Chaotic behaviors of the (2+1)-dimensional generalized Breor–Kaup system*

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With the help of the Maple symbolic computation system and the projective equation approach, a new family of variable separation solutions with arbitrary functions for the (2+1)-dimensional generalized Breor–Kaup (GBK) system is derived. Based on the derived solitary wave solution, some chaotic behaviors of the GBK system are investigated.

Keywords: projective equation approach, generalized Breor–Kaup system, variable separation solution, chaotic behavior

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1. Introduction

Many dynamical problems in physics and other natural fields are usually characterized by nonlinear evolution of partial differential equations known as governing equations.[1−6] Searching for an analytical exact solution to a nonlinear system has been an important and interesting topic in nonlinear science both for physicists and mathematicians, and various methods for obtaining exact solutions of a nonlinear system have been proposed, for example, the bilinear method, the standard Painlevé truncated expansion, the method of "coalescence of eigenvalue" or "wavenumbers", the homogenous balance method, and the projective equation method.[7−13] The projective equation approach is a kind of classic, efficient, and well-developed method to solve nonlinear evolution equations, the remarkable characteristic of which is that we can have many different ansatzs, and thus a large number of solutions.[14−24] In this paper, by virtue of a projective equation ($\phi' = \sigma\phi + \phi^2$) and a linear variable separation approach, a new family of exact solutions with arbitrary functions of the (2+1)-dimensional generalized Broek–Kaup (GBK) system is derived. Based on the derived solutions, we find a few novel chaotic line solitons and chaotic patterns.

The (2+1)-dimensional GBK system is

$$H_t - H_{xx} + 2HH_x + U_x + aU + bG = 0, \quad (1)$$

$$G_t + 2(GH)_x + G_{xx} + 4a(G_x - H_{xy}) + 4b(G_y - H_{yy}) + c(G - 2H_y) = 0, \quad (2)$$

$$U_y - G_x = 0, \quad (3)$$

where a, b, and c are arbitrary constants.

The GBK system is recently derived from a typical (1+1)-dimensional Broer–Kaup (BK) system by means of the Painlevé analysis.[25] Obviously, when $a = b = c = 0$, the GBK system will be degenerated to the usual (2+1)-dimensional Broer–Kaup (BK) system,[26−28] which can be derived from the inner parameter-dependent symmetry constraint of Kadomtsev–Petviashvili model.[3] Using some suitable dependent and independent variable transformations, Chen and Li[29] have proved that the (2+1)-dimensional BK system can be transformed to the (2+1)-dimensional dispersive long-water wave equation[30] and (2+1)-dimensional Ablowitz–Kaup–Newell–Segur system.[31] Actually, the (2+1)-dimensional BK system has been widely investigated by many researchers.[32−34] However, to the best of our knowledge, the GBK system has not been understood very well, though Huang *et al.*[35] derived some special soliton-like solitons to a simplified GBK system (b=0) with a variable-coefficient projective Riccati equation method, and Zhang *et al.*[36] obtained certain semi-folded structures via a Painlevé–Bäcklund transformation. In what follows, we will derive some new soliton solutions to the GBK system with a projective equation and variable separation method. The basic idea of the algorithm is as follows. For a given nonlinear partial differential equation (NPDE) with respect to the

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independent variables $x = (x_0 = t, x_1, x_2, \ldots, x_m)$, and the dependent variable u in the form of

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, \ldots) = 0, \tag{4}$$

where P is conventionally a polynomial function of its arguments, and the subscripts denote the partial derivatives. The solution can be assumed to be in the form of

$$u = A(x) + \sum_{i=1}^{n} \{B_i(x)\phi^i[q(x)] + C_i(x)\phi^{i-1} \times [q(x)]\sqrt{\sigma\phi + \phi^2[q(x)]}\}, \tag{5}$$

with

$$\phi' = \sigma\phi + \phi^2, \tag{6}$$

where σ is a constant and the prime denotes the differentiation with respect to q. To determine u explicitly, one can substitute Eqs. (5) and (6) for the given NPDE and collect the coefficients of polynomials of ϕ, and then eliminate each coefficient to derive a set of partial differential equations of $A(x)$, $B_i(x)$, $C_i(x)$, and q, which are solved to obtain $A(x)$, $B_i(x)$, $C_i(x)$, and q. Finally, as Eq. (6) possesses the general solution

$$\phi = \begin{cases} -\frac{1}{2}\sigma\left[1 + \tanh\left(\frac{1}{2}\sigma q\right)\right], & \sigma \neq 0, \\ -\frac{1}{2}\sigma\left[1 + \coth\left(\frac{1}{2}\sigma q\right)\right], & \sigma \neq 0, \end{cases} \tag{7}$$

one can obtain the exact solutions of the given NPDE by substituting $A(x)$, $B_i(x)$, $C_i(x)$, q, and Eq. (7) into Eq. (5).

2. Solitary wave solutions of the (2+1)-dimensional GBK system

Now, we apply the projective equation and the variable separation method to the (2+1)-dimensional GBK system. First, let us transform Eqs. (1), (2), and (3). Differentiating Eq. (1) with respect to y once and substituting Eq. (3) to Eq. (1), we obtain

$$(H_t - H_{xx} + 2HH_x)_y + G_{xx} + aG_x + bG_y = 0, \tag{8}$$

$$G_t + 2(GH)_x + G_{xx} + 4a(G_x - H_{xy}) + 4b(G_y - H_{yy}) + c(G - 2H_y) = 0. \tag{9}$$

Next, we apply the following Bäcklund transformation to Eqs. (8) and (9):[36]

$$H = \frac{\varphi_x}{\varphi} + H_0, \quad G = 2\frac{\varphi_{xy}}{\varphi} - 2\frac{\varphi_x\varphi_y}{\varphi^2}, \tag{10}$$

where $\varphi = \varphi(x, y, t)$ and $H_0 = H_0(x, t)$ are arbitrary functions of indicated arguments. Based on Eqs. (3) and (10), we have

$$G = 2H_y, \quad U = 2H_x + A(x, t), \tag{11}$$

where $A(x, t)$ is an arbitrary integral function of (x, t). Substituting Eq. (11) into Eqs. (8) and (9) yields an identical nonlinear partial differential equation

$$(H_t + H_{xx} + 2aH_x + 2bH_y + 2HH_x)_y = 0, \tag{12}$$

which is equivalent to the following equation:

$$H_{yt} + H_{xxy} + 2aH_{xy} + 2bH_{yy} + 2(HH_x)_y = 0. \tag{13}$$

By the balancing procedure for Eq. (13), ansatz (5) becomes

$$H = f(x, y, t) + g(x, y, t)\phi(q(x, y, t)) + h\sqrt{\sigma\phi(q(x, y, t)) + \phi^2(q(x, y, t))}, \tag{14}$$

where f, g, and q are functions of (x, y, t) to be determined. Substituting Eqs. (14) and (6) into Eq. (13), collecting the coefficients of polynomials of ϕ, and then setting each coefficient zero, we have

$$f = -\frac{1}{4}\frac{q_x^2\sigma + 2q_{xx} + 2q_t + 4bq_y + 4aq_x}{q_x}, \quad g = -\frac{1}{2}q_x, \quad h = \frac{1}{2}q_x, \tag{15}$$

with the function q in a linear variable separation form similar to that in Ref. [37]

$$q = \chi(x, t) + \varphi(y - 2bt), \tag{16}$$

where $\chi \equiv \chi(x, t)$ and $\varphi \equiv \varphi(y - 2bt)$ are two arbitrary variable separation functions of (x, t) and (y, t), respectively. Based on the solutions of Eq. (6), one can obtain the exact solutions for the GBK system.

For $\sigma \neq 0$, we can derive the following solitary wave solutions and periodic wave solutions of the GBK system:

$$H_1 = -\frac{1}{4}\frac{2\chi_{xx} + 2\chi_t + 2\varphi_t + 4a\chi_x + 4b\varphi_y - \chi_x^2\sigma\tanh[\frac{1}{2}\sigma(\chi + \varphi)] - I\sigma\chi_x^2\mathrm{sech}[\frac{1}{2}\sigma(\chi + \varphi)]}{\chi_x}, \tag{17}$$

$$G_1 = -\frac{1}{4}\frac{(8b\varphi_{yy}+4\varphi_{yt})\{\cosh[\frac{1}{2}\sigma(\chi+\varphi)]^2\}-(\sigma^2\chi_x^2\varphi_y)\{1-I\sinh[\frac{1}{2}\sigma(\chi+\varphi)]\}}{\cosh[\frac{1}{2}\sigma(\chi+\varphi)]^2\chi_x},$$

$$\begin{aligned}U_1 = {} & \frac{1}{4\chi_x^2}(-4\chi_x\chi_{xxx}-4\chi_x\chi_{xt}+8\chi_{xx}b\varphi_y+4\chi_{xx}^2+4\chi_{xx}^2\chi_{xt})+\frac{1}{4\cosh[\frac{1}{2}\sigma(\chi+\varphi)]^2}\\ & \times\left\{\chi_x^2\sigma^2+2\sigma\sinh\left[\frac{1}{2}\sigma(\chi+\varphi)\right]\coth\left[\frac{1}{2}\sigma(\chi+\varphi)\right]\chi_{xx}+2I\sigma\chi_{xx}\cosh\left[\frac{1}{2}\sigma(\chi+\varphi)\right]\right.\\ & \left.-I\sigma\chi_x^2\sinh\left[\frac{1}{2}\sigma(\chi+\varphi)\right]\right\}+A(x,t),\end{aligned} \tag{18}$$

$$H_2 = -\frac{1}{4}\frac{2\chi_{xx}+2\chi_t+2\varphi_t+4a\chi_x+4b\varphi_y-\chi_x^2\sigma\coth[\frac{1}{2}\sigma(\chi+\varphi)]+\sigma\chi_x^2\mathrm{csch}[\frac{1}{2}\sigma(\chi+\varphi)]}{\chi_x}, \tag{19}$$

$$G_2 = -\frac{1}{4}\frac{8b\varphi_{yy}+4\varphi_{yt}-\chi_x^2\sigma^2\varphi_y+(4\varphi_{yt}+8b\varphi_{yy})\cosh[\frac{1}{2}\sigma(\chi+\varphi)]}{\{\cosh[\frac{1}{2}\sigma(\chi+\varphi)]+1\}\chi_x}, \tag{20}$$

$$\begin{aligned}U_2 = {} & \frac{1}{4\chi_x^2(\cosh[\frac{1}{2}\sigma(\chi+\varphi)]+1)}\left\{-4\chi_x\chi_{xxx}+4\chi_{xx}\chi_t+8\chi_{xx}b\varphi_y+4\chi_{xx}^2+4\chi_{xx}\varphi_t+\chi_{xx}^4\sigma^2\right.\\ & -4\chi_x\chi_{xxx}+\cosh\left[\frac{1}{2}\sigma(\chi+\varphi)\right](-4\chi_x\chi_{xt}-4\chi_x\chi_{xxx}+8\chi_{xx}b\varphi_y+4\chi_{xx}\chi_{xt}+4\chi_{xx}\chi_{yt}+4\chi_{xx})\\ & \left.+2\sinh\left[\frac{1}{2}\sigma(\chi+\varphi)\right]\chi_{\mathrm{x}}^2\right\}+\mathrm{A(x,t)},\end{aligned} \tag{21}$$

with three arbitrary functions $\chi(x,t)$, $\varphi(y-2bt)$, and $A(x,t)$.

3. Chaotic behaviors in the GBK system

Chaos is an important part of nonlinear science and has been widely applied in many natural sciences. In this section, we mainly discuss some localized coherent excitations with chaotic behavior. For simplification, we only discuss the field G_2 of Eq. (21)

$$\begin{aligned}G = G_2 = {} & -\frac{1}{4}\left\{8b\varphi_{yy}+4\varphi_{yt}-\chi_x^2\sigma^2\varphi_y\right.\\ & \left.+(4\varphi_{yt}+8b\varphi_{yy})\cosh\left[\frac{1}{2}\sigma(\chi+\varphi)\right]\right\}\\ & \times\left\{\cosh\left[\frac{1}{2}\sigma(\chi+\varphi)\right]+1\right\}^{-1}\chi_x^{-1}.\end{aligned} \tag{22}$$

Recently, Wu and Guo have introduced a new chaotic system (named WG) of the three-dimensional quadratic autonomous ordinary differential equations[37]

$$\begin{aligned}& m_\xi = 0.2m-0.2n, \quad n_\xi = m-0.15n+l,\\ & l_\xi = m+n^2-0.6l,\end{aligned} \tag{23}$$

where m, n, and l are functions of ξ ($\xi=(x+kt)$ or $\xi=(y-2bt)$). A novel chaotic attractor for the WG system (24) is shown in Fig. 1 when

$$m(0)=0.02, \quad n(0)=0.13, \quad l(0)=0.06. \tag{24}$$

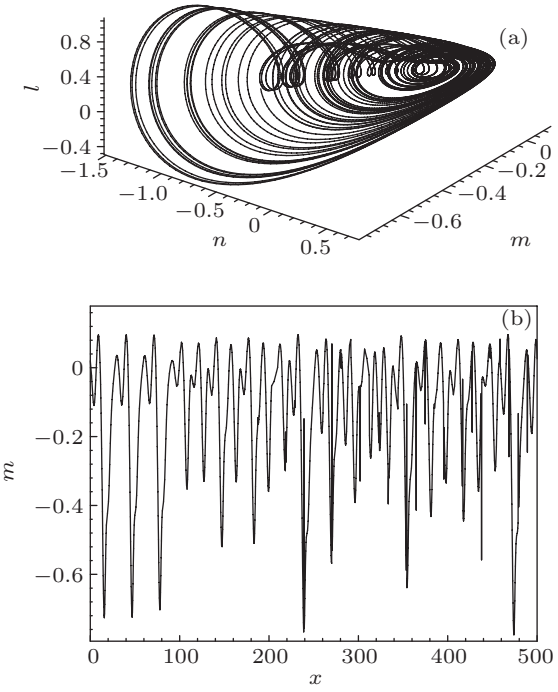


Fig. 1. (a) A novel attractor plot of the chaotic WG system (24) with the initial condition (25). (b) A typical plot of the chaotic solution m of (24) related to (a).

If the functions χ and/or φ are assumed to be solutions of a chaotic dynamical system, we can derive some localized excitations with chaotic behavior. For example, χ is defined as a solution of the WG system (24), and satisfies

$$\chi = 1+m(x+kt), \quad \varphi = 1+\exp(y-2bt), \tag{25}$$

where $m(x+kt)$ is a solution of the WG system (24) in the initial condition (25). Based on this choice, the dromion localized in all directions is changed into a chaotic line soliton, which presents the chaotic behavior in the x-direction and is still localized in y-direction. Figure 2 shows the corresponding plot of the chaotic line soliton for the field (23) with $b = 1$, $k = 1$, and $\sigma = 1$ at time $t = 0$.

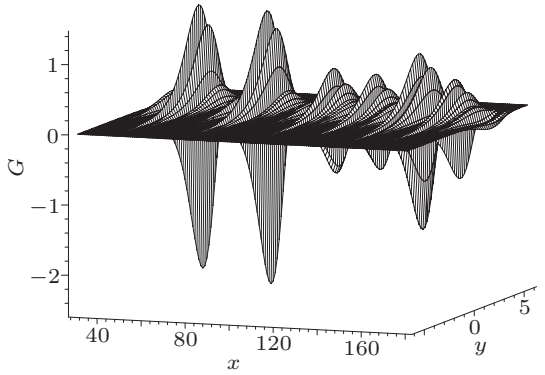


Fig. 2. Chaotic line soliton for the field G determined by Eq. (23) in condition (26) with $b = 1$, $k = 1$, $\sigma = 1$, and $t = 0$.

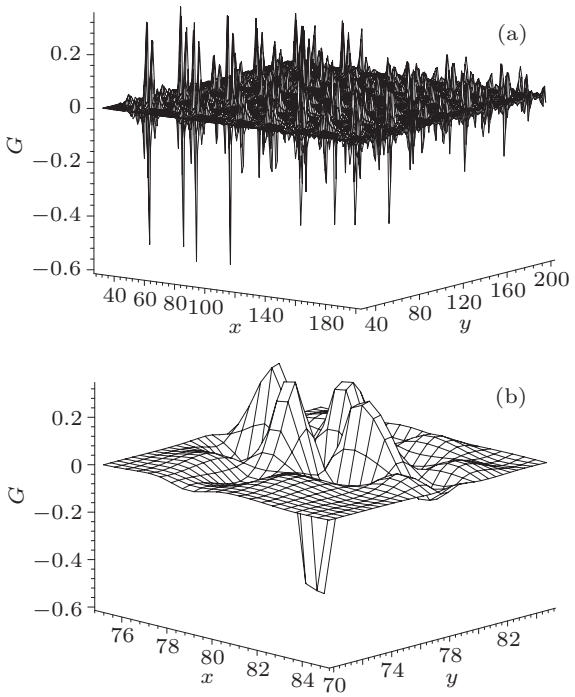


Fig. 3. (a) Chaotic patten for the field G expressed by Eq. (23) in condition (27) with $b = 1$, $k = 1$, $\sigma = 1$, and $t = 0$. (b) Enlargement of the center area ($x \in [75, 85]$, $y \in [70, 85]$) of Fig. 3(a).

Furthermore, if χ and φ are all selected as chaotic solutions of the WG system, the field G given by Eq. (23) will behave chaotically in all directions and yield a chaotic pattern. For example, χ and φ can be chosen as

$$\chi(x,t) = 1 + 0.1m(x+kt),$$

$$\varphi(y) = 1 + 0.1m(y-2bt), \tag{26}$$

where $m(x+kt)$ and $m(y-2bt)$ are the solutions of the WG system (24) in the initial condition (25). Figure 3a shows the special chaotic pattern for the field G expressed by Eq. (23) in condition (27) at $b = 1$, $k = 1$, $\sigma = 1$, and $t = 0$. In order to show that the chaotic behavior is due to the peak value of solitons, we enlarge a small region ($x \in [75, 85]$, $y \in [70, 85]$) of Fig. 3(a). The result is shown in Fig. 3(b), which presents a kind of dromion with a chaotic structure clearly.

4. Conclusion and discussion

In summary, with the help of a projective equation and a linear variable separation method, the (2+1)-dimensional generalized Breor–Kaup system is derived. Additionally, using the WG chaotic system, Yang recently obtained some chaotic solitons of the (2+1)-dimensional Broek–Kaup equation with variable coefficients.[38] Along with the above line, we use the WG system to get new chaotic patterns of the (2+1)-dimensional GBK system, which are different from the ones in the previous work. Since the widespread applications of the soliton theory, it is worthwhile to learn more about the localized excitation and its applications in the future.

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