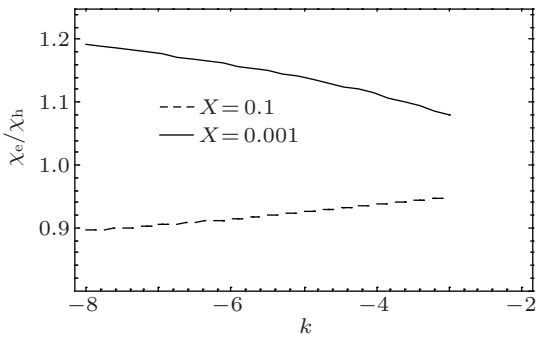


Fig. 5. Effective nonlinear dielectric response contrast radio $\chi_{\rm e}/\chi_{\rm h}$ is plotted against parameter k at $f_{\rm i} = 0.1$, where $\chi_{\rm e}/\chi_{\rm h}$ and X are $\chi_{\rm i}/\chi_{\rm h}$ and $(\chi_{\rm e}/\chi_{\rm h})/\max(\chi_{\rm e}/\chi_{\rm h})$, respectively, and $c_k/\varepsilon_{\rm h} = 1$.

Our results can be reduced to the effective nonlinear responses of weakly nonlinear composites when we

take the limit of $k \to 0$ (i.e., $\varepsilon_{\rm i} = c_0$) for the power-law dielectric profile. Taking $k \to 0$, we have

$$\begin{aligned}
&\lim_{k\to 0} H_0 = 2\varepsilon_{\rm h}/(\varepsilon_{\rm h} + c_0) = c,\\
&\lim_{k\to 0} b_0 = (\varepsilon_{\rm h} - c_0)/(\varepsilon_{\rm h} + c_0) = M_0,\\
&\lim_{k\to 0}(s+1)/(s+k+1) = 1,\\
&\lim_{k\to 0}(3s^3 + s^2 + s + 3)/(3s-1) = 4,\\
&\lim_{k\to 0}\{H^{\rm i,1}_{1,0}[(s+1)/(s+k+1) - \varepsilon_{\rm h}]\\
&- H^{\rm i}_0[c_k(3s-k-1)/(3s-1) - \varepsilon_{\rm h}]\}\\
&= (c_0 - \varepsilon_{\rm h})(1 - 2M_0 - M_0^3/3 - \beta_{\rm i}c^3)/(c_0 + \varepsilon_{\rm h})\\
&= B^{\rm i,1}_{\rm i,0}(c_0 - \varepsilon_{\rm h}),
\end{aligned}$$

where

$$\begin{aligned}
&B^{\rm i,1}_{\rm i,0} = 2(1 - 2M_0 - 1/3M_0^3 - \beta_{\rm i}c^3)/5(\varepsilon_{\rm h} + \varepsilon_{\rm i}),\\
&M_0 = (\varepsilon_{\rm h} - c_0)/(\varepsilon_{\rm h} + c_0).
\end{aligned}$$

Substituting these limits into Eqs. (18) and (19), we have the following effective responses:

$$\varepsilon_{\rm e} = \varepsilon_{\rm h} + f_{\rm i}(\varepsilon_{\rm i} - \varepsilon_{\rm h})c, \tag{20}$$

$$\chi_{\rm e} = \chi_{\rm h} + f_{\rm i}[(\varepsilon_{\rm i} - \varepsilon_{\rm h})\chi_{\rm h}B^{\rm i,1}_{\rm i,0} + (\chi_{\rm i} - \chi_{\rm h})c^3]. \tag{21}$$

Let us compare our results of a cylindrical particle under the AC and DC field with the previous ones derived in Ref. [13] under a DC external field. For constitutive relationship Eq. (1), the effective responses of nonlinear composites in the dilute limit are[13,31]

$$\varepsilon^* = \varepsilon_{\rm h} + f_{\rm i}(\varepsilon_{\rm i} - \varepsilon_{\rm h})c, \tag{22}$$

$$\chi^* = \chi_{\rm h} + f_{\rm i}[(\varepsilon_{\rm i} - \varepsilon_{\rm h})\chi_{\rm h}b_3 + (\chi_{\rm i} - \chi_{\rm h})c^3], \tag{23}$$

where ε^* and χ^* are the linear and the nonlinear effective dielectric responses of Kerr nonlinear composites under the external DC electric field, respectively, and $b_3 = (1 - 2b_0 - 1/3b_0^3 - \beta_{\rm i}c^3)/(\varepsilon_{\rm h} + \varepsilon_{\rm i})$. Comparing $B^{\rm i,1}_{\rm i,0}$ with b_3, we have $b_3 = 5/2B^{\rm i,1}_{\rm i,0}$. Substituting the relation into Eqs. (21) and (22), we have $\varepsilon_{\rm e} = \varepsilon^*$ and $\chi_{\rm e} = \chi^*$. As shown, our results can be exactly reduced to the effective responses of nonlinear homogeneous composites.

8. Conclusion

For nonlinear graded composites under DC and AC external electric field, we have developed a perturbation method to calculate the effective nonlinear response in the dilute limit. The local potentials are derived, and the controls of local electric fields by cylindrical graded particles with power-law dielectric profiles have been studied. By varying the graded composite parameters k and C_k, we can confine the field to the center of the graded cylinder. It will be useful for investigating the electrical field control in more general graded materials.[31,32] For instance, the localization of electrical fields can be used in electric control and printing technologies.

Meanwhile, the relationships between the nonlinear effective DC and AC responses at harmonics and the general effective nonlinear response are derived in the dilute limit. These relationships can be used to estimate the effective nonlinear DC and AC responses at harmonics if we have obtained the general effective nonlinear response of nonlinear graded composites under the DC electric field in experiment. Our result is different from that in Ref. [25], because the effective responses vanish at the even harmonic frequencies and the zero frequency in the Kerr composites with external field $\boldsymbol{E}_0 \sin \omega t$. Our results can be reduced to be exactly the effective response of nonlinear homogeneous composites. On the basis of our method, we can also deal with the effective nonlinear response of spherical nonlinear graded composites with the general power-law dielectric and the complex dielectric profiles under the DC and AC external electric field. For higher concentrations, the effective response of nonlinear graded composites can be studied by EMA and TFM methods combined with our potential solution of graded composites under the DC and AC electric field.[19,33,34] In engineering applications of nonlinear materials, the formulae can be used to predict the nonlinear effective DC and AC responses and the general effective nonlinear response of nonlinear graded composites. It is note that there are some other methods for dealing with nonlinear graded composites in the dilute limit, for example, the differential effective dipole approximation (NDEDA).[35–38]

References

[1] Zeng X C, Bergman D J, Hu P M and Stroud D 1988 *Phys. Rev.* B **38** 10970

[2] Amada S 1995 *Mater. Res. Bull.* **20** 35

[3] Holt J B, Koizumi K, Hirai T and Munir Z A 1993 *Ceramic Transaction: Functionally Graded Materials* (Westerville: American Ceramic Cociety) p. 34

[4] Milton G W 2002 *The Theory of Composites* (Cambridge: Cambridge University Press)

[5] Jin Z H and Noda N J 1994 *Therm. Stresse.* **17** 591

[6] Noda N J and Jin Z H 1993 *Int. J. Solids Struc.* **30** 1039

[7] Haus J W, Inguva R and Bowden M C 1989 *Phys. Rev.* A **40** 5729
[8] Delale F and Erdogan F 1983 *J. Appl. Mech.* **50** 609
[9] Suresh S 2001 *Science* **292** 2447
[10] Huang J P and Yu K W 2004 *Appl. Phys. Lett.* **85** 94
[11] Wei E B, Sun Lei and Yu K W 2010 *Chin. Phys.* B **19** 107802
[12] Pendry J B, Schurig D and Smith D R 2006 *Science* **312** 1780
[13] Gu G Q and Yu K W 1992 *Phys. Rev.* B **46** 4502
[14] Levy O, Bergman D J and Stroud D G 1995 *Phys. Rev.* E **52** 3184
[15] Huang J P 2005 *J. Phys. Chem.* B **109** 4824
[16] Wei E B, Poon Y M and Shin F G 2005 *Phys. Lett.* A **336** 264
[17] Wei E B and Wu Z K 2004 *J. Phys.*: *Condens. Matter* **16** 5377
[18] Gu G Q and Yu K W 2005 *J. Composite. Matter.* **39** 2
[19] Wei E B and Song J B 2004 *J. Appl. Phys.* **37** 1377
[20] Gu G Q and Yu K W 2003 *J. Appl. Phys.* **94** 3376
[21] Li D, Huang J P, Yu K W and Gu G Q 2004 *J. Appl. Phys.* **95** 621
[22] Li G, Huang J P and Yu K W 2004 *Phys. Rev.* B **69** 075105
[23] Shen Y Y, Chen X G, Li Q Q, Cui W and Hao Y H 2009 *Commun. Theor. Phys.* **51** 743
[24] Wei E B, Poon Y M and Yu K W 2006 *Phys. Lett.* A **350** 159
[25] Ding X, Jia X Y and Wang X 2008 *Phys. Lett.* A **372** 4247
[26] Gu G Q, Hui P M and Yu K W 2000 *Phys.* B **279** 62
[27] Huang J P, Li D and Yu K W 2004 *Europhys. Lett.* **67** 854
[28] Huang J P and Yu K W 2005 *J. Appl. Phys.* **97** 013102
[29] Huang J P 2004 *Phys. Rev.* E **70** 011403
[30] Gu G Q, Yu K W and Wang B H 1995 *J. Appl. Phys.* **78** 1737
[31] Fung T H, Leung L L, Xiao J J and Yu K W 2009 *Opt. Commun.* **282** 1028
[32] Wei E B and Yu K W 2009 *Phys. Rev.* E **80** 046607
[33] Liu Y and Li Z 2008 *Commun. Theor. Phys.* **49** 231
[34] Wei E B, Gu G Q, Poon Y M and Shin F G 2010 *Commun. Theor. Phys.* **53** 377
[35] Huang J P, Li G, Yu K W and Gu G Q 2004 *Phys. Rev.* E **69** 036605
[36] Wei E B, Gu G Q and Yu K W 2007 *Phys. Rev.* B **76** 134206
[37] Ma J, Shi Z, Lin Y H and Nan C W 2009 *Acta Phys. Sin.* **58** 5852 (in Chinese)
[38] Xu H, Wang S L, Liu A P, Chen B Y and Tang W H 2010 *Acta Phys. Sin.* **59** 3601 (in Chinese)