Quantum phase distribution and the number–phase Wigner function of the generalized squeezed vacuum states associated with solvable quantum systems

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The quantum phase properties of the generalized squeezed vacuum states associated with solvable quantum systems are studied by using the Pegg–Barnett formalism. Then, two nonclassical features, i.e., squeezing in the number and phase operators, as well as the number–phase Wigner function of the generalized squeezed states are investigated. Due to some actual physical situations, the present approach is applied to two classes of generalized squeezed states: solvable quantum systems with discrete spectra and nonlinear squeezed states with particular nonlinear functions. Finally, the time evolution of the nonclassical properties of the considered systems has been numerically investigated.

Keywords: squeezed vacuum states, solvable quantum systems, phase distribution, number–phase Wigner function

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1. Introduction

Squeezed states are nonclassical states of electromagnetic fields, in which certain observables fluctuate more slightly than that of the vacuum.[1−6] These states are very useful in various branches of physics, especially in quantum teleportation and quantum cryptography.[7,8] The squeezed vacuum states can be generated by using a parametric down-converter.[9] The generalization of these states has been frequently introduced in the literature.[10−13] Using the properties of the inverse of annihilation and creation operators, the f-analogy of the squeezed vacuum states were constructed.[11] Later, Kwek and Kiang considered a nonlinear extension of the single-mode squeezed states based on the deformation of the generators of the oscillator algebra, and introduced nonlinear squeezed states.[12] Next, the nonlinear two-mode squeezed vacuum states were studied.[14] Then, nonlinear squeezed coherent states were introduced in Ref. [15]. Recently, the Gazeau–Klauder squeezed states associated with solvable quantum systems were constructed by one of us.[16] The phase properties of a few classes of coherent states have been studied in Refs. [17]–[19]. More recently, the phase distribution, squeezing in number or phase operators, and Wigner function for the generalized coherent states associated with solvable quantum systems with known discrete spectrum have been reported by us.[20−22] In the present paper, we will further apply this approach to the generalized squeezed states associated with solvable quantum systems. In particular, by using the simple relation between the nonlinear function of nonlinear coherent states and the nondegenerate discrete spectra of solvable quantum systems,[23] the formalism may also work for the nonlinear squeezed states. The outlined structure is presented in a general regime and may recover many special cases. To the best of our knowledge, the presented formalism and its new and interesting results have not been discussed up to now. In particular, due to the fact that the nonclassical features of quantum states have attracted much interest in recent years, we will pay attention to this subject in the near future.

The paper is organized as follows. In Section 2 the nonlinear squeezed states associated with solvable quantum systems with nondegenerate spectra are briefly reviewed. Next, the phase distribution of the introduced states is investigated by using the Pegg–Barnett formalism in Section 3. In Section 4 the number–phase squeezing of the nonlinear squeezed state associated with arbitrary solvable quantum systems is discussed. In Section 5, the number–phase

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Wigner function of the generalized squeezed state is studied. In Section 6, the present approach is applied to two solvable quantum systems, as well as a class of nonlinear squeezed states to be introduced. The nonclassicality of the obtained states is numerically studied by investigating the number–phase squeezing and number–phase Wigner function, in addition to their phase distributions. In Section 7 the dynamical evolution of the generalized squeezed state is introduced, and then the time evolution of nonclassical features is discussed. Finally, we conclude the paper in Section 8.

2. Generalized squeezed states associated with solvable quantum systems

The generalized annihilation and creation operators associated with nonlinear coherent and squeezed states are given by $A = af(N)$ and $A^\dagger = f(N)a^\dagger$, where a, $a^\dagger$, and $N = a^\dagger a$ are the bosonic annihilation, creation, and number operators, respectively. These operators satisfy the following nonlinear algebra:

$$[N, A] = -A, \quad [N, A^\dagger] = A^\dagger,$$
$$\left[A, A^\dagger\right] = (N+1)f^2(N+1) - Nf^2(N). \quad (1)$$

Note that we have confined ourselves to the special case of real-value function $f^\dagger(N) = f(N)$. The canonical conjugates of the operators A and $A^\dagger$ are given by $B^\dagger = f^{-1}(N)a^\dagger$ and $B = af^{-1}(N)$, respectively.[23–25] Hence, one has

$$[A, B^\dagger] = 1, \quad [B, A^\dagger] = 1. \quad (2)$$

Nonlinear squeezed states are then defined by applying the deformed squeezing operator, $S(\xi)$, to the vacuum as follows:[12]

$$|\xi, f\rangle = S(\xi)|0\rangle = \exp\left[\frac{1}{2}(\xi A^{\dagger 2} - \xi^* B^2)\right]|0\rangle. \quad (3)$$

The explicit form of these states has been found by superposing even Fock states as follows:[12,16]

$$|\xi, f\rangle = \mathcal{N}_f(r^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{(2n)!!}[f(2n)]! \times (\mathrm{e}^{\mathrm{i}\phi}\tanh r)^n\,|2n\rangle, \quad (4)$$

where $[f(2n)]! \doteq f(2n)\ f(2n-1)\ \cdots\ f(1)$, by definition $[f(0)]! = 1$, $(2n)!! = (2n)(2n-2)\cdots 2$ and ξ is replaced by $r\exp(\mathrm{i}\phi)$. The normalization constant $\mathcal{N}_f(r^2)$ can be fixed by using $\langle\xi, f|\xi, f\rangle = 1$, i.e.,

$$\mathcal{N}_f(r^2) = \sum_{n=0}^{\infty} \frac{(2n)!}{[(2n)!!]^2}\,[f(2n)]!^2\,(\tanh r)^{2n}. \quad (5)$$

The class of the dual state of (4) can be obtained by acting $\tilde{S}(\xi) = \exp\left[(\xi B^{\dagger 2} - \xi^* A^2)/2\right]$ on the vacuum. The result is simply expressed by replacing $f(2n)$ by $f^{-1}(2n)$ in state (4).

In addition, Roknizadeh *et al.* introduced a Hamiltonian associated with a nonlinear oscillator system, that is, $H = A^\dagger A = Nf^2(N)$.[26] Therefore, the eigenvalue equation for a physical system with a known discrete spectrum, e_n, can be given by[26,27]

$$H|n\rangle = e_n|n\rangle = nf^2(n)|n\rangle, \quad (6)$$

and it is obvious that $f(n) = \sqrt{e_n/n}$. Thus, by inserting $f(2n) = \sqrt{e_{2n}/2n}$ in Eq. (4), the generalized squeezed states associated with an arbitrary solvable quantum system with nondegenerate spectrum may be introduced as[16]

$$|\xi, e_n\rangle = \mathcal{N}_e(r^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\sqrt{[e_{2n}]!}}{(2n)!!} \times (\mathrm{e}^{\mathrm{i}\phi}\tanh r)^n\,|2n\rangle, \quad (7)$$

where

$$\mathcal{N}_e(r^2) = \sum_{n=0}^{\infty} \frac{[e_{2n}]!}{[(2n)!!]^2}\,(\tanh r)^{2n}, \quad (8)$$

$[e_{2n}]! \doteq e_{2n}\ e_{2n-1}\ \ldots\ e_1$, and $[e_0]! = 1$ ($e_0 = 0$). Our motivation for working with such states is mainly due to the fact that they are more directly related to real physical systems. It also requires that $0 < \mathcal{N}_e(r^2) < \infty$, which implies that the parameter ξ should be restricted by the inequality

$$\tanh^2 r < \lim_{n\to\infty} \frac{4(n+1)^2}{e_{2n+2}e_{2n+1}}. \quad (9)$$

Setting $f(n) = 1$ and equivalently $e_n = n$ in Eqs. (4) and (7) recovers the standard set of squeezed vacuum states.

3. Phase distribution of the generalized squeezed states associated with solvable quantum systems

The Pegg–Barnett phase operator is defined in an $(s+1)$-dimensional Hilbert space as[28]

$$\phi_\theta = \sum_{m=0}^{s} \theta_m |\theta_m\rangle\langle\theta_m|, \tag{10}$$

where

$$\theta_m = \theta_0 + \frac{2\pi m}{s+1}, \qquad m = 0, 1, ..., s, \tag{11}$$

and θ_0 is an arbitrary value. The orthonormal phase states $|\theta_m\rangle$ are expressed in terms of Fock states by

$$|\theta_m\rangle = \frac{1}{\sqrt{s+1}} \sum_{n=0}^{s} \exp(\mathrm{i}n\theta_m)|n\rangle. \tag{12}$$

The Pegg–Barnett phase probability distribution of nonlinear squeezed states associated with solvable quantum systems can be defined as[29]

$$P_e(\theta) = \lim_{s\to\infty} \frac{s+1}{2\pi} \left|\langle\theta_m|\xi, e_n\rangle\right|^2. \tag{13}$$

Using Eq. (12), we can also express Eq. (7) in the phase space basis as

$$\begin{aligned} |\xi, e_n\rangle \;=\; & \frac{\mathcal{N}_e(r^2)^{-1/2}}{\sqrt{s+1}} \sum_{n=0}^{\infty}\sum_{m=0}^{s} \frac{\sqrt{[e_{2n}]!}}{(2n)!!} \\ & \times (\mathrm{e}^{\mathrm{i}\phi}\tanh r)^n \mathrm{e}^{-2\mathrm{i}n\theta_m}|\theta_m\rangle. \end{aligned} \tag{14}$$

Note that

$$\begin{aligned} \langle\theta_m|\xi, e_n\rangle \;=\; & \frac{\mathcal{N}_e(r^2)^{-1/2}}{\sqrt{s+1}} \sum_{n=0}^{\infty} \frac{\sqrt{[e_{2n}]!}}{(2n)!!} \\ & \times (\mathrm{e}^{\mathrm{i}\phi}\tanh r)^n \mathrm{e}^{-2\mathrm{i}n\theta_m}, \end{aligned} \tag{15}$$

the phase distribution (13) for the generalized squeezed state associated with a solvable quantum system can be obtained as

$$\begin{aligned} & P_e(\theta) \\ &= \frac{\mathcal{N}_e(r^2)^{-1}}{2\pi} \left| \sum_{n=0}^{\infty} \frac{\sqrt{[e_{2n}]!}\,(\mathrm{e}^{\mathrm{i}\phi}\tanh r)^n}{(2n)!!} \mathrm{e}^{-2\mathrm{i}n\theta_n} \right|^2 \\ &= \frac{\mathcal{N}_e(r^2)^{-1}}{2\pi} \sum_{n=0}^{\infty} \frac{[e_{2n}]!}{[(2n)!!]^2} (\tanh r)^{2n} \\ &\quad + \frac{2\mathcal{N}_e(r^2)^{-1}}{2\pi} \sum_{n=0}^{\infty}\sum_{k<n} \frac{\sqrt{[e_{2n}]![e_{2k}]!}}{(2n)!!\,(2k)!!} \\ &\quad \times (\mathrm{e}^{\mathrm{i}\phi}\tanh r)^n (\mathrm{e}^{-\mathrm{i}\phi}\tanh r)^k \cos[2(n-k)\theta], \end{aligned} \tag{16}$$

where the terms corresponding to $n = k$ and $n \neq k$ are separately written. Thus, with the help of Eq. (8), we finally obtain

$$\begin{aligned} P_e(\theta) \;=\; & \frac{1}{2\pi}\{1 + 2\mathcal{N}_e(r^2)^{-1} \sum_{n=0}^{\infty}\sum_{k<n} \frac{\sqrt{[e_{2n}]![e_{2k}]!}}{(2n)!!(2k)!!} \\ & \times (\mathrm{e}^{\mathrm{i}\phi}\tanh r)^n (\mathrm{e}^{-\mathrm{i}\phi}\tanh r)^k \\ & \times \cos[2(n-k)\theta]\}. \end{aligned} \tag{17}$$

It is worth mentioning that one can easily obtain the phase distribution associated with a nonlinear (f-deformed) squeezed state (17), by inserting $e_{2n} = 2nf^2(2n)$ from proposal in Eq. (6). Putting $e_n = n$ in Eq. (17) leads to the phase distribution for standard squeezed states.

4. Number-phase squeezing of the generalized squeezed states associated with solvable quantum systems

Two operators N and ϕ_θ obey the following uncertainty relation

$$\left\langle(\Delta\phi_\theta)^2\right\rangle \left\langle(\Delta N)^2\right\rangle \geq \frac{1}{4}|\langle[N, \phi_\theta]\rangle|^2, \tag{18}$$

where it is proved that[28]

$$[N, \phi_\theta] = \mathrm{i}[1 - 2\pi P_e(\theta_0)]. \tag{19}$$

If $\langle(\Delta N)^2\rangle < |\langle[N, \phi_\theta]\rangle|/2$ or $\langle(\Delta\phi_\theta)^2\rangle < |\langle[N, \phi_\theta]\rangle|/2$, the squeezing occurs in the number or phase operator, respectively.

With the help of Eqs. (17) and (7), the phase and number variances for the squeezed states can be obtained, respectively, as follows:

$$\begin{aligned} \langle(\Delta\phi_\theta)^2\rangle \;=\; & \int \theta^2 P_e(\theta)\mathrm{d}\theta - \left(\int \theta P_e(\theta)\mathrm{d}\theta\right)^2 \\ =\; & \frac{\pi^2}{3} + \mathcal{N}_e(r^2)^{-1} \sum_{n=0}^{\infty}\sum_{k<n} \frac{\sqrt{[e_{2n}]![e_{2k}]!}}{(2n)!!\,(2k)!!} \\ & \times \frac{(\mathrm{e}^{\mathrm{i}\phi}\tanh r)^n (\mathrm{e}^{-\mathrm{i}\phi}\tanh r)^k}{(n-k)^2}, \end{aligned} \tag{20}$$

and

$$\begin{aligned} & \langle(\Delta N)^2\rangle \\ &= \langle N^2\rangle - \langle N\rangle^2 \\ &= \mathcal{N}_e(r^2)^{-1} \sum_{n=0}^{\infty} 4n^2 \frac{[e_{2n}]!}{[(2n)!!]^2} (\tanh r)^{2n} \end{aligned}$$

$$- \left(\mathcal{N}_e(r^2)^{-1} \sum_{n=0}^{\infty} 2n \frac{[e_{2n}]!}{[(2n)!!]^2} (\tanh r)^{2n} \right)^2 . \quad (21)$$

To examine the number or phase squeezing behavior, the following squeezing parameters are introduced:

$$S_N = \frac{2\langle (\Delta N)^2 \rangle}{|\langle [N, \phi_\theta] \rangle|} - 1, \quad (22)$$

$$S_\phi = \frac{2\langle (\Delta \phi_\theta)^2 \rangle}{|\langle [N, \phi_\theta] \rangle|} - 1. \quad (23)$$

If $S_N < 0$ ($S_\phi < 0$), then the associated state is number (phase) squeezed. Again, by setting $e_n = n$ in Eqs. (20) and (21), the phase or number squeezing for standard squeezed vacuum states will be recovered.

5. Number-phase Wigner function of the generalized squeezed state associated with solvable quantum systems

The number–phase Wigner function for an arbitrary state is defined as the expectation value of the number–phase Wigner operator which can be expressed as[30,31]

$$\begin{aligned} & \mathcal{W}_{\rm NP}(n, \theta) \\ = & \frac{1}{2\pi} \Bigg\{ \sum_{p=-n}^{n} \mathrm{e}^{2\mathrm{i}p\theta} |n+p\rangle\langle n-p| \\ & + \sum_{p=-n}^{n-1} \mathrm{e}^{\mathrm{i}(2p+1)\theta} |n+p\rangle\langle n-p-1| \Bigg\}, \end{aligned} \quad (24)$$

where $n = 0, 1, ...$, θ is chosen to be real, and the second summation is taken to be zero for $n = 0$. Thus, the number–phase Wigner function for the squeezed states may be defined as follows:

$$W_{\rm NP}(n, \theta) = \langle \xi, e_n | \mathcal{W}_{\rm NP} | \xi, e_n \rangle. \quad (25)$$

Substituting the state (7) in function (25), the number–phase Wigner function takes the following form:

$$\begin{aligned} & W_{\rm NP}(n, \theta) \\ = & \frac{\mathcal{N}_e(|\xi|^2)^{-1}}{2\pi} \Bigg\{ \sum_{p=-n}^{n} \frac{\sqrt{[e_{n+p}]![e_{n-p}]!}}{(n+p)!!(n-p)!!} \\ & \times (\tanh r)^n \, \mathrm{e}^{-\mathrm{i}p\phi} \, \mathrm{e}^{2\mathrm{i}p\theta} \\ & + \sum_{p=-n}^{n-1} \frac{\sqrt{[e_{n+p}]![e_{n-p-1}]!}}{(n+p)!!(n-p-1)!!} (\tanh r)^{n-1/2} \\ & \times \mathrm{e}^{-\mathrm{i}(p+1/2)\phi} \, \mathrm{e}^{\mathrm{i}(2p+1)\theta} \Bigg\}. \end{aligned} \quad (26)$$

This distribution function may become negative in some regions, which can be interpreted as the nonclassical feature of physical states.

We end this section by emphasizing the fact that by replacing $f(n)$ and e_n, respectively, with $f^{-1}(n)$ and n^2/e_n in the resultant formulas, one can easily investigate the physical quantities for the dual states (4) and (7). In any case, the domain of the states should be checked firstly.

6. Some physical realizations of the formalism and their nonclassical properties

Now, we are ready to apply the present formalism to arbitrary quantum systems with a known spectrum e_n. For this purpose, the hydrogen-like spectrum[26,32] and isotonic oscillator[31] are considered. Also, the present approach can be applied to a series of nonlinear squeezed states with particular nonlinear functions. For the latter case, we will deal with the nonlinear function of Gilmore–Perelomov (GP) type of SU(1,1) coherent states,[35] which is well-established as a nonlinear coherent state in Refs. [20] and [23].

6.1. Hydrogen-like spectrum

The hydrogen-like spectrum, as an important physical system, are considered. Their shifted energy eigenvalues are given by

$$e_n = 1 - \frac{1}{(n+1)^2} . \quad (27)$$

The associated squeezed states can be obtained explicitly by substituting Eq. (27) in Eq. (7) as follows:

$$\begin{aligned} |\xi, e_n\rangle = & \mathcal{N}_e(r^2)^{-1/2} \sum_{n=0}^{\infty} \frac{1}{(2n)!!} \sqrt{\frac{n+1}{2n+1}} \\ & \times (\, \mathrm{e}^{\mathrm{i}\phi} \tanh r)^n \, |2n\rangle. \end{aligned} \quad (28)$$

These states are defined in the whole complex plane (see Eq. (9)). The Pegg–Barnett phase distribution against θ for a hydrogen-like spectrum is plotted in Fig. 1(a) for different values of r (we set $\phi = 0$ throughout this paper). From Fig. 1(a), it is seen that $P_e(\theta)$ has three peaks at $\theta = 0$ and $\theta = \pm\pi$, and the peaks become sharper and higher as r increases.

Thus, the generalized squeezed states associated with a hydrogen-like spectrum show quantum interference effects.

To observe the squeezing behavior of the number or phase operator for the squeezed states of hydrogen-like spectrum, we plot the curves of S_N and S_ϕ against r in Fig. 2(a). From this figure, it is found that the squeezing parameters have opposite behaviors. For $r < 1$, $S_n < 0$ ($S_\phi > 0$) implies that the squeezing occurs in the number component in this interval. For $r > 1$ no squeezing effect can be observed in the number and phase components.

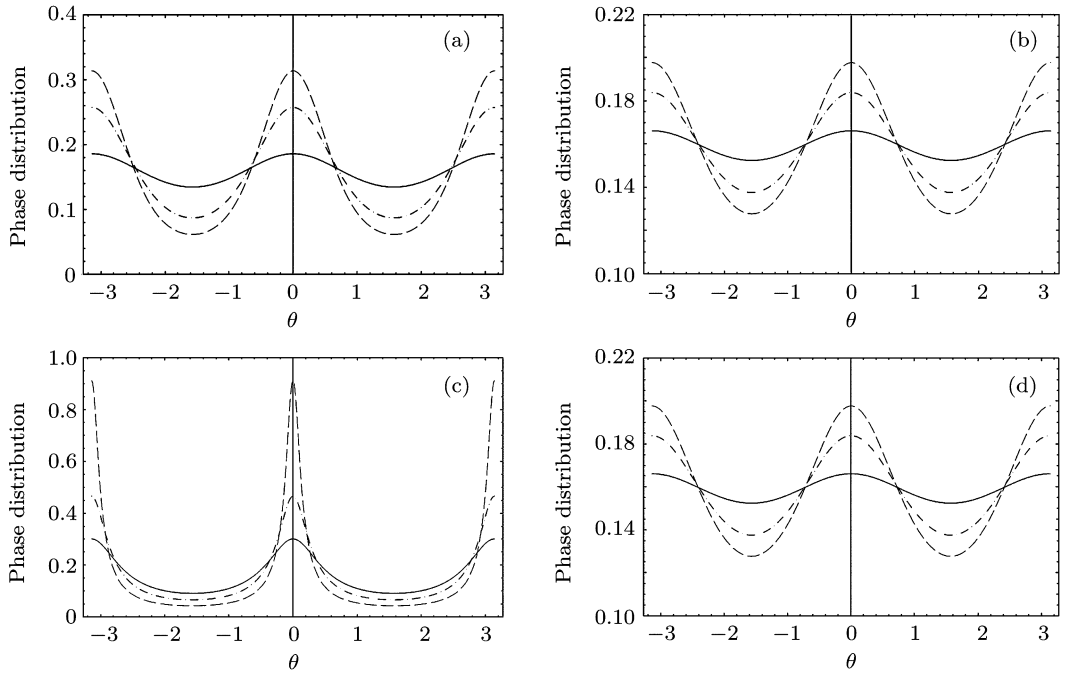


Fig. 1. Pegg–Barnett phase distribution against θ for different values of r for (a) generalized squeezed states associated with a hydrogen-like spectrum, (b) generalized squeezed states associated with an isotonic oscillator, (c) GP squeezed states, and (d) standard squeezed states.

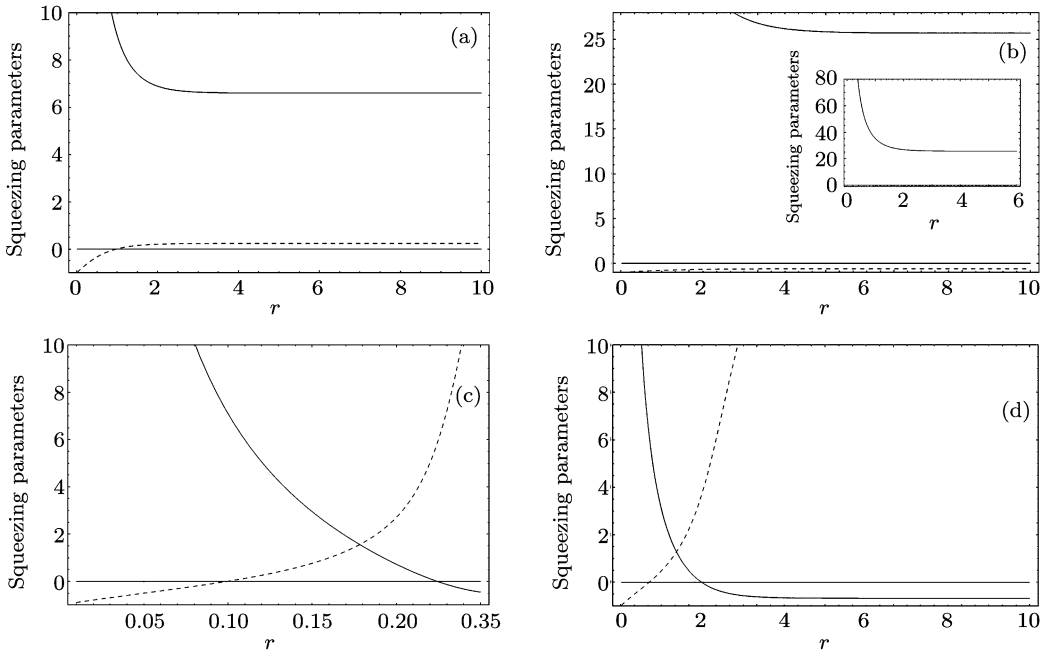


Fig. 2. Plots of S_N (the dashed curve) and S_ϕ (the solid curve) against r for (a) the generalized squeezed states associated with a hydrogen-like spectrum, (b) generalized squeezed states associated with an isotonic oscillator, (c) GP squeezed states, and (d) standard squeezed states.

The number–phase Wigner function for the squeezed states of this spectrum is shown in Fig. 3(a). It is represented in the cylindrical coordinates as a surface at a height of $W_{\rm NP}(n,\theta)$ above the point (x,y) ($x = n\cos\theta$, $y = n\sin\theta$). The surface is drawn as curves of the constant integer n (concentric rings incremented by one unit) which are crossed by curves of the constant θ (radial lines). It is seen that a peak appears on the $n = 1$ curve and along the $\theta = 0$ direction. As can be observed, in some regions the number–phase Wigner function is negative, indicating the nonclassical behavior of these states.

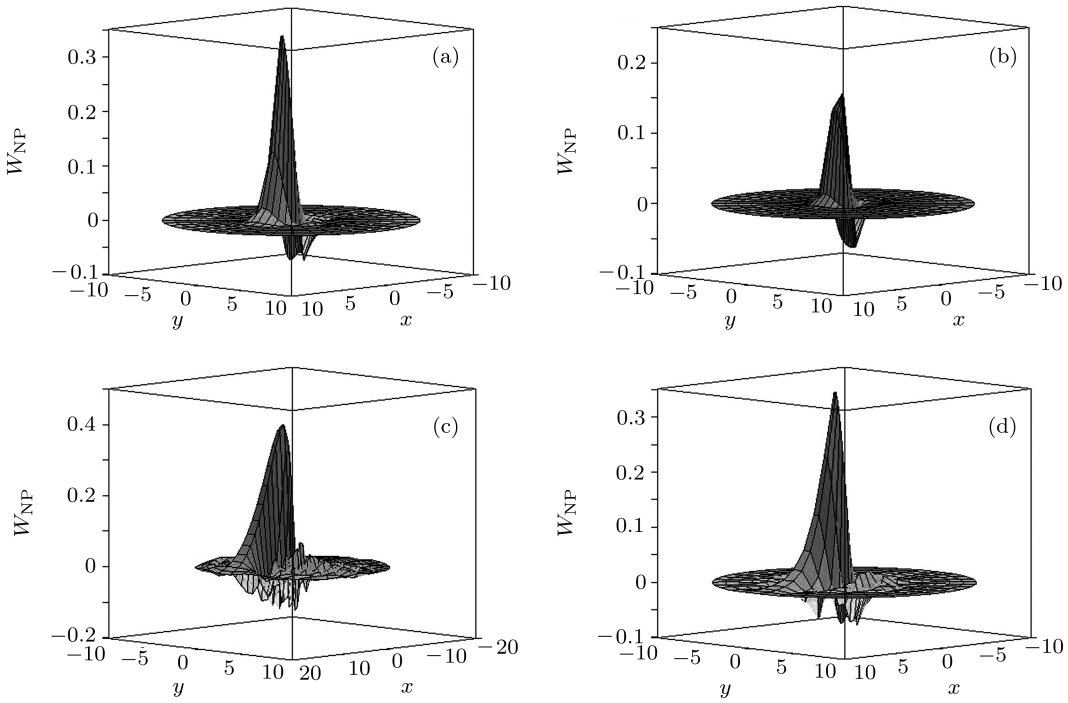


Fig. 3. Plot of the number–phase Wigner function for (a) the generalized squeezed states associated with the hydrogen-like spectrum, (b) generalized squeezed states associated with isotonic oscillator, (c) GP squeezed states, and (d) standard squeezed states.

6.2. Isotonic oscillator

Another interesting model of solvable systems is the isotonic oscillator with the following Hamiltonian:[33]

$$H = -\frac{{\rm d}^2}{{\rm d}x^2} + x^2 + \frac{A}{x^2}, \qquad A \geq 0. \tag{29}$$

It is known that Hamiltonian (29) admits exact eigenvalues given by $\epsilon_n = 2(2n+\gamma)$, where $\gamma = 1 + \sqrt{1+4A}/2$.[34] To satisfy the condition that $e_0 = 0$, we have to use the shifted energy

$$e_n = \epsilon_n - 2\gamma = 4n. \tag{30}$$

The associated squeezed states may be obtained explicitly by putting Eq. (30) in Eq. (7) as follows:

$$|\xi, e_n\rangle = \mathcal{N}_e (r^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\sqrt{16^n (2n)!}}{(2n)!!} \times ({\rm e}^{{\rm i}\phi} \tanh r)^n \, |2n\rangle. \tag{31}$$

From Eq. (9), it is seen that these states are restricted in a disk centered at the origin with $r < \tanh^{-1}(1/4) \approx 1/4$. In Fig. 1(b), we plot the Pegg–Barnett phase distribution against θ for different values of r for the squeezed states associated with the isotonic oscillator. The behavior of $P_e(\theta)$ in this case is qualitatively similar to Fig. 1(a). We plot the squeezing parameters against r for the isotonic oscillator in Fig. 2(b). From this figure, we find that the number squeezing occurs for $r < 0.1$. We also observe that in the interval of $0.1 < r < 0.22$ both S_N and S_ϕ are positive, which implies the absence of squeezing in both components in the mentioned region. Moreover, for $r > 0.22$, S_ϕ is negative, indicating the squeezing in the phase component. The number–phase Wigner function for the considered system is plotted in Fig. 3(b). It is observed that a peak occurs on the $n = 2$ curve and along the $\theta = 0$ direction. Again, it is seen that in some regions the number–phase Wigner function is negative, showing the nonclassicality of the states.

6.3. GP squeezed states of SU(1,1) group

The GP coherent states of the SU(1,1) group in the number-state representation read:[35]

$$|z,\kappa\rangle_{\rm GP}^{\rm SU(1,1)} = \mathcal{N}_f(|z|^2)^{-1/2}\sum_{n=0}^{\infty}\sqrt{\frac{\Gamma(n+2\kappa)}{n!}}z^n|n\rangle, \quad |z|<1, \tag{32}$$

where $\mathcal{N}_f(|z|^2)$ is the normalization constant and the label κ takes discrete values of $1/2, 1, 3/2, 2, \ldots$. The nonlinear function in this case can be described as[20,23]

$$f_{\rm GP}(n,\kappa) = \frac{1}{\sqrt{n+2\kappa-1}}. \tag{33}$$

Thus, the associated squeezed states can be obtained explicitly by putting Eq. (33) in Eq. (4)

$$|\xi,f\rangle = \mathcal{N}_f(r^2)^{-1/2}\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{(2n)!!}\sqrt{\frac{\Gamma(2k)}{\Gamma(2n+2k)}} \times (\,{\rm e}^{{\rm i}\phi}\tanh r)^n\,|2n\rangle. \tag{34}$$

These states are defined in the whole complex plane. In Fig. 1(c) we plot the Pegg–Barnett phase distribution for GP squeezed states against θ for different values of r, with $\kappa = 3$. From the figure it is seen that there are three peaks at $\theta = 0$ and $\theta = \pm\pi$, and the peaks become sharper and higher as r increases, which is similar to the previous examples shown in Figs. 1(a) and 1(b). We plot the squeezing parameters against r for GP squeezed states in Fig. 2(c). From the figure we find that the number squeezing occurs for all values of r. The number–phase Wigner function for GP squeezed states is shown in Fig. 3(c). It is seen that a peak appears on the $n = 1$ curve and along the $\theta = 0$ direction. As can be observed, the number–phase Wigner function is negative in some regions, indicating the nonclassical behavior of these states.

We also perform numerical calculations for the standard squeezed states with $e_n = n$ (i.e., $f(n) = 1$) for comparing it with our new states. In Figs. 1(d) and 2(d), we plot the Pegg–Barnett phase distribution and the squeezing parameters for the standard squeezed states, respectively. The behavior of the phase distribution and the squeezing parameters for the standard squeezed states are qualitatively similar to the hydrogen-like spectrum, isotonic oscillator, and GP squeezed states except that the distance with no squeezing ranges from 0.4 to 1.2. The number–phase Wigner function for the standard squeezed states is plotted in Fig. 3(d). It is seen that a peak appears on the $n = 1$ curve and along the $\theta = 0$ direction. Again, the number–phase Wigner function is negative in some regions, and thus the state exhibits nonclassical behavior.

7. The time evolution of the generalized squeezed states associated with solvable quantum systems

Applying the evolution operator ($\hbar = 1 = \omega$)

$$U(t) = {\rm e}^{-{\rm i}Ht} \tag{35}$$

to the state (7), one can readily obtain the time evolution of the generalized squeezed states associated with solvable quantum systems as

$$|\xi,e_n,t\rangle = \mathcal{N}_e(r^2)^{-1/2}\sum_{n=0}^{\infty}{\rm e}^{-{\rm i}e_{2n}t}\frac{\sqrt{[e_{2n}]!}}{(2n)!!} \times (\,{\rm e}^{{\rm i}\phi}\tanh r)^n|2n\rangle. \tag{36}$$

Using Eq. (36), one can investigate the nonclassical properties of state (7) as time goes on. Considering Eq. (36) and following the same procedure which leads us to Eq. (21), an exactly same result for $\langle(\Delta N)^2\rangle$ is obtained again. Moreover, for the phase distribution of the temporally stable states (36), the relation (17) directly becomes

$$P_e(\theta,t) = \frac{1}{2\pi}\{1+2\mathcal{N}_e(r^2)^{-1}\sum_{n=0}^{\infty}\sum_{k<n}\frac{\sqrt{[e_{2n}]![e_{2k}]!}}{(2n)!!(2k)!!} \times (\,{\rm e}^{{\rm i}\phi}\tanh r)^n(\,{\rm e}^{-{\rm i}\phi}\tanh r)^k \times \cos[2(n-k)\theta+(e_{2n}-e_{2k})t]\}. \tag{37}$$

By combining Eq. (37) with Eqs. (20) and (25), the following expressions for the phase variance and the number–phase Wigner function can be obtained, respectively,

$$\begin{aligned}&\langle(\Delta\phi_\theta)^2\rangle\\ &= \frac{\pi^2}{3}+\mathcal{N}_e(r^2)^{-1}\sum_{n=0}^{\infty}\sum_{k<n}\frac{\sqrt{[e_{2n}]![e_{2k}]!}}{(2n)!!\,(2k)!!}\\ &\times\frac{(\,{\rm e}^{{\rm i}\phi}\tanh r)^n(\,{\rm e}^{-{\rm i}\phi}\tanh r)^k}{(n-k)^2}\\ &\times\cos[(e_{2n}-e_{2k})t]-(\mathcal{N}_e(r^2)^{-1}\\ &\times\sum_{n=0}^{\infty}\sum_{k<n}\frac{\sqrt{[e_{2n}]![e_{2k}]!}}{(2n)!!\,(2k)!!}\\ &\times\frac{(\,{\rm e}^{{\rm i}\phi}\tanh r)^n(e^{-i\phi}\tanh r)^k}{n-k}\\ &\times\sin[(e_{2n}-e_{2k})t])^2,\end{aligned} \tag{38}$$

$W_{NP}(n,\theta)$

$$= \frac{\mathcal{N}_e (r^2)^{-1}}{2\pi} \Bigg\{ \sum_{p=-n}^{n} \frac{\sqrt{[e_{n+p}]![e_{n-p}]!}}{(n+p)!!(n-p)!!} \times (\tanh r)^n \, \mathrm{e}^{-\mathrm{i}p\phi} \, \mathrm{e}^{2\mathrm{i}p\theta} \times \mathrm{e}^{\mathrm{i}(e_{n+p}-e_{n-p})t} + \sum_{p=-n}^{n-1} \frac{\sqrt{[e_{n+p}]![e_{n-p-1}]!}}{(n+p)!!(n-p-1)!!} \times (\tanh r)^{n-1/2} \times \mathrm{e}^{-\mathrm{i}(p+1/2)\phi} \, \mathrm{e}^{\mathrm{i}(2p+1)\theta} \, \mathrm{e}^{\mathrm{i}(e_{n+p}-e_{n-p-1})t} \Bigg\}. \quad (39)$$

Therefore, we can study the time evolution of the generalized squeezed states corresponding to any quantum system with a known nondegenerate discrete spectrum, as well as nonlinear squeezed states with a particular nonlinear function (by setting $e_n = nf^2(n)$ in Eqs. (37), (38), and (39)) and investigate their nonclassical properties as time goes on. For this purpose, we pay attention to the Pegg–Barnet phase distribution and the number–phase squeezing of the generalized squeezed states. Figs. 4(a) and 4(b) show the time evolution of $P_e(\theta, t)$ for the squeezed states of a hydrogen-like spectrum and GP states, respectively. In addition, in Figs. 5(a) and 5(b) we plot the squeezing parameters against t for the squeezed states of hydrogen-like spectrum and GP states, respectively. It is seen that both S_N and S_ϕ oscillate with time.

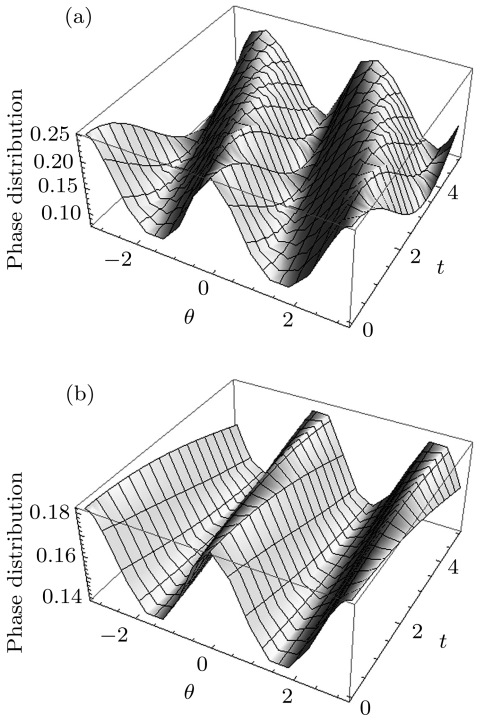


Fig. 4. Three-dimensional plot of the Pegg–Barnett phase distribution for (a) the generalized squeezed states associated with hydrogen-like spectrum, and (b) GP squeezed states.

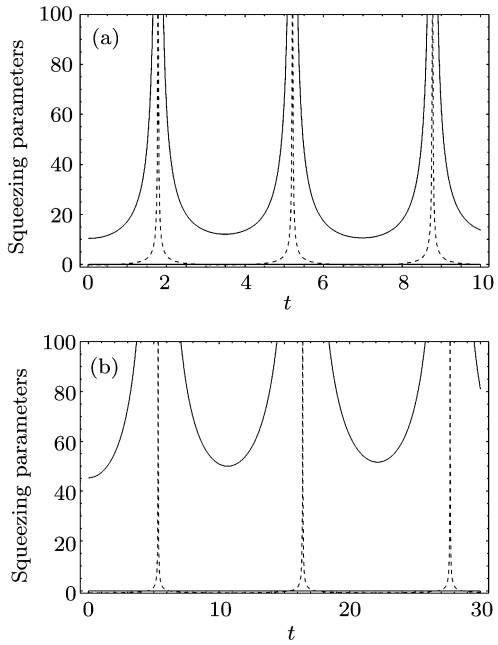


Fig. 5. Plots of S_N (the dashed curve) and S_ϕ (the solid curve) against t for (a) the hydrogen-like spectrum, and (b) GP squeezed states.

8. Conclusion

In summary, we have shown that the Pegg–Barnett formalism can be applied to the generalized squeezed states associated with solvable quantum systems with discrete spectra. Then, the phase probability distribution, the phase and number squeezing of two physical systems with a known discrete nondegenerate spectrum (e_n), and the nonlinear squeezed states with known nonlinear functions ($f(n)$), have been investigated. It is found that the obtained states which correspond to a few known physical systems exhibit phase or number squeezing for different ranges of real values of ξ. In addition, the number–phase Wigner function of the generalized squeezed states for the considered systems is studied. Consequently, all of the introduced states possess nonclassical features. Finally, the time evolution of some nonclassical properties for the generalized squeezed states is discussed. We arrive at a conclusion that in all of the considered squeezed states associated with the physical systems, the mentioned nonclassical features can be observed.

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