Evolution of emergent C-points, L-lines, and C-lines in free space*

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Polarization singularities, which emerge from the incoherent superposition of two vector electric fields with the same frequency, and their evolution in free space are studied analytically and illustrated by numerical examples. It is shown that there exist C-points, L-lines, in particular, C-lines in incoherently superimposed two-dimensional wavefields. Usually, the C-lines are unstable and disappear during the free-space propagation. The motion, pair creation–annihilation process of the emergent C-points, as well as the distortion of the L-lines may take place, and the degree of polarization of the emergent C-points varies upon propagation and may be less than 1.

Keywords: vector singular optics, emergent C-points, L-lines and C-lines, incoherent superposition, degree of polarization

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1. Introduction

Studies of optical singular phenomena have been extended from scalar wavefields to vector wavefields.[1−18] In vector singular optics, the polarization singularities in monochromatic vector wavefields have been extensively studied theoretically, numerically, and experimentally since Nye's work.[6−15,18] Nye, Berry, and Dennis showed that in the two-dimensional (2D) case the polarization pattern may contain singularities, such as C-points, which are isolated points of circular polarization, where the orientations of the major and the minor axes of the ellipse are undermined, and L-lines, which are continuous lines of linear polarization, where the handedness of the ellipse is undefined.[3,6] Angelsky *et al.* proposed the interferometric experimental methods for the diagnostics of polarization singularities.[7] Based on the detection of polarization singularities, Soskin *et al.* developed the singular Stokes polarimetery, which can be used for metrology and inspection of polarized speckle fields.[10] Kessler and Freund exposed polarization singularities in polychromatic vector wavefields, which are referred to as Lissajous singularities.[11−13] Furthermore, Lissajous singularities are shown to emerge from the incoherent superposition of optical fields whose individual frequency components are uncorrelated.[14] Therefore, an interesting question arises: can emergent polarization singularities be obtained by incoherently superimposing two beams in monochromatic vector wavefields? The motivation of the present paper is to study the emergent C-points and L-lines, as well as C-lines and their propagation and dynamic behaviors in free space.

2. Theoretical formulation

Assume that a monochromatic 2D vector field $\boldsymbol{E}$ in the plane z=0 is formed by the incoherent superposition of two vector fields $\boldsymbol{E}(A)$ and $\boldsymbol{E}(B)$ with the same frequency ω

$$\boldsymbol{E}(A,t) = [E_x(A)\,\hat{\boldsymbol{x}} + E_y(A)\,\hat{\boldsymbol{y}}]\exp(-\mathrm{i}\omega t), \quad (1a)$$

$$\boldsymbol{E}(B,t) = [E_x(B)\,\hat{\boldsymbol{x}} + E_y(B)\,\hat{\boldsymbol{y}}]\exp(-\mathrm{i}\omega t), \quad (1b)$$

where $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ are unit vectors, and E_x, E_y denote the spatial distributions of the field in the x and the y directions, respectively.

In the incoherent superposition, the intensities are added incoherently, $\langle I\rangle = \langle I_A\rangle + \langle I_B\rangle$, where $\langle I\rangle$ and $\langle I_A\rangle = |E_x(A)|^2 + |E_y(A)|^2$, $\langle I_B\rangle = |E_x(B)|^2 + |E_y(B)|^2$ represent the total intensity and the intensities of two electric field components A and B, respectively. The Stokes parameter S_j of the resulting field

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$\boldsymbol{E}$ is a sum of the individual components[14]

$$S_j = S_j(A) + S_j(B), \quad (j = 0, 1, 2, 3), \tag{2}$$

where

$$S_0(l) = |E_x(l)|^2 + |E_y(l)|^2, \tag{3a}$$
$$S_1(l) = |E_x(l)|^2 - |E_y(l)|^2, \tag{3b}$$
$$S_2(l) = 2\,\mathrm{Re}\left[E_x^*(l)\,E_y(l)\right], \tag{3c}$$
$$S_3(l) = 2\,\mathrm{Im}\left[E_x^*(l)\,E_y(l)\right], \tag{3d}$$

Re and Im denote real and imaginary parts of the function, respectively, $l = A, B$, and the asterisk denotes the complex conjugate.

The C-points and the L-lines are determined by[8,9]

$$S_{12} = S_1 + \mathrm{i}S_2 = 0, \tag{4}$$

and

$$S_3 = 0, \tag{5}$$

respectively. The degree of polarization is defined as[13,14]

$$P = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0} = \frac{\sqrt{\left[S_1(A) + S_1(B)\right]^2 + \left[S_2(A) + S_2(B)\right]^2 + \left[S_3(A) + S_3(B)\right]^2}}{S_0(A) + S_0(B)}. \tag{6}$$

In the following, we consider three typical cases of spatial distributions $\boldsymbol{E}(A)$ and $\boldsymbol{E}(B)$.

Case 1

$$E_x(A) = \frac{1 + x_0}{2}\exp\left(-\frac{x_0^2 + y_0^2}{w_0^2}\right), \tag{7a}$$
$$E_y(A) = \mathrm{i}\frac{1 - x_0}{2}\exp\left(-\frac{x_0^2 + y_0^2}{w_0^2}\right), \tag{7b}$$
$$E_x(B) = \frac{1 + \mathrm{i}(y_0 - a)}{2}\exp\left(-\frac{x_0^2 + y_0^2}{w_0^2}\right), \tag{7c}$$
$$E_y(B) = \frac{\mathrm{i} + y_0 - a}{2}\exp\left(-\frac{x_0^2 + y_0^2}{w_0^2}\right), \tag{7d}$$

where w_0 denotes the waist width of the background Gaussian beam, and a is a parameter.

From Eq. (7), we obtain

$$S_{12}|_{z=0} = \left[x_0 + \mathrm{i}(y_0 - a)\right]\exp\left(-2\frac{x_0^2 + y_0^2}{w_0^2}\right), \tag{8}$$
$$S_3|_{z=0} = \frac{1}{2}\left[2 - x_0^2 - (y_0 - a)^2\right] \times \exp\left(-2\frac{x_0^2 + y_0^2}{w_0^2}\right). \tag{9}$$

Equations (8) and (9) indicate that the resulting electric field $\boldsymbol{E}$ in the plane $z = 0$ contains an emergent C-point $(0, a)$ and an emergent L-line located on the circle of $x_0^2 + (y_0 - a)^2 = 2$.

In the paraxial approximation, electric field components $E_x(A)$, $E_y(A)$, $E_x(B)$, $E_y(B)$ in the z plane can be derived by using the Fresnel diffraction integral (see Appendix A), and S_{12}, S_3 are given by

$$S_{12}|_z = 4|Q|^2\left[x - \xi y + \mathrm{i}\left(\xi x + y - a - \xi^2 a\right)\right], \tag{10}$$
$$S_3|_z = 2|Q|^2\left[2\xi^2 + 2 - x^2 - (y - a)^2 - a^2\xi^2\right], \tag{11}$$

where

$$Q = \frac{\exp\left\{-\mathrm{i}k\left[\dfrac{kw_0^2\xi}{2} + \dfrac{x^2 + y^2}{kw_0^2(\mathrm{i} + \xi)}\right]\right\}}{2(\mathrm{i} + \xi)^2}, \tag{12}$$
$$\xi = \frac{z}{z_0}, \quad z_0 = \frac{kw_0^2}{2} - \text{Rayleigh range}, \tag{13}$$

and k is the wave number related to wavelengths λ ($k = 2\pi/\lambda$). From Eqs. (10)–(13), we can see that during the free-space propagation, the C-point moves from $(0,\ a)$ in the plane $z = 0$ to $(\xi a,\ a)$ in the z plane. The L-line in the z plane is located on the circle of $x^2 + (y - a)^2 = 2\xi^2 + 2 - a^2\xi^2$. For the special case of $2\xi^2 + 2 - a^2\xi^2 = 0$, the L-line degenerates into a point located at $(0,\ a)$.

Case 2

$$E_x(A) = \frac{x_0 + 1}{2}\exp\left(-\frac{x_0^2 + y_0^2}{w_0^2}\right), \tag{14a}$$
$$E_y(A) = \mathrm{i}\frac{1 + x_0}{2}\exp\left(-\frac{x_0^2 + y_0^2}{w_0^2}\right), \tag{14b}$$
$$E_x(B) = \frac{y_0 - b + 1}{2}\exp\left(-\frac{x_0^2 + y_0^2}{w_0^2}\right), \tag{14c}$$
$$E_y(B) = \mathrm{i}\frac{1 - y_0 + b}{2}\exp\left(-\frac{x_0^2 + y_0^2}{w_0^2}\right). \tag{14d}$$

From Eq. (14), we obtain

$$S_{12}|_{z=0} = (y_0 - b)\exp\left(-2\frac{x_0^2 + y_0^2}{w_0^2}\right), \tag{15}$$
$$S_3|_{z=0} = \frac{1}{2}\left[(x_0 + 1)^2 + 1 - (y_0 - b)^2\right] \times \exp\left(-2\frac{x_0^2 + y_0^2}{w_0^2}\right). \tag{16}$$

It is noted that for such a case, instead of C-point, a C-line is present, which is located on y_0=b. In ad-

dition, there is an L-line located on the hyperbola of $(y_0-b)^2-(x_0+1)^2=1$.

The electric field components in the z plane are expressed as

$$S_{12}|_z = 4|Q|^2\left(y-b-\xi^2 b+\mathrm{i}\xi y\right), \quad (17)$$

$$S_3|_z = 2|Q|^2\left[2\xi^2-b^2\xi^2+(x+1)^2 - (y-b)^2+1\right]. \quad (18)$$

Equations (17) and (18) imply that during the free-space propagation, the C-line is unstable and vanishes when $b\neq 0$. The L-line is located on the hyperbola of $2\xi^2-b^2\xi^2+1=(y-b)^2-(x+1)^2$, and degenerates into two straight lines $y-b=\pm(x+1)$ when $2\xi^2-b^2\xi^2+1=0$.

Case 3

$$E_x(A) = \left(\frac{1+x_0^2-x_0 b}{2}\right)\exp\left(-\frac{x_0^2+y_0^2}{w_0^2}\right), \quad (19a)$$

$$E_y(A) = \mathrm{i}\frac{1-x_0^2+x_0 b}{2}\exp\left(-\frac{x_0^2+y_0^2}{w_0^2}\right), \quad (19b)$$

$$E_x(B) = \frac{1+\mathrm{i}\,(x_0y_0-by_0-ax_0+ab)}{2} \times \exp\left(-\frac{x_0^2+y_0^2}{w_0^2}\right), \quad (19c)$$

$$E_y(B) = \frac{\mathrm{i}+x_0y_0-by_0-ax_0+ab}{2} \times \exp\left(-\frac{x_0^2+y_0^2}{w_0^2}\right). \quad (19d)$$

From Eq. (19), we obtain

$$S_{12}|_{z=0} = (x_0-b)\left[x_0+\mathrm{i}\,(y_0-a)\right] \times \exp\left(-2\frac{x_0^2+y_0^2}{w_0^2}\right), \quad (20)$$

$$S_3|_{z=0} = \frac{1}{2}\left[2-\left(x_0^2-x_0b\right)^2-(x_0-b)^2(y_0-a)^2\right] \times \exp\left(-2\frac{x_0^2+y_0^2}{w_0^2}\right). \quad (21)$$

From Eqs. (20) and (21), we can see that there are a C-poin at $(0,\ a)$, a C-line on $x_0=b$, and an L-line on $2=(x_0-b)^2\left[x_0^2+(y_0-a)^2\right]$.

During the free-space propagation, the electric field components in the z plane are given by

$$E_x(A) = M\frac{\pi}{2p^2}\exp\left(\frac{q_1^2+q_2^2}{p}\right) \times\left(\frac{1}{2}+\frac{q_1^2}{p}-bq_1+p\right), \quad (22a)$$

$$E_y(A) = M\frac{\pi}{2p^2}\exp\left(\frac{q_1^2+q_2^2}{p}\right) \times\mathrm{i}\left(p-\frac{1}{2}-\frac{q_1^2}{p}+bq_1\right), \quad (22b)$$

$$E_x(B) = M\frac{\pi}{2p^2}\exp\left(\frac{q_1^2+q_2^2}{p}\right) \times\left(p+\mathrm{i}\frac{q_1q_2}{p}-\mathrm{i}bq_2-\mathrm{i}aq_1+\mathrm{i}abp\right), \quad (22c)$$

$$E_y(B) = M\frac{\pi}{2p^2}\exp\left(\frac{q_1^2+q_2^2}{p}\right) \times\left(\mathrm{i}p+\frac{q_1q_2}{p}-bq_2-aq_1+abp\right), \quad (22d)$$

where

$$M=\frac{\mathrm{i}}{\lambda z}\exp\left[-\mathrm{i}kz-\frac{\mathrm{i}k}{2z}(x^2+y^2)\right], \quad (23)$$

$$p=\frac{1}{w_0^2}+\frac{\mathrm{i}k}{2z}, \quad (24)$$

$$q_1=\frac{\mathrm{i}kx}{2z},\quad q_2=\frac{\mathrm{i}ky}{2z}. \quad (25)$$

The evolution behaviors of the emergent polarization singularities for case 3 will be illustrated using numerical examples in the next section.

3. Numerical examples

In numerical calculations, parameters $\lambda = 632.8$ nm and $w_0 = 1$ mm are kept fixed. Figures 1(a)–1(d) give contour lines of phase of S_{12} and L-line ($S_3=0$) in the free-space propagation by using Eqs. (19)–(25) for (a) $\xi=0$, (b) $\xi=0.5$, (c) $\xi=1$, and (d) $\xi=10$, where $a=0$ mm, $b=0.5$ mm. As can be expected, an on-axis C-point A at $(0,\ a=0)$, an off-axis C-line on $x_0=b=0.5$ mm, and a closed L-line on $2=(x_0-0.5)^2\left[x_0^2+y_0^2\right]$ (see the enlargement of the L-line) are present in the plane of $z=0$. The singularity index of the C-point is equal to $I_{\mathrm{C}}=+1/2$ according to the sign principle,[19] and the C-point A is right-handed determined by the sigh of S_3.[9,20] According to Eq. (6), the degree of polarization of the C-point A is $P=1$. When the propagation distance is increased to $\xi=0.5$ (Fig. 1(b)), the C-line is absent, and two C-points with the same right handedness but opposite topological charge $I_{\mathrm{C}}=+1/2,-1/2$ at (0.75 mm, 0.25 mm) and (0.25 mm, −0.75 mm) respectively are present because of the interaction between the C-point and the C-line, which is similar to the vortex–edge dislocation in scalar wavefields.[21−23] Meanwhile, the C-point A moves to (0, −0.5 mm), and its handedness and singularity index remain unchanged. Interestingly, the degrees of polarization of C-points A, B, and C become $P=0.905, 0.990, 0.914$, respectively. It means that for the case of incoherent superposition, the degree of polarization of C-point is variable depending upon the propagation and may be less than 1. The proof is shown in Appendix B. With

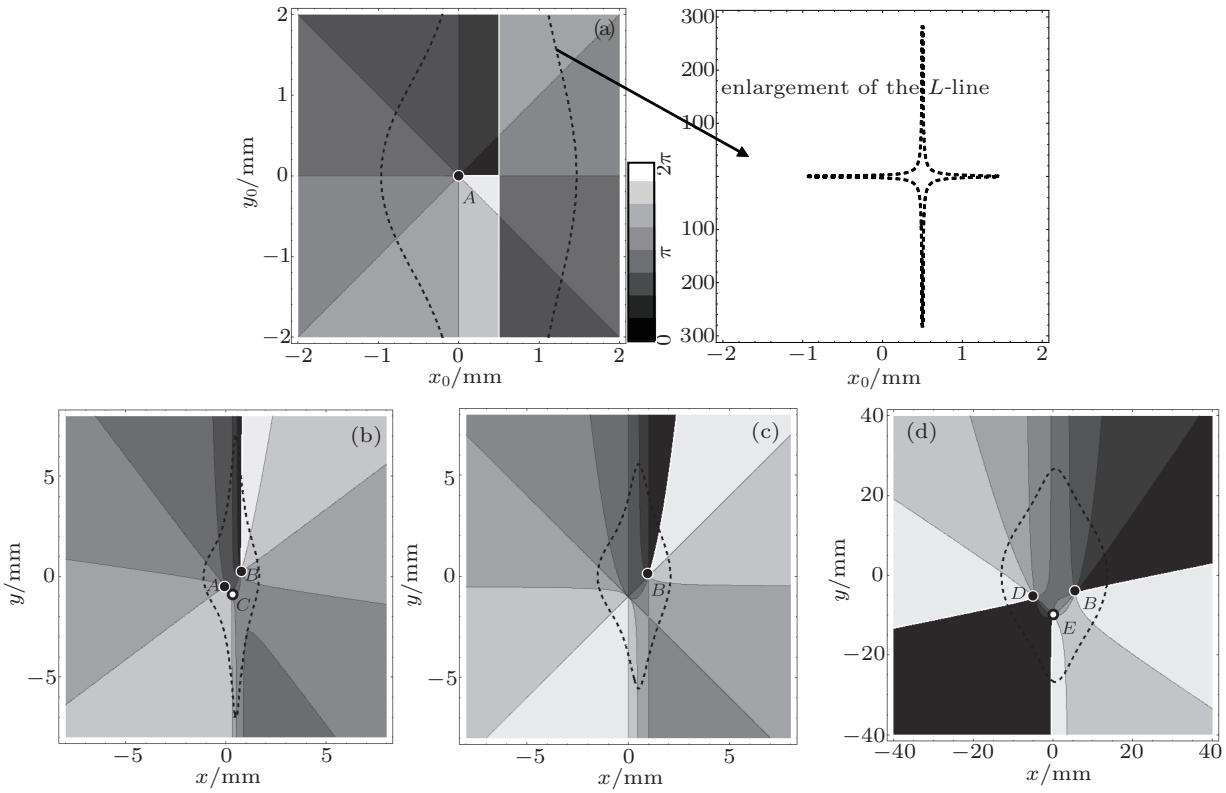


Fig. 1. Contour lines of phase of S_{12} and L-line at (a) $\xi = 0$, (b) $\xi = 0.5$, (c) $\xi = 1$, and (d) $\xi = 10$. In a filled black circle with white rims, $I_{\rm C} = +1/2$; in a filled white circle with black rims, $I_{\rm C} = -1/2$; the dashed line is the L-line.

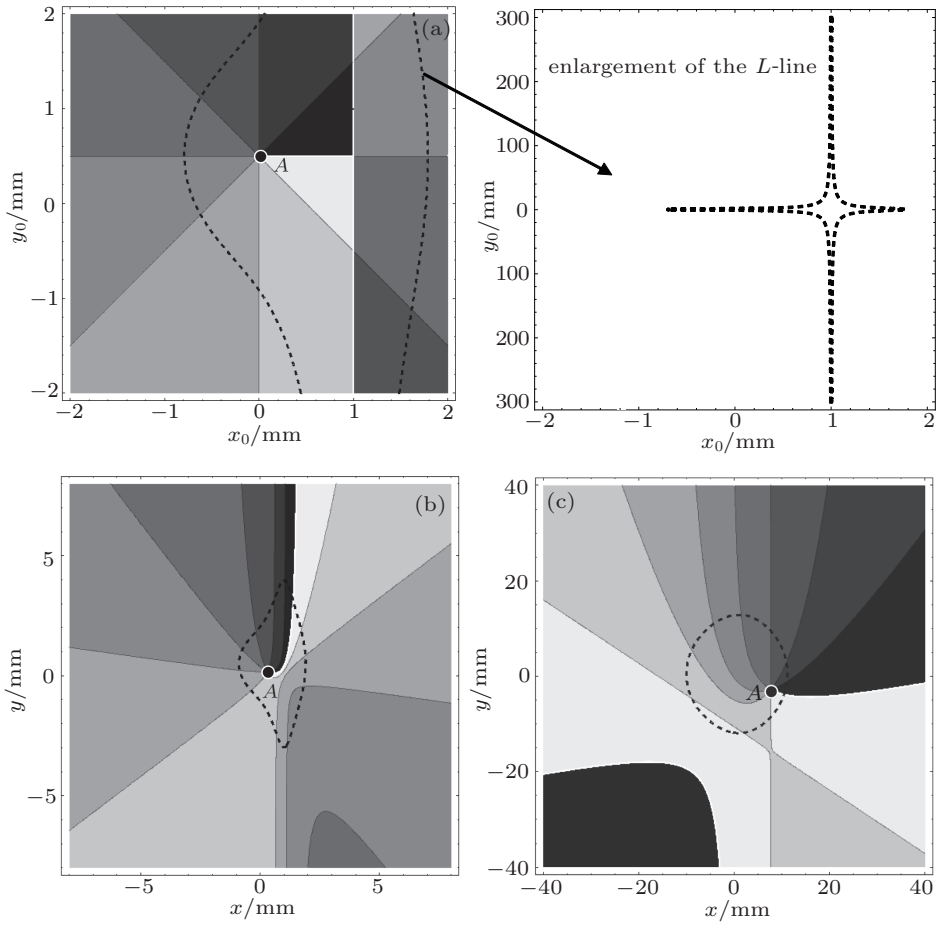


Fig. 2. Contour lines of the phase of S_{12} and L-line at (a) $\xi = 0$, (b) $\xi = 0.5$, and (c) $\xi = 10$.

z increasing to the propagation distance $\xi = 1$ in Fig. 1(c), a pair of C-points A and C with the same right handedness and opposite charge $I_{\rm C} = \pm 1/2$ annihilate each other, and only the C-point B at (1 mm, 0) remains, whose degree of polarization becomes $P = 1$. With a further increase ($\xi = 10$ in Fig. 1(d)), a pair of oppositely charged C-points D and E with the same right handedness and $P = 0.742$, 0.603 are present, and the C-point B shifts to (5.5 mm, −4.5 mm), whose degree of polarization becomes $P = 0.819$. Additionally, from Figs. 1(a)–1(d), we can see that the L-line remains during the free-space propagation, but its distortion is obvious.

The contour lines of the phase of S_{12} and L-line ($S_3 = 0$) are depicted in Figs. 2(a)–2(c) by using Eqs. (19)–(25) for (a) $\xi = 0$, (b) $\xi = 0.5$, and (c) $\xi = 10$, where $a = 0.5$ mm, $b = 1$ mm. As can be seen, there are an off-axis C-point with right handedness $I_{\rm C} = +1/2$ and $P = 1$ at (0, $a = 0.5$ mm), an off-axis C-line on $x_0 = b = 1$ mm, and an L-line on a closed curve of $2 = (x_0 - 1)^2 \left[x_0^2 + (y_0 - 0.5)^2\right]$ in the plane of z=0. Owing to their interaction, at the propagation distance of $\xi = 0.5$ in Fig. 2(b), the C-line vanishes. The C-point A moves to (0.301 mm, 0.179 mm), and its degree of polarization becomes $P = 0.855$. In Fig. 2(c), $\xi = 10$, the C-point A shifts to (7.652 mm, −2.736 mm), and its degree of polarization is decreased to $P = 0.332$. The right handedness of the C-point A remains unchanged, and the L-line is distorted upon the propagation.

4. Conclusion

In this paper, the polarization singularities formed by the incoherent superposition of two vector electric fields with the same frequency and their evolution in free space have been studied analytically and numerically. We have found that besides C-points and L-lines, there exist C-lines, which emerge from two incoherently superimposed monochromatic 2D wavefields. Usually, the C-lines are unstable and vanish during the free-space propagation. The motion, pair creation and annihilation of emergent C-points, as well as the distortion of L-lines may appear, and the degree of polarization of the emergent C-points varies upon propagation and may be less than 1, which is different from that of C-points in a fully coherent vector electric field. For the latter case, the Stokes parameters obey $S_1^2 + S_2^2 + S_3^2 = S_0^2$, so that the degree of polarization is $P \equiv 1$. The interaction of the C-point and the C-line in vector wavefields is similar to the vortex–edge dislocation interaction in scalar wavefields. As compared with Ref. [14], where the emergent Lissajous singularities in incoherently superimposed polychromatic vector wavefields were discovered, in our paper, we have shown the existence of emergent C-points, L-lines, and C-lines in monochromatic vector wavefields and studied their evolution behavior in free space. The results obtained are beneficial to a deep understanding of the propagation and the dynamic behavior of polarization singularities in superimposed wavefields, which may enhance their usefulness in their applications.

Appendix A

Within the framework of the paraxial approximation, the electric field components $E_x(A)$, $E_y(A)$, $E_x(B)$, $E_y(B)$ in the z plane can be derived by using the Fresnel diffraction integral, and are given by

$$E_x(A) = M[q_1 + p], \quad E_y(A) = \mathrm{i}M[p - q_1], \tag{A1}$$

$$E_x(B) = M[p + \mathrm{i}q_2 - \mathrm{i}ap],$$

$$E_y(B) = M[\mathrm{i}p + q_2 - ap], \tag{A2}$$

where

$$M = \frac{\mathrm{i}\pi}{2\lambda p^2 z} \exp\left[-\mathrm{i}kz - \frac{\mathrm{i}k\left(x^2 + y^2\right)}{2z} + \frac{q_1^2 + q_2^2}{p}\right], \tag{A3}$$

$$p = \frac{1}{w_0^2} + \frac{\mathrm{i}k}{2z}, \quad q_1 = \frac{\mathrm{i}kx}{2z}, \quad q_2 = \frac{\mathrm{i}ky}{2z}, \tag{A4}$$

and wave number k is related to wavelength λ by $k = 2\pi/\lambda$.

Appendix B

For the incoherent superposition of two vector wavefields, the degree of polarization is expressed in Eq. (6), which can be rewritten as

$$P = \sqrt{\frac{S_0(A)^2 + S_0(B)^2 + 2S_1(A)S_1(B) + 2S_2(A)S_2(B) + 2S_3(A)S_3(B)}{S_0(A)^2 + S_0(B)^2 + 2S_0(A)S_0(B)}}. \tag{B1}$$

It is readily shown that

$$\begin{aligned}&[S_0(A)S_0(B)]^2\\&-[S_1(A)S_1(B)+S_2(A)S_2(B)+S_3(A)S_3(B)]^2\\=&[S_1(A)S_2(B)-S_2(A)S_1(B)]^2\\&+[S_1(A)S_3(B)-S_3(A)S_1(B)]^2\\&+[S_2(A)S_3(B)-S_3(A)S_2(B)]\geq 0.\end{aligned}\tag{B2}$$

Additionally,

$$S_0(A)>0 \quad \text{and} \quad S_0(B)>0, \tag{B3}$$

from Eqs. (B2)–(B3), we obtain

$$\begin{aligned}S_0(A)S_0(B)>&S_1(A)S_1(B)+S_2(A)S_2(B)\\&+S_3(A)S_3(B).\end{aligned}\tag{B4}$$

Therefore

$$0\leq P\leq 1, \tag{B5}$$

and P may vary because $S_0(l)$, $S_1(l)$, $S_2(l)$, and $S_3(l)$ ($l=A$, B) are variable upon propagation.

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