Sudden birth and sudden death of thermal fidelity in a two-qubit system*

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We study the energy level crossing and the thermal fidelity in a two-qubit system with the presence of a transverse inhomogeneous magnetic field. With the help of contour plots, we clearly identify the ground states of the system in different regions of parameter space, and discuss the corresponding energy level crossing. The fidelity between the ground state of the system and the state of the system at temperature T is calculated. The result shows that the fidelity is very sensitive to the magnetic field anisotropic factor, indicating that this factor may be used as a controller of the fidelity. The influence of the Yangian transition operators on the fidelity of the system is discussed. We find that the Yangian operators can change the fidelity dramatically and give rise to sudden birth and sudden death phenomena of the thermal fidelity. This makes the corresponding Yangian operators possible candidates for switchers to turn the fidelity on and off.

Keywords: the ground state, energy level crossing, fidelity, Yangian

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1. Introduction

Energy levels and eigenvalues of a Hamiltonian, especially the ground states, play an important role in determining the properties of a quantum system.[1] When the energy gap between the ground state and the lowest excited state tends to zero, the energy level crossing will emerge.[2] It is well known that the degeneracy will lead to quantum phase transitions and various peculiar phenomena.[1−3] Hence, it is very important to investigate level crossings for understanding the change in the properties of the system. Although Bhattacharya and Raman presented an effective algebraic method for finding level crossings,[4,5] it is worthwhile to explore an alternative way to understand how the eigenvalues of the Hamiltonian change at the crossing.

The fidelity, a new concept originated from quantum information theory to detect the overlap between states, has been under extensive investigation.[6−19] In particular, as a geometric measure,[20,21] the fidelity may represent an effective approach to identify the quantum state overlap,[22−25] and is a good indicator to energy level crossings.[26,27] At present, most efforts have been devoted to the ground-state fidelity originating from slightly different values of the controlling parameters.

In this work, we will study the evolution of the fidelity between the ground state and the state of the system at temperature T. And we will investigate a two distinguishing qubits system in a transverse inhomogeneous magnetic field, and calculate the energy level crossings and the quantum fidelity. For the XY spin chain model in a homogeneous magnetic field, the fidelity has been studied.[6,11] Here, we would like to calculate the fidelity of the two-qubit system in a non-homogeneous magnetic field, which acts on the two qubits differently. By taking advantage of this nonuniform characteristic, we will study its effects on the energy level crossings and the thermal fidelity, and try to find new interesting physical phenomena, such as the sudden birth and the sudden death of thermal

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fidelity. Actually, similar behaviors have already been found in multipartite systems.[28]

On the other hand, due to the particular characteristics of the Yangian algebra[29,30] in dealing with physical models, i.e., the property of symmetry and the ability for introducing state transitions, the Yangian algebra has been widely studied[31−33] in recent years. Researchers have found the Yangian symmetry in many physical models.[34−36] It is quite important and helpful to explore physical systems by using the transition characteristic of Yangian. In this paper, by making use of the transition effect of Yangian operators in the tensor product space, we will investigate their effects on the thermal fidelity of the non-homogeneous two-qubit system and explore an algebra method for controlling the fidelity of the system.

The paper is organized as follows. In Section 2, we introduce a two-qubit XY spin model. The system is divided into two subspaces: the isotropy and the anisotropy spaces. Then the energy level crossings are studied for different cases in detail in Section 3. In Section 4, the thermal fidelity between the ground state and the state of the system at temperature T is studied in the parametric space. In Section 5, we present the effects of Yangian operators on the thermal fidelity. Finally, Section 6 is devoted to the conclusion.

2. Model

The Hamiltonian of a two-qubit XY spin model with an inhomogeneous magnetic field in the z direction reads

$$H = -\frac{1+\gamma}{2}\sigma_1^x\sigma_2^x - \frac{1-\gamma}{2}\sigma_1^y\sigma_2^y - B(\sigma_1^z + \lambda\sigma_2^z), \quad (1)$$

where σ_i^α are the Pauli matrices of the i-th qubit, $\alpha = x, y, z$, B is the strength of the external magnetic field, λ is the anisotropic factor of the magnetic field and denotes the non-uniformity degree of the magnetic field, and γ is the anisotropic factor in the interaction.

The eigenstates and the corresponding eigenvalues of the Hamiltonian can be obtained as

$$\begin{aligned}
&H|\psi_1\rangle = E_1|\psi_1\rangle = -\sqrt{\xi^2+1}|\psi_1\rangle,\\
&H|\psi_2\rangle = E_2|\psi_2\rangle = \sqrt{\xi^2+1}|\psi_2\rangle,\\
&H|\psi_3\rangle = E_3|\psi_3\rangle = \sqrt{\eta^2+\gamma^2}|\psi_3\rangle,\\
&H|\psi_4\rangle = E_4|\psi_4\rangle = -\sqrt{\eta^2+\gamma^2}|\psi_4\rangle, \quad (2)\\
&|\psi_1\rangle = \frac{1}{N_1}(|10\rangle + a_1|01\rangle),\\
&|\psi_2\rangle = \frac{1}{N_2}(|10\rangle + a_2|01\rangle),\\
&|\psi_3\rangle = \frac{1}{N_3}(\gamma|11\rangle + a_3|00\rangle),\\
&|\psi_4\rangle = \frac{1}{N_4}(\gamma|11\rangle + a_4|00\rangle), \quad (3)
\end{aligned}$$

where $a_1 = \xi + \sqrt{\xi^2+1}$, $a_2 = \xi - \sqrt{\xi^2+1}$, $a_3 = \eta - \sqrt{\eta^2+\gamma^2}$, $a_4 = \eta + \sqrt{\eta^2+\gamma^2}$, $\xi = B(1-\lambda)$, $\eta = B(1+\lambda)$, $|0\rangle$ and $|1\rangle$ stand for the spin down and the spin up states, respectively, and N_i is the normalization coefficient for $|\psi_i\rangle$ $(i = 1, 2, 3, 4)$.

The Hamiltonian H can acquire a more effective form by rewriting Eq. (1) in the matrix form[37,38]

$$H = H_{\rm e} \oplus H_{\rm o}, \quad (4)$$

where $H_{\rm e} = -\begin{pmatrix} \eta & \gamma \\ \gamma & -\eta \end{pmatrix}$ and $H_{\rm o} = -\begin{pmatrix} \xi & 1 \\ 1 & -\xi \end{pmatrix}$ acting on the isotropy subspace spanned by $\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ and the anisotropy subspace spanned by $\{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$, respectively. From Eqs. (2) and (4), we can recognize that $E_{3,4}$ and $|\psi_{3,4}\rangle$ are the corresponding eigenvalues and eigenvectors of the Hamiltonian $H_{\rm e}$, and $|\psi_4\rangle$ is the ground state in the isotropy subspace. Similarly, $E_{1,2}$ and $|\psi_{1,2}\rangle$ are the eigenvalues and the eigenvectors of the Hamiltonian $H_{\rm o}$, and the ground state is $|\psi_1\rangle$. The two subspaces are also closely related to the Yangian algebra as will be shown below. With the two subspaces, the energy level crossings and the thermal fidelity will be studied in detail as follows.

3. Energy level crossings of the ground state

From Eq. (2), we learn that the ground state can only be $|\psi_1\rangle$ or $|\psi_4\rangle$, however, which one is the ground state depends on the choice of the parameters. When the energy difference $\Delta E = E_1 - E_4 < 0$, $|\psi_1\rangle$ is the ground state; when $\Delta E > 0$, $|\psi_4\rangle$ is the ground state. And the ground states are degenerate when $\Delta E = 0$. In order to express clearly the evolution of the energy difference ΔE in the parameter space, we show the evolution graphically.

Figure 1 shows the contour plots of the energy difference as a function of the magnetic field strength B and the anisotropic factor λ or γ for different cases. The energy level crossing of the system occurs at the zero-contour lines, on which $E_1 = E_4$. It is clear that the ground state changes abruptly at $\Delta E = 0$ in Fig. 1.

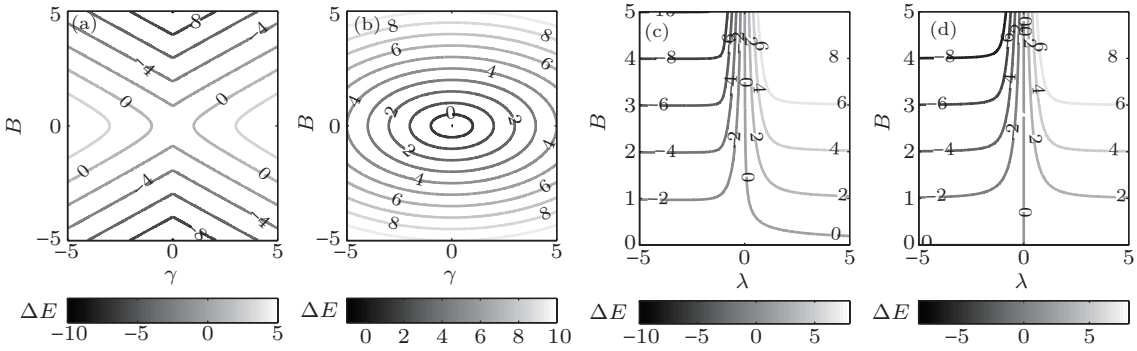


Fig. 1. Contour plots of energy difference. The model is in an inhomogeneous magnetic field with $\lambda = -1$ in panel (a) and in a homogeneous magnetic field with $\lambda = 1$ in panel (b). The model is the ordinary XY model in panel (c), and reduces to the Ising model in panel (d).

It is quite obvious from Fig. 1 that the behaviors of contours of ΔE versus B and γ in a non-homogeneous magnetic field with $\lambda = -1$ and a homogeneous magnetic field with $\lambda = 1$ are very different. In Fig. 1(a), the contours of $\Delta E < 0$ are located in the upper and the lower parts, which indicates that in these parameter regions the ground state is $|\psi_1\rangle$. In the right and the left parts, $\Delta E > 0$ shows that the ground is $|\psi_4\rangle$. The contours in Fig. 1(b) are quite different from those in Fig. 1(a), they are a series of concentric ellipses, and the energy difference becomes larger as the radius increases. Especially, $\Delta E < 0$ is within the zero contour. By comparing these two figures, we can find that the probability of $|\psi_1\rangle$ as the ground state, i.e., the areas of $\Delta E < 0$ in the (γ, B) plane, is comparatively larger in the inhomogeneous magnetic field and almost disappears in the case of the homogeneous magnetic field.

Now we investigate the evolution of the energy difference versus λ and B at some fixed γ. Without loss of generality, we consider the cases of $\gamma = 0.5$ and $\gamma = \pm 1$. In the former case, the system is an XY model, while the latter one corresponds to a transverse-field Ising model. Figures 1(c) and 1(d) show the contours of ΔE for these two models. In the case of $\gamma = 0.5$, the probability of $|\psi_1\rangle$ as the ground state is larger than that of $|\psi_4\rangle$. In the case of $\gamma = \pm 1$, the zero contour is a straight line at $\lambda = 0$, and the contour plots are bilateral. The probability of $|\psi_1\rangle$ as the ground state is equal to the case of $|\psi_4\rangle$.

Figure 1 shows that we can obtain different ground states by adjusting the parameters. It is worth recalling that $|\psi_4\rangle$ is in the isotropy subspace while $|\psi_1\rangle$ is in the anisotropy subspace. If the ground state of the system, $|\psi_1\rangle$ or $|\psi_4\rangle$, is identified, we can confirm that the system is in the isotropy or the anisotropy subspace at zero temperature.

4. Evolution of fidelity

It is well-known that the ground state plays an important role in determining the properties of a system. Hence, the relation between the ground state and the other states of the system becomes very important.

In this section, we turn our attention to probe the overlap between the thermal-equilibrium state of the system and the corresponding ground state of the anisotropy subspace. The fidelity between arbitrary state ρ and pure state $|\psi\rangle$ is defined as[39−41]

$$F = \langle\psi|\rho|\psi\rangle, \tag{5}$$

which is a good indicator for the overlap of the two states. We will study the fidelity between the ground state of the anisotropy subspace and the equilibrium state of the system at temperature T. The density matrix $\rho(T)$ of the model at temperature T is given by

$$\rho(T) = \sum_{i=1}^{4} p_i |\psi_i\rangle\langle\psi_i|, \tag{6}$$

where $p_i = \exp(-E_i/kT)/Z$ is the probability distribution, and the partition function reads $Z = \mathrm{Tr}[\exp(-H/kT)]$. For simplicity, we set $k = 1$ in all following equations. On the base of $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$, $\rho(T)$ of the system is obtained as

$$\rho(T) = \frac{1}{Z}\begin{pmatrix} v_1 & 0 & 0 & u \\ 0 & w_1 & y & 0 \\ 0 & y & w_2 & 0 \\ u & 0 & 0 & v_2 \end{pmatrix}, \tag{7}$$

where $v_1 = (b_3 + b_4)\gamma^2$, $v_2 = b_3 a_3^2 + b_4 a_4^2$, $u = (b_3 a_3 + b_4 a_4)\gamma$, $y = b_1 a_1 + b_2 a_2$, $w_1 = b_1 + b_2$, and $w_2 = b_1 a_1^2 + b_2 a_2^2$, with $b_i = \exp(-E_i/T)/N_i$. By making use of Eq. (5), the fidelity between the state of the system at temperature T and the ground state $|\psi_1\rangle$ of the anisotropy subspace can be expressed as

$$F = \frac{w_1 + 2ya_1 + w_2 a_1^2}{Z(1 + a_1^2)}. \tag{8}$$

In Fig. 2, we show the evolution behaviors of the thermal fidelity versus T for the XY model ($\gamma = 0.2$) and the Ising model ($\gamma = 1$) with three different magnetic fields. It is not difficult to recognize from Fig. 2 that in both cases, the fidelity at zero temperature is 1 for $\lambda = -1$ and 0 for $\lambda = 1$, which indicates that the ground state of the system with $\lambda = -1$ is $|\psi_1\rangle$ and it changes to $|\psi_4\rangle$ for $\lambda = 1$. Due to the orthogonality of $|\psi_1\rangle$ and $|\psi_4\rangle$, $F = 0$ for $\lambda = 1$. However, for the case of $\lambda = 0$, the fidelity at zero temperature is 1 in Fig. 2(a), showing that $|\psi_1\rangle$ is still the ground state. And it becomes 0.5 in Fig. 2(b), which indicates that the ground state is a degenerate state of $|\psi_1\rangle$ and $|\psi_4\rangle$ for the case of $\gamma = 1$. As T increases, the ground state mixes with the excited states, hence the fidelities vary correspondingly. The change trends of the fidelity are shown in Fig. 2.

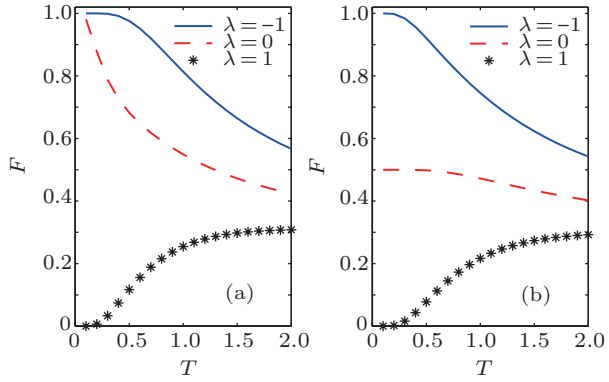


Fig. 2. (colour online) Fidelity F as a function of temperature T under three different magnetic fields in the two cases of (a) $\gamma = 0.2$ and (b) $\gamma = 1$.

Figure 3 displays the fidelity contours in the (B, λ) plane at $T = 0.2$ for $\gamma = 0.2$ and $\gamma = 1$ respectively. The fidelities are very susceptible to λ and vary dramatically from zero to maxima around the critical points. The high thermal fidelities mainly appear in the third and the fourth quadrants of the (B, λ) plane. Especially for the reversed inhomogeneous magnetic field with $\lambda < 0$ and near $B = 0$, the fidelity is enhanced, as shown in Fig. 3(a). When the XY model is reduced to the Ising model, the fidelity declines around $B = 0$ because of the degeneracy, as displayed in Fig. 3(b).

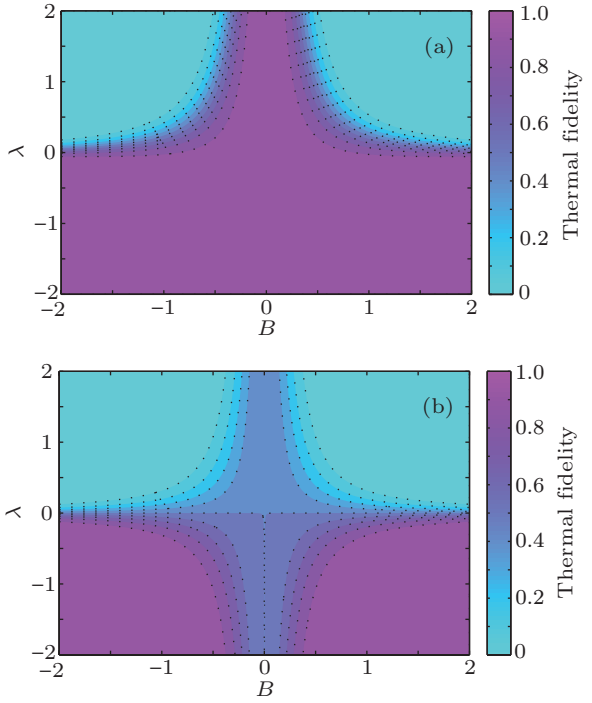


Fig. 3. (colour online) Contours of the thermal fidelity in the (B, λ) plane at $T = 0.2$ for (a) $\gamma = 0.2$ and (b) $\gamma = 1$.

In order to illustrate the evolutions of the thermal fidelity around the critical points, we plot the graphs of F versus λ at $T = 0.2$ for $\gamma = 0.2$ and $\gamma = 1$ in Figs. 4(a) and 4(b), respectively. At low temperature, the system is primarily in the ground state. According to the expressions of energy levels in Eq. (2), the ground state is $|\psi_1\rangle$ for $|\gamma| \leq 1$ and $\lambda < 0$, and becomes $|\psi_4\rangle$ in the other cases. Hence we obtain $F = 1$ for the cases with $|\psi_1\rangle$ being the ground state and $F = 0$ for the cases with $|\psi_4\rangle$ being the ground state. For the other cases with a degenerate ground state of $|\psi_1\rangle$ and $|\psi_4\rangle$, the corresponding fidelity is in the region of $0 < F < 1$, as shown in Fig. 4. In Fig. 4(a), we can see that especially when the magnetic field is relatively strong, F quickly decays from 1 to 0 around the critical point ($\lambda = 0$) as λ increases. From the expression of the Hamiltonian of the system (Eq. (1)), it is very easy to recognize that the ground state of the system with $\gamma = \pm 1$ and $\lambda = 0$ is an equal-weight mixture of $|\psi_1\rangle$ and $|\psi_4\rangle$ for any B, hence the three curves in Fig. 4(b) intersect at the point of $F = 0.5$ and $\lambda = 0$. Since the fidelity is sensitive to the change of λ, λ may be used as a switch to turn on or off the fidelity and this may provide us a new possibility to build quantum logical gates.

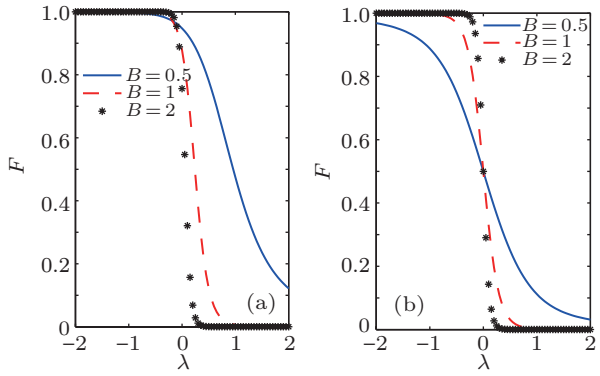


Fig. 4. (colour online) Two-dimensional plots of the fidelity as a function of λ under the three different magnetic fields at $T = 0.2$ in the cases of (a) $\gamma = 0.2$ and (b) $\gamma = 1$.

5. Evolution of the fidelity with the transition effect of Yangian generators

The Yangian algebra was presented by Drinfeld in 1985.[27] Since then, it has attracted much attention. Not only can the Yangian algebra describe the symmetry of integrable quantum models, but also can present the transitions between different states. For a bi-spin system, the realization of Yangian $Y(sl(2))$ depends on the following generators:[42,43]

$$\begin{aligned} \boldsymbol{I} &= \boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2, \\ \boldsymbol{J} &= \mu \boldsymbol{S}_1 + \nu \boldsymbol{S}_2 + \mathrm{i}\lambda \boldsymbol{S}_1 \times \boldsymbol{S}_2, \end{aligned} \tag{9}$$

where $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ are the spin-$\frac{1}{2}$ operators, and μ, ν, λ are arbitrary parameters. These generators can also be used as transition operators. From these operators, we can define two other useful transition operators, $I_\pm = I_1 \pm \mathrm{i}I_2$ and $J_\pm = J_1 \pm \mathrm{i}J_2$. Here $\boldsymbol{I}$ is the total spin operator satisfying $[I_i^a, I_j^b] = \mathrm{i}\epsilon_{abc} I_i^c \delta_{ij}, (i, j = 1, 2)$, namely, the generators of the Lie algebra SU(2). By making use of the transition, $\boldsymbol{J}$ can transit mutually to different sites and jump between states.[33]

Figure 5 sketches the transition characteristic of the Yangian algebra in the two subspaces of the Hamiltonian (1). Here J_3 can only make transitions between states in the same subspaces, while $J_\pm$ can transit the system from one state in one subspace to another state in the other subspace. This transition property of the Yangian algebra is valid for any bi-spin systems, not only for the XY model. Due to these properties, the Yangian operators may be used to boost the fidelity of the system.

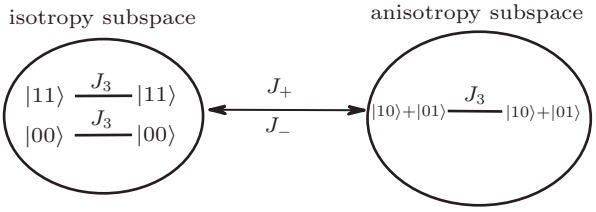


Fig. 5. Transition graph of Yangian generators between states in two subspaces of H_e and H_o.

The effect of the Yangian algebra on the fidelity is illustrated as follows. When applying transition operators $J_\pm$ on the initial state $|\psi_1\rangle$, we obtain the corresponding final states with suitable normalization conditions. It is worth noting that the effects of $J_\pm$ can be considered as an external environment. We can make use of Eq. (3) to measure the overlap. After the calculation, the corresponding fidelities can be obtained as

$$F' = \frac{w_1}{Z}, \quad F'' = \frac{w_2}{Z}. \tag{10}$$

To show the behavior clearly, we plot the contours of the fidelity after the transitions for $\gamma = 0.2$ and $\gamma = 1$ in Figs. 6 and 7, respectively. By comparing Figs. 6 and 7 with Fig. 3, it is quite clear that the transitions can dramatically change the behavior of the fidelity. In concrete, operator J_+ will transit the high fidelity only into the region of $\lambda > 0$ and $B < 0$ in Figs. 6(a) and 7(a), while under the transition of J_- , the high fidelity only goes to the region of $\lambda > 0$ and $B > 0$ in Figs. 6(b) and 7(b). This indicates that after the transitions by J_+ and J_-, the high fidelity regions in both cases are more concentrated on one quadrant in the upper plane where the fidelities are zero before the transitions. However for the region in the lower plane, upon the transitions, the

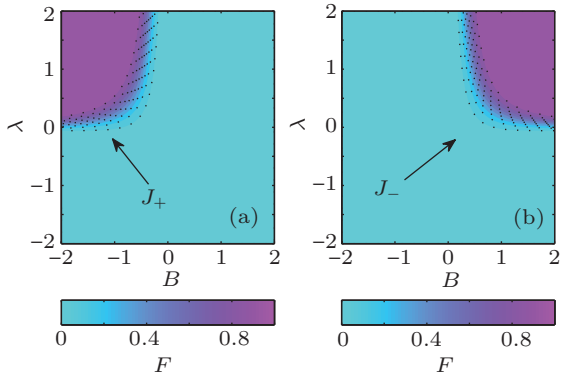


Fig. 6. (colour online) Two-dimensional contours of the fidelity under the transition effects of (a) J_+ and (b) J_- for $\gamma = 0.2$ in the (λ, B) plane.

fidelities of the systems suddenly change from 1 to 0. In other words, the Yangian transition operation may give rise to the sudden birth and the sudden death of the thermal fidelity, and we may use the corresponding Yangian transition operators as a type of switcher to control the thermal fidelity of the system.

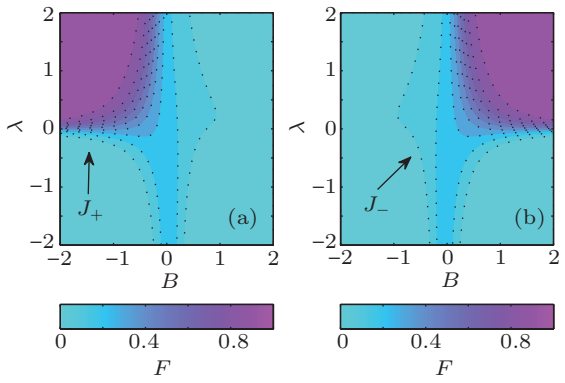


Fig. 7. (colour online) Two-dimensional contours of the fidelity under the transition effects of (a) J_+ and (b) J_- for $\gamma = 1$ in the (λ, B) plane.

6. Conclusion

In summary, we have investigated the properties of the energy level crossings and the evolution of the thermal fidelity in a two-qubit system with a transverse non-homogeneous magnetic field. We have also discussed the effects of the Yangian transition operators on the quantum states and studied the behavior of the fidelity under the transition operation.

With the help of contour plots of the energy difference, we have illustrated the energy level crossing and also identified the ground states for different regions in the parameter spaces. The thermal fidelity has been calculated in both the XY model ($\gamma = 0.2$) and the Ising model ($\gamma = 1$). In these two models, the evolution tendency of the thermal fidelity versus different parameters, such as temperature, λ, and B, has been discussed extensively. We find that the fidelity is very sensitive to the anisotropic factor of the magnetic field, which implies that it can be used to control the thermal fidelity of the system.

The effect of the Yangian transition operators has also been investigated. Interestingly, we find that the Yangian transition operators can change the fidelity dramatically. In concrete, after the transition, the high fidelity only appears in one branch ($\lambda > 0$) of the equilateral hyperbola in the (B, λ) plane, i.e., the sudden birth and the sudden death phenomena of thermal fidelity can be realized by the use of the Yangian transition operators. In other words, the corresponding Yangian transition operators can be used as switchers to turn the fidelity on and off. These results may shed new light on fidelity controlling by the use of the Yangian operators.

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