# Analytical evaluation of the plasma dispersion function for a Fermi–Dirac distribution

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An efficient method for the analytic evaluation of the plasma dispersion function for the Fermi–Dirac distribution is proposed. The new method has been developed using the binomial expansion theorem and the Gamma functions. The general formulas obtained for the plasma dispersion function are utilized for the evaluation of the response function. The resulting series present better convergence rates. Several acceleration techniques are combined to further improve the efficiency. The obtained results for the plasma dispersion function are in good agreement with the known numerical data.

**Keywords:** Fermi–Dirac distribution, plasma dispersion function, binomial coefficients, incomplete gamma functions

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## 1. Introduction

The importance of the plasma dispersion function (PDF) for the Fermi–Dirac (FD) distribution arises in the derivation of the longitudinal electron susceptibility for a nonrelativistic thermal electron gas, with both the quantum recoil and the quantum degeneracy included.[1−3] The calculations of the longitudinal response function for the FD distribution is reduced to the computation of the PDF, which was denoted as $Z(y,\xi)$ in Refs. [4]–[7]. Therefore, choosing reliable formulas for the evaluation of the PDF for the FD distribution is of prime importance for the accurate calculations of the response function.[4] Recently, several efficient analytical and numerical approximations have been formulated for finding the PDFs for the FD distributions with different structures.[4] Unfortunately, for most of these purely numerical methods, there are some limitations in their applicability despite the huge increase in the computational resources. Here we present firstly a more general analytical approach to the calculation of the PDF for the FD distribution, which is valid for a wide range of integral parameters. Note that all particle acceleration techniques are based on the classical plasma not the quantum one.[8,9] But it seems that in the near future, a theoretical method to evaluate the gas laser theory can be constructed by using the plasma dispersion function for the FD distribution.[9]

The main purpose of this paper is to derive new analytical formulas for the plasma dispersion function, which can be used for arbitrary values of $y$ and $\xi$. The obtained formulas could be useful in dealing with a range of applications in physics engineering where the plasma dispersion function emerges. Several results are presented to demonstrate the improvements in the convergence rates obtained with the new decomposition.

## 2. Definition and unified expansion relations of plasma dispersion function for the Fermi–Dirac distribution

In Ref. [4], the longitudinal electron susceptibility for a nonrelativistic thermal electron gas with both the quantum recoil and the quantum degeneracy included called as the response function has the following form:

$$\chi_{\mathrm{e}}^{\mathrm{L}}(\omega,\boldsymbol{k}) = -\frac{2\pi\,\mathrm{e}^2 m_{\mathrm{e}} T_{\mathrm{e}}\xi}{\varepsilon_0(2\pi\hbar)^3 k\Delta_{\mathrm{e}}}\int_0^\infty \frac{t}{\mathrm{e}^{t^2}+\xi} \times \ln\frac{(t-y_+)(t+y_-)}{(t-y_-)(t+y_+)}\,\mathrm{d}t, \tag{1}$$

where $k=|\boldsymbol{k}|$, and

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$$y_{\pm} = \frac{\omega \pm \Delta_{\rm e}}{\sqrt{2}kV_{\rm e}}, \quad \Delta_{\rm e} = \frac{\hbar k^2}{2m_{\rm e}}. \tag{2}$$

Here, parameter $\Delta_{\rm e}$ decribes the quantum recoil. The exact definitions of quantities used in these equations can be found in Refs. [1] and [2]. Equation (1) can be easily rewritten in terms of PDF in the following form:

$$\chi_{\rm e}^{\rm L}(\omega, \boldsymbol{k}) = -\frac{\pi^{3/2}\,{\rm e}^2 m_{\rm e} T_{\rm e}\xi}{\varepsilon_0(2\pi\hbar)^3 k\Delta_{\rm e}}(Z(y_+,\xi) - Z(y_-,\xi)). \tag{3}$$

Now, we start with the general definition of the PDF for the FD distribution $Z(y,\xi)$ occurring in Eq. (3)

$$\begin{aligned} Z(y,\xi) &= \frac{2}{\sqrt{\pi}}\int_0^{\infty} \frac{t}{{\rm e}^{t^2}+\xi} \ln\left(\frac{t-y}{t+y}\right) {\rm d}t \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{t\ln(t-y)}{{\rm e}^{t^2}+\xi}\,{\rm d}t, \end{aligned} \tag{4}$$

where $\xi = {\rm e}^{\mu_{\rm e}/T_{\rm e}}$, $\mu_{\rm e}$ and $T_{\rm e}$ are the chemical potential and the temperature of the electrons, respectively. In the completely degenerate limit, we have $\mu_{\rm e} \to T_{\rm F} \gg T_{\rm e}$, and $\xi \gg 1$, where $T_{\rm F} = \frac{1}{2}m_{\rm e}v_{\rm F}^2$ is the Fermi temperature (see Ref. [4]). Function $Z(y,\xi)$ in the special case of $\xi = 0$ gives $Z(y) = Z(y,\xi)$

$$Z(y) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{{\rm e}^{t^2}}{t-y}\,{\rm d}t. \tag{5}$$

In Ref. [4], the series expression is derived for the PDF in terms of $Z(y)$

$$Z(y,\xi) = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}\xi^s Z(\sqrt{s}y)}{s}, \tag{6}$$

where the real part of $Z(y)$ is determined as

$$\text{Re } Z(y) = \begin{cases} -\dfrac{1}{y}\left(1 + \dfrac{1}{2y^2} + \dfrac{1}{4y^4} + \cdots\right), & (y^2 \gg 1), \\ -\dfrac{1}{y}\left(2y - \dfrac{4y^4}{3} + \cdots\right), & (y^2 \ll 1). \end{cases} \tag{7}$$

Recently, an expression based on the binomial expansion theorem for $Z(y)$ was developed in Ref. [10].

As can be seen from Eqs. (1) and (2), the choice of reliable formulas for the evaluation of the PDF has prime importance in the accurate calculation of the response function. In order to establish expressions for the PDF, we shall first consider the well known binomial expansion theorem and the series relationship of $\ln x$ respectively[11]

$$(x \pm y)^n = \lim_{N\to\infty} \sum_{m=0}^{N} (\pm 1)^m F_m(n)\, x^{n-m}\, y^m, \tag{8}$$

$$\ln x = 2\lim_{N'\to\infty} \sum_{k=1}^{N'} \frac{1}{2k-1}\left(\frac{x-1}{x+1}\right)^{2k-1}, \quad (x > 0), \tag{9}$$

where $N$ and $N'$ are the upper limits of the summations, and $F_m(n)$ are binomial coefficients defined by

$$F_m(n) = \begin{cases} \dfrac{n(n-1)\cdots(n-m+1)}{m!} & \text{for integer } \ n, \\ \dfrac{(-1)^m\Gamma(m-n)}{m!\Gamma(-n)} & \text{for noninteger } n. \end{cases} \tag{10}$$

Taking into account Eq. (8), we obtain for function $({\rm e}^{t^2}+\xi)^{-1}$ occurring in Eq. (4) the following series expansion relations:

$$\frac{1}{({\rm e}^{t^2}+\xi)} = \lim_{N\to\infty}\sum_{i=0}^{N} F_i(-1) \begin{cases} \xi^i\,{\rm e}^{-(1+i)t^2}, & ({\rm e}^{t^2} \geq \xi), \\ \xi^{-(1+i)}\,{\rm e}^{it^2}, & ({\rm e}^{t^2} \leq \xi). \end{cases} \tag{11}$$

Substituting Eqs. (11) and (9) into Eq. (4), we obtain the following series expansion formulas for the PDF in terms of the binomial coefficients: for $\xi = 0$,

$$Z(y,0) = \begin{cases} -\dfrac{2}{\sqrt{\pi}}\lim\limits_{L\to\infty}\sum\limits_{k=1}^{\rm L}\left[\dfrac{\gamma(k+1/2,y^2)}{(2k-1)y^{2k-1}} + \dfrac{y^{2k-1}\Gamma(-k+3/2,y^2)}{2k-1}\right], & (y>1), \\ \dfrac{2}{\sqrt{\pi}}\lim\limits_{L\to\infty}\sum\limits_{k=1}^{\rm L}\dfrac{(-1)^{2k-1}y^{2k-1}\Gamma(3/2-k)}{2k-1}, & (y<1), \end{cases} \tag{12}$$

for $\xi \leq 1$,

$$Z(y,\xi) = \begin{cases} \dfrac{2}{\sqrt{\pi}} \lim\limits_{\substack{L\to\infty \\ L'\to\infty}} \sum\limits_{k=1}^{\mathrm{L}} \dfrac{(-1)^{2k-1} y^{2k-1}}{2k-1} \sum\limits_{i=0}^{L'} F_i(-1) \\ \quad \times \left[ \dfrac{M_i(k-1/2, \ln\xi)}{\xi^{i+1}} + \dfrac{\xi^i \Gamma(3/2-k, (i+1)\ln\xi)}{(i+1)^{3/2-k}} \right], & (y<1), \\ -\dfrac{2}{\sqrt{\pi}} \lim\limits_{\substack{L\to\infty \\ L'\to\infty}} \sum\limits_{k=1}^{\mathrm{L}} \sum\limits_{i=0}^{L'} F_i(-1)\xi^i \\ \quad \times \left[ \dfrac{\gamma(k+1/2, (i+1)y^2)}{(2k-1)y^{2k-1}(i+1)^{k+1/2}} + \dfrac{y^{2k-1}\Gamma(3/2-k, (i+1)y^2)}{(2k-1)(i+1)^{3/2-k}} \right], & (y \geq 1), \end{cases} \tag{13}$$

where $\Gamma(\alpha)$ and $\Gamma(\alpha, x)$ are the well known Gamma functions[11]

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\,\mathrm{e}^{-t}\,\mathrm{d}t, \tag{14}$$

$$\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1}\,\mathrm{e}^{-t}\,\mathrm{d}t, \tag{15}$$

and

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1}\,\mathrm{e}^{-t}\,\mathrm{d}t. \tag{16}$$

We can evaluate the plasma dispersion function $Z(y,\xi)$ using an alternative formula

$$Z(y,\xi) = \frac{2}{\sqrt{\pi}} \lim_{\substack{L\to\infty \\ L'\to\infty}} \sum_{k=1}^{\mathrm{L}} \frac{(-1)^{2k-1} y^{2k-1} \Gamma(3/2-k)}{2k-1} \times \sum_{i=0}^{L'} F_i(-1) \frac{\xi^i}{(i+1)^{3/2-k}}, \quad (\xi < 1, y \leq 1). \tag{17}$$

In Eq. (13), function $M_i(p,q)$ is defined as

$$M_n(p,q) = \int_0^q x^{-p}\,\mathrm{e}^{nx}\,\mathrm{d}x. \tag{18}$$

After evaluating the integration and simplification, we obtain the following formula:

$$M_n(p,q) = \begin{cases} \dfrac{q^{1-p}}{1-p}, & (n=0), \\ \lim\limits_{L''\to\infty} \sum\limits_{j=0}^{L''} \dfrac{n^j q^{j-p+1}}{j!(j-p+1)}, & (n \neq 0). \end{cases} \tag{19}$$

For those values of parameters with which the $M_n(p,q)$ functions exist, the program Mathematica gives the following result:

$$M_n(p,q) = \begin{cases} \dfrac{q^{1-p}}{1-p}, & (n=0), \\ \dfrac{2\sqrt{nq}\,(15-10nq+4(nq)^2)\,\mathrm{e}^{nq} - 15\sqrt{\pi}\,\mathrm{Erfi}(\sqrt{nq})}{8n^{7/2}}, & (n \neq 0). \end{cases} \tag{20}$$

where $\mathrm{Erfi}\,(x)$ is the complementary error function.[11] In Eqs. (8), (9), (11)–(13), 17, and (19), indices $N$, $N'$, $L$, $L'$, and $L''$ are the upper limits of the summations. In our previous paper,[12] for the direct calculation of the complete and the incomplete gamma functions $\Gamma(\alpha)$ and $\Gamma(\alpha, x)$, the upward and downward recursion and analytical relations have been derived.

## 3. Numerical results and discussion

We have proposed an alternative method for the analytical evaluation of the plasma dispersion function for the Fermi–Dirac distribution. The obtained formulas are practically simple, and they offer some advantages over the currently available methods. Based on the formulas obtained in this paper, we build a program for computing the plasma dispersion function for the Fermi–Dirac distribution by making use of Mathematica 7.0. The results are listed in Table 1. As can be seen in Table 1, the values of the plasma dispersion function for the Fermi–Dirac distribution calculated with these two calculation (numerical) methods are in very good agreement with each other. A comparison of our results with the exact numerical solutions shows that the explicit analytical expressions that we obtained for the plasma dispersion function is more accurate. Some computational results with Eq. (15) are much lower than the accuracies required in the present paper, which are then not shown in Table 1. Table 2

lists the partial summations corresponding to progressively increasing the upper summation limit (denoted as $L$, $L'$) for this expression. As can be seen from Table 2, equation (13) displays the most rapid convergence to the numerical result by reducing the values of $\xi$.

**Table 1.** Values of plasma dispersion function $Z(y,\xi)$ for $L = L' = 70$ obtained with different methods.

| $y$ | $\xi$ | Eqs. (12) and (13) | Eq. (15) | Mathematica numerical integration results |
|---|---|---|---|---|
| 0.6 | 0.3 | –0.8146435819130297 | –0.8146435819130297 | –0.8146435454670678 |
| 0.8 | 0.4 | –0.8949237153151383 | –0.8949237153151378 | –0.8949290751490729 |
| 0.3 | 0.9 | –0.3625200131131346 | –3625200131131347 | –0.3625206114987278 |
| 0.5 | 0.1 | –0.8009678881183464 | –8009678881183456 | –0.8009630251056242 |
| 1 | 0.2 | –0.9942993448900502 | –0.9942993448900505 | –0.9942992885169313 |
| 0.03 | 0.7 | –0.0409086708245991 | –0.0409086708245991 | –0.0409085358581634 |
| 0.5 | 0.08 | –0.810038868059551 | –0.810043802376962 | –0.810043802376961 |
| 0.07 | 0.06 | –0.13391650176523864 | –0.13391650176523845 | –0.1339156657129376 |
| 5.3 | 0.5 | –0.17479985126499775 | | –0.1747998512653207 |
| 2.5 | 0.6 | –0.3778425362927977 | | –0.3781341494422134 |
| 8.2 | 0.8 | –0.098295467237332 | | –0.0982954672225952 |
| 13.8 | 0.9 | –0.0568109994638712 | | –0.0568109639119558 |
| 21.3 | 0.05 | –0.04619176163625185 | | –0.04619176163625185 |
| 32.1 | 0.007 | –0.03109096156154856 | | –0.03109096156154856 |
| 43.2 | 1.0 | -0.01773626159886021 | | –0.01771711952393453 |
| 22.6 | 0.0 | -0.04429123108207397 | | –0.04429123108207397 |

**Table 2.** Convergence of the derived expression for $Z(y,\xi)$ as a function of summation limits $L$ and $L'$.

| $L$, $L'$ | $y = 8.5,\ \xi = 0.4$ | $y = 63.4,\ \xi = 0.06$ |
|---|---|---|
| 10 | –0.10467837954923018822 | –0.015450739306462027632 |
| 20 | –0.10467829182405125863 | –0.015450739306462014561 |
| 30 | –0.10467829182040066596 | –0.015450739306462014561 |
| 40 | –0.10467829182040044903 | |
| 50 | –0.10467829182040044901 | |
| 60 | –0.10467829182040044901 | |

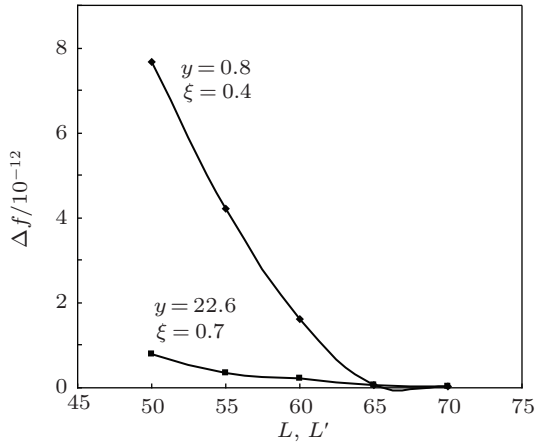



**Fig. 1.** The convergence rates of the series in Eqs. (12) and (13) for $Z(y,\xi)$.

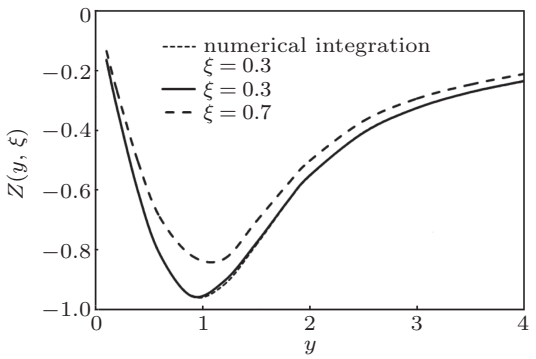



**Fig. 2.** Values of plasma dispersion function $Z(y,\xi)$ for different $y$ and $L = L' = 100$.

In Fig. 1, we present the convergence of the series in Eqs. (12) and (13) for $L_1 < L$ and $L_2 < L'$. Here

$L$ and $L'$ are the upper limits of indices $k$ and $i$, respectively. The series accuracy $\Delta f = f_{L,L'} - f_{L_1,L_2}$ is shown in Fig. 1, where the quantities $f = f_{L,L'}$ are the values of integral for $L = L_1$ and $L' = L_2$. We can see that the convergence of our established formulas with respect to $y$ and $\xi$ is rapid. A greater accuracy is attainable by the use of more terms in expansions (12), (13), and (17). In Fig. 2, we plot the PDF for the FD distribution as a function of $y$ for $\zeta = 0.3$, 0.7 and the numerical integration results. The results in Fig. 2 show that the series representation agrees well with the results of the numerical integration. So, we provide a simple and efficient analytical approach as an alternative to evaluate the $Z(y, \xi)$ plasma dispersion function in the range of $\xi \leq 1$. The application of the present method to the evaluation of the PDF for the FD distribution is straightforward.

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