Level crossing in a two-photon Jaynes–Cummings model*

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In this paper, the energy spectrum of the two-photon Jaynes–Cummings model (TPJCM) is calculated exactly in the non-rotating wave approximation (non-RWA), and we study the level-crossing problem by means of fidelity. A narrow peak of the fidelity is observed at the level-crossing point, which does not appear at the avoided-crossing point. Therefore fidelity is perfectly suited for detecting the level-crossing point in the energy spectrum.

Keywords: level crossing, fidelity, two-photon Jaynes–Cummings model

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1. Introduction

Quantum phase transitions occur at zero temperature and are driven purely by quantum fluctuation.[1] In recent years, quantum phase transitions have drawn considerable attention due to their connections with quantum information[2] and quantum computing.[3] Ground-state fidelity has been introduced recently as a tool to investigate quantum phase transitions.[4–8] As in the proximity of the critical regions, small changes of the Hamiltonian parameters can cause dramatic ground state variations due to the large difference in the ground-state structure between different phases. Moreover, fidelity is proposed as a reliable tool to detect the level-crossing points in the energy spectrum.[9,10]

The rotating wave approximation (RWA) is widely used in the realm of quantum optics, where the rapidly oscillating terms are neglected, therefore analytical solutions can be obtained.[11] This approximation is invalid if the coupling of the cavity field to the atom is strong enough.[12–14] In this work, the energy spectrum of the two-photon Jaynes–Cummings model (TPJCM) is calculated exactly in the non-RWA, and the level-crossing problem in the energy spectrum is also discussed by means of fidelity.

2. Model and solutions

We consider the TPJCM with a single-mode field. The two-photon transition is considered as a nonlinear physical process. Since Brune *et al.*[15] successfully invented the two-photon micromaser, people have shown a great deal of interest in the TPJCM in the past thirty years. The Hamiltonian of the system can be written as ($\hbar = 1$)

$$H = \frac{\omega_0}{2}(|E\rangle\langle E| - |G\rangle\langle G|) + \omega a^+ a + g(a^{+2} + a^2)(|E\rangle\langle G| + |G\rangle\langle E|), \quad (1)$$

where ω_0 is the transition frequency of the two levels, ω is the frequency of the field, $|E\rangle$ ($|G\rangle$) is the excited (ground) atomic state, a and a^+ are the Bose annihilation and creation operators, respectively, and g is the atom–field coupling coefficient. Applying transitions $|e\rangle = (|E\rangle + |G\rangle)/\sqrt{2}$ and $|g\rangle = (|E\rangle - |G\rangle)/\sqrt{2}$ to Eq. (1), we obtain

$$H = \frac{\omega_0}{2}(|e\rangle\langle g| + |g\rangle\langle e|) + \omega a^+ a + g(a^{+2} + a^2)(|e\rangle\langle e| - |g\rangle\langle g|). \quad (2)$$

The wavefunction of the system can be written as

$$|\psi\rangle = |\varphi_1\rangle|e\rangle + |\varphi_2\rangle|g\rangle. \quad (3)$$

Substituting Eqs. (2) and (3) into the time-independent Schrödinger equation yields

$$\frac{\omega_0}{2}|\varphi_2\rangle + \omega_1 A^+ A|\varphi_1\rangle - \frac{\omega - \omega_1}{2}|\varphi_1\rangle = E|\varphi_1\rangle, \quad (4)$$

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$$\frac{\omega_0}{2}\left|\varphi_1\right\rangle+\omega_1 B^+B\left|\varphi_2\right\rangle-\frac{\omega-\omega_1}{2}\left|\varphi_2\right\rangle=E\left|\varphi_2\right\rangle, \quad (5)$$

where $\omega_1=\sqrt{\omega^2-4g^2}$, and two Bogoliubov transformations $A=a\cosh\lambda+a^+\sinh\lambda$ and $B=a\cosh\lambda-a^+\sinh\lambda$ are introduced. The wavefunctions of the system, $|\varphi_1\rangle$ and $|\varphi_2\rangle$, can be expanded in complete bases $|n\rangle_A$ and $|n\rangle_B$, respectively,

$$\begin{aligned}\left|\varphi_1\right\rangle&=\sum_{n=0}^{N}c_n\left|n\right\rangle_A\\&=\sum_{n=0}^{N}c_n\frac{(A^+)^n}{\sqrt{n!}}(1-\tanh^2\lambda)^{1/4}\,\mathrm{e}^{-\frac{\tanh\lambda}{2}a^{+2}}\left|0\right\rangle, \quad (6)\\\left|\varphi_2\right\rangle&=\sum_{n=0}^{N}d_n\left|n\right\rangle_B\\&=\sum_{n=0}^{N}d_n\frac{(B^+)^n}{\sqrt{n!}}(1-\tanh^2\lambda)^{1/4}\,\mathrm{e}^{\frac{\tanh\lambda}{2}a^{+2}}\left|0\right\rangle, \quad (7)\end{aligned}$$

where $|n\rangle_A$ and $|n\rangle_B$ are the squeezed number states[16] of the new squeezed boson operators A (A^+) and B (B^+). Substituting Eqs. (6) and (7) into Eqs. (4) and (5) yields

$$\begin{aligned}&\frac{\omega_0}{2}\sum_{n=0}^{N}d_n\left|n\right\rangle_B+\omega_1A^+A\sum_{n=0}^{N}c_n\left|n\right\rangle_A\\&\quad-\frac{\omega-\omega_1}{2}\sum_{n=0}^{N}c_n\left|n\right\rangle_A=E\sum_{n=0}^{N}c_n\left|n\right\rangle_A, \quad (8)\\&\frac{\omega_0}{2}\sum_{n=0}^{N}c_n\left|n\right\rangle_A+\omega_1B^+B\sum_{n=0}^{N}d_n\left|n\right\rangle_B\\&\quad-\frac{\omega-\omega_1}{2}\sum_{n=0}^{N}d_n\left|n\right\rangle_B=E\sum_{n=0}^{N}d_n\left|n\right\rangle_B. \quad (9)\end{aligned}$$

We left multiply Eqs. (8) and (9) with states ${}_A\langle m|$ and ${}_B\langle m|$, respectively. Because $A^+A\left|n\right\rangle_A=n\left|n\right\rangle_A$, $B^+B\left|n\right\rangle_B=n\left|n\right\rangle_B$, ${}_A\langle m|n\rangle_A=\delta_{mn}$, ${}_B\langle m|n\rangle_B=\delta_{mn}$, we can obtain the following equations:

$$\begin{aligned}&\frac{\omega_0}{2}\sum_{n=0}^{N}d_{nA}\left\langle m|n\right\rangle_B+\left(\omega_1m-\frac{\omega-\omega_1}{2}\right)c_m\\&=Ec_m, \quad (10)\\&\frac{\omega_0}{2}\sum_{n=0}^{N}c_{nB}\left\langle m|n\right\rangle_A+\left(\omega_1m-\frac{\omega-\omega_1}{2}\right)d_m\\&=Ed_m, \quad (11)\end{aligned}$$

where

$${}_A\langle m|n\rangle_B=\frac{\left(\frac{g}{\omega}\right)^{\frac{n}{2}}}{\sqrt{m!n!}}\sqrt{\frac{\omega_1}{\omega}}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k n!}{k!(n-2k)!}\left(\sqrt{\frac{\omega_1^2}{g\omega}}\right)^{n-2k}\left(\frac{g}{\omega}\right)^{\frac{m-n+2k}{2}}\frac{m!}{\left(\frac{m-n+2k}{2}\right)!}, \quad (12)$$

$${}_B\langle m|n\rangle_A=\frac{\left(\frac{g}{\omega}\right)^{\frac{m}{2}}}{\sqrt{m!n!}}\sqrt{\frac{\omega_1}{\omega}}\sum_{k=0}^{[\frac{m}{2}]}\frac{(-1)^k m!}{k!(m-2k)!}\left(\sqrt{\frac{\omega_1^2}{g\omega}}\right)^{m-2k}\left(\frac{g}{\omega}\right)^{\frac{n-m+2k}{2}}\frac{n!}{\left(\frac{n-m+2k}{2}\right)!}. \quad (13)$$

Substituting Eqs. (12) and (13) into Eqs. (10) and (11), we can obtain $2N+2$ equations. Solving these equations, we can obtain the eigenvalues $\{E^{(i)}\}$ and the eigenvectors with coefficients $\{c_n^{(i)}\}$ and $\{d_n^{(i)}\}$ $(i=1,\ldots,2N+2)$.

3. Energy spectrum

Figure 1 shows the energy spectra of the TPJCM for $\omega_0=2\omega$. The dotted lines are obtained by using the method given in Ref. [13], and the solid lines are our results. From Fig. 1, we can see that the energy spectra obtained using the two methods fit well with each other. However when the coupling coefficient is in the proximity of $g/\omega_0=0.25$, the results from the two methods are different.

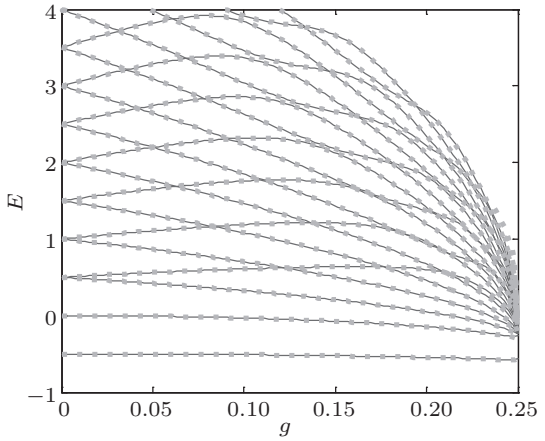


Fig. 1. Energy E of TPJCM versus coupling coefficient g for $\omega_0=2\omega$, the dotted lines represent the results obtained by using the method given in Ref. [13], and the solid lines denote our results.

4. Level crossing and avoided crossing

There are many level-crossing and avoided-crossing points in Fig. 1, which are difficult to detect when the distance between two energy levels is very small. In the following, we use fidelity as a tool to investigate the level crossing. Fidelity[17,18] for eigenstates with slightly different coupling parameters is defined as

$$F(g,\delta g) = |\langle\psi(g)|\psi(g+\delta g)\rangle|^2 . \tag{14}$$

In our calculations, $\delta g = 10^{-4}$ is used. As is well known, in the quantum phase transition system, the wave functions from different sides of the level-crossing points are almost orthogonal,[19,20] therefore the values of peaks of fidelity are zero at the level-crossing points.

Figures 2 and 3 show the energy spectra of the TPJCM and $F' = 1 - F$ of the different eigenstates. From Fig. 2, we can see that the peaks of F' correspond to the level-crossing points, which are marked by small circles. Particularly, there are many points, which look like crossings but are actually avoided crossings in the energy spectrum. These points can also be detected by the fidelity, and the values of F' at these points are around zero rather than one. In Fig. 3 the upper energy levels are shown with the same parameters as those in Fig. 2. The level-crossing points are marked by small rectangles, and the values of F' are zero. Therefore, the fidelity is useful for solving the crossover problem.

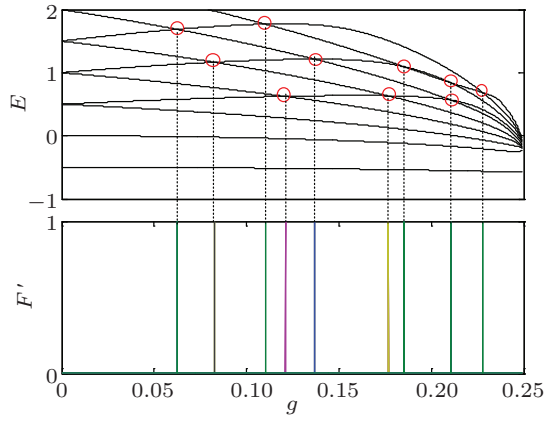


Fig. 2. (colour online) Energy E of the TPJCM and F' versus coupling coefficient g for $\omega_0 = 2\omega$. The small circles show the level-crossing points.

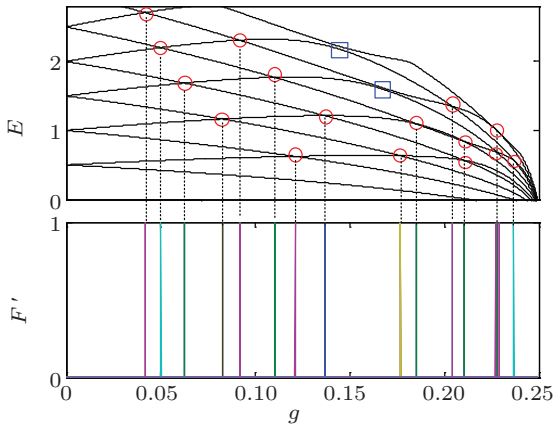


Fig. 3. (colour online) Energy E of the TPJCM and F' versus coupling coefficient g for $\omega_0 = 2\omega$. The small circles indicate the level-crossing points, and the small rectangles show the avoided-crossing points.

5. Conclusion

The energy spectrum of the TPJCM is calculated exactly in the non-RWA. We use fidelity to detect the level-crossing points in the energy spectrum. In the vicinity of the level-crossing point, F' is nearly one, otherwise F' is around zero. Therefore the fidelity is suited for detecting the level-crossing and the avoided-crossing points in the energy spectrum. Moreover, the fidelity is accessible experimentally for non-interacting cold atoms in optical potentials[20,21] and magnetic quantum phase transitions.[22]

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