Solitons for the cubic–quintic nonlinear Schrödinger equation with varying coefficients*

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In this paper, by means of similarity transfomations, we obtain explicit solutions to the cubic–quintic nonlinear Schrödinger equation with varying coefficients, which involve four free functions of space. Four types of free functions are chosen to exhibit the corresponding nonlinear wave propagations.

Keywords: cubic–quintic nonlinear Schrödinger equation, similarity transformation, explicit solutions

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1. Introduction

The nonlinear Schrödinger (NLS) equation and its extensions have attracted much extensive attention due to their numerous physical applications in many branches of nonlinear science, such as nonlinear optics[1−3] and the Bose–Einstein condensates (BECs) (where the NLS equation is also known as the Gross–Pitaevskii equation[4]), biomolecular dynamics,[2] and many others.[5−11]

One of the simplest extensions of the cubic NLS equation is the so-called cubic–quintic nonlinear Schrödinger (CQNLS) model, which is

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\frac{\partial^2\psi}{\partial x^2} + G_1|\psi|^2\psi + G_2|\psi|^4\psi. \tag{1}$$

The CQNLS equation is another universal mathematical model describing many situations of physical interest and approximating many other more complicated models.[8]

The aim of this paper is to study explicit solutions of the CQNLS equation with varying coefficients, which can be written in the following dimensionless form:

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -f(t,x)\frac{\partial^2\psi}{\partial x^2} + h(t,x)\psi + g_1(t,x)|\psi|^2\psi + g_2(t,x)|\psi|^4\psi, \tag{2}$$

where $\psi(t,x)$ represents the macroscopic wavefunction, $f(t,x)$ is the group-velocity dispersion coefficient, $h(t,x)$ is an external poternal potential, $g_1(t,x)$ and $g_2(t,x)$ are the cubic and the quintic nonlinear coefficients corresponding to the two-body and the three-body interactions, respectively. The signs of $g_1(t,x)$ and $g_2(t,x)$ can be positive or negative, indicating that the interactions are repulsive or attractive, respectively. Thus, in this paper, we construct different types of explicit solutions, such as bright and dark soliton solutions.

2. Main method

We consider the CQNLS with varying coefficients. Our first goal is to transform Eq. (2) into the stationary CQNLS equation[8]

$$E\Psi = -U_{XX} + G_1|\Psi|^2\Psi + G_2|\Psi|^4\Psi, \tag{3}$$

where both $\Psi \equiv \Psi(X)$ and $X \equiv X(t,x)$ are real functions, E denotes the eigenvalue of the nonlinear equation, and G_1 and G_2 are the (constant) nonlinearity parameters.

To connect solutions of Eq. (2) with those of Eq. (3), we will use the the following transformation:

$$\psi(t,x) = r(t,x)\Psi[X(t,x)], \tag{4}$$

where $\Psi(X)$ satisfies Eq. (3), and $\psi(t,x)$ is a solution of Eq. (2). The group-velocity dispersion coefficient $f(t,x)$, the potential $h(t,x)$, and the nonlinearities $g_1(t,x)$ and $g_2(t,x)$ are determined after the transformation has been applied. Although the methodology of this transformation has been explained in

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Ref. [12] for the cubic nonlinear Schrödinger (CNLS) equation, for the sake of completeness, we show here the method. Moreover, the CQNLS equation presents an extra term with respect to the CNLS equation, which implies extra complications in the treatment of the solutions. Other similarity transformations have been studied for the NLS equations in different contexts.[13−21]

In our calculations, it is convenient to introduce the polar form for the complex modulating function $r = r(t, x)$,

$$r(t,x) = \rho(t,x)\,\mathrm{e}^{\mathrm{i}\,\phi(t,x)}, \tag{5}$$

where ρ and ϕ are real functions (ρ is also non-negative). Substituting Eq. (4) into Eq. (2), and after some simple algebra, we obtain the following system of conditions for g_1, g_2, f, and h:

$$g_1(t,x) = \frac{G_1 X_x}{\gamma\sqrt{\frac{\gamma}{X_x}}}, \tag{6}$$

$$g_2(t,x) = \frac{G_2 X_x^2}{\gamma^2\sqrt{\frac{\gamma}{X_x}}}, \tag{7}$$

$$f(t,x) = \frac{1}{\sqrt{\frac{\gamma}{X_x}}X_x^2}, \tag{8}$$

$$h(t,x) = \frac{\rho_{xx} - \rho^2\phi_t X_x^2 - \rho\phi_x^2 - E\rho X_x^2}{\rho^2 X_x^2}, \tag{9}$$

together with

$$\begin{cases} (\rho^2 X_x)_x = 0, \\ \rho(\rho)_t + f(t,x)(\rho^2\phi)_x = 0, \\ X_t + 2f(t,x)\phi_x X_x = 0, \end{cases} \tag{10}$$

from Eq. (10), and

$$\rho(t,x) = \sqrt{\frac{\gamma(t)}{X_x}}, \tag{11}$$

where $\gamma(t) \neq 0$ is an arbitrary differentiable function of time, and $\gamma X_x > 0$. The above-mentioned solutions can make sure Eq. (2) be integrable or non-integrable. Therefore, based on the similarity transformation (4), we can obtain the solutions of Eq. (2) from those of Eq. (3). Moreover, these free functions $X(t,x)$, $\phi(t,x)$, and $\gamma(t)$ lead Eq. (2) to admit the abundant structures.

3. Solutions for different choices of free functions

It follows from Eqs. (6)–(9) and (11) that there are many types of choices for free functions $X(t,x)$ and $\phi(t,x)$ and integration function $\gamma(t)$, which will make Eq. (2) admit various types of structures. To gain insight into the wave propagations of the solutions, we choose several types of functions $X(t,x)$, $\phi(t,x)$, and $\gamma(t)$ to exhibit the dynamics of the solutions of Eq. (2).

3.1. $X(t,x)$ being a linear function of x

Here we choose $X(t,x)$ and $\phi(t,x)$ to be linear functions of x expressed as

$$\begin{aligned} X(t,x) &= m_1(t)x + m_2(t), \\ \phi(t,x) &= n_1(t)x + n_2(t), \end{aligned} \tag{12}$$

where m_1, m_2, n_1, and n_2 are all functions of time, which are substituted into Eqs. (6)–(9) and (11) to obtain the following functions:

$$\begin{cases} \rho(t) = \sqrt{\frac{\gamma(t)}{m_1(t)}}, \\ f(t,x) = \frac{1}{\sqrt{\gamma(t)m_1(t)^3}}, \\ g_1(t,x) = \frac{G_1\sqrt{m_1(t)^3}}{\sqrt{\gamma(t)^3}}, \\ g_2(t,x) = \frac{G_2\sqrt{m_1(t)^5}}{\sqrt{\gamma(t)^5}}, \\ h(t,x) = -(n_1(t)^2 + E)\sqrt{\frac{m_1(t)}{\gamma(t)}} \\ \qquad\qquad - (n_1'(t)x + n_2'(t)). \end{cases} \tag{13}$$

Since m_1, m_2, n_1, and n_2 are all free parameters, we can further choose them to exhibit the dynamics of the solutions of Eq. (2). Based on the similarity transformation (4), the intensity of the solution of Eq. (2) is expressed as $|\psi(t,x)|^2 = \rho^2|\Psi(X)|^2$, thus we only take some functions related to ρ and X to investigate their intensities.

Case 1 $m_1(t) = t$, $m_2(t) = m_0 t^3$. These expressions are from Ref. [8]. On the other hand, many solutions of Eq. (3) are known. In this paper, we use the bright soliton solution given in Ref. [12]

$$\Psi = \frac{\eta}{\sqrt{\sqrt{1-\beta}\cosh(2\sqrt{-E}X) + 1}}, \tag{14}$$

where

$$\eta = \sqrt{\frac{4E}{G_1}}, \quad \beta = \frac{-16EG_2}{3G_1^2}, \tag{15}$$

with $G_2 > 0$, $E < 0$, and $G_1 < 0$. The restriction on β is that $0 \leq \beta < 1$. The dark soliton is given by the following expression:

$$\Psi = \sqrt{a_1(\mathrm{sech}(\mu X) - 1)}, \tag{16}$$

where

$$\mu^2 = -\frac{4E}{5}, \tag{17}$$

$$G_1 = \frac{8E}{5a_1}, \quad G_2 = \frac{3E}{5a_1^2}, \tag{18}$$

with $a_1 < 0$ being an arbitrary constant.

For the CNLS Eq. (3) with $G_1 = -4$ and $G_2 = 1$, we further take $\gamma(t) = \gamma_0 t^3$, $(\gamma_0 > 0)$, so the bright soliton intensity of Eq. (2) can be written as

$$|\psi_{\mathrm{bs}}(t,x)|^2 = 4t^2\gamma_0 \left| \frac{E}{-4\left(\sqrt{\frac{2}{3}}\cosh(2\sqrt{-E}(tx + m_0 t^3)) + 1\right)} \right|. \tag{19}$$

The corresponding wave propagation is illustrated in Figs. 1(a) and 1(b), in which it is found that the amplitude of the solution becomes larger when space variable z becomes larger.

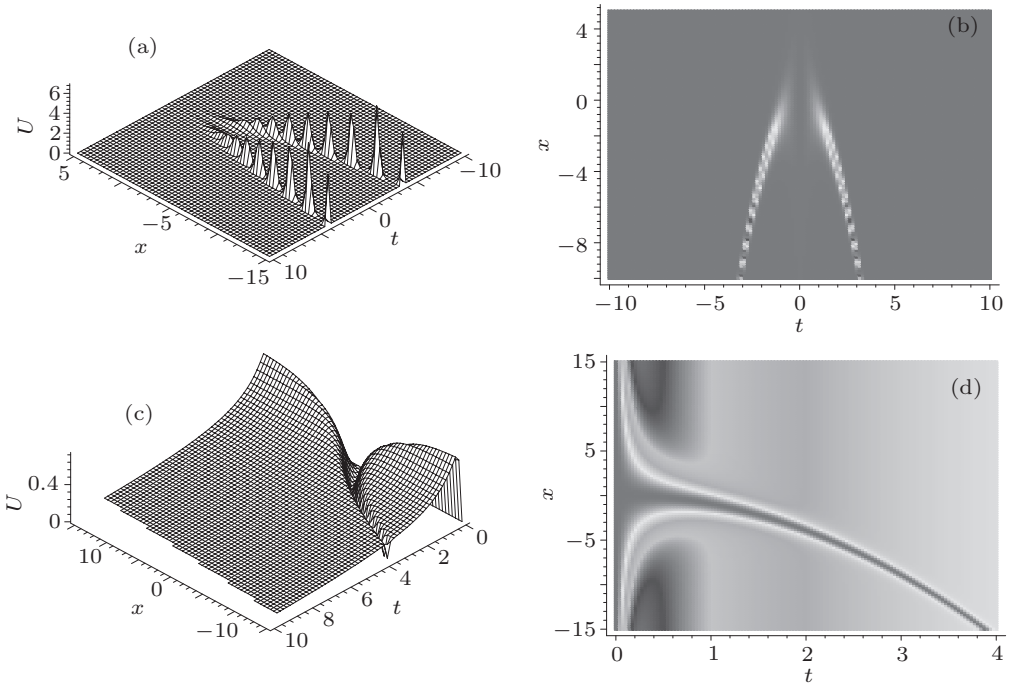


Fig. 1. Intensity $U = |\psi|^2$ distributions of ((a), (b)) the bright soliton given by Eq. (19) with $E = -1.0$, and ((c), (d)) the dark soliton given by Eq. (20) with $E = 1.0$. The other parameters are $m_0 = 0.1$ and $\gamma_0 = 0.5$.

For the periodic wave and the dark soliton solutions, we may take the bounded function $\gamma(t) = \frac{\gamma_0 t}{1+t}$, $(\gamma_0 > 0)$, which makes sure ρ be bounded. The intensity of the dark soliton of Eq. (2) is given by

$$|\psi_{\mathrm{ds}}(t,x)|^2 = \frac{\gamma_0}{1+t} \left| \frac{a_1(-1 + \cosh(\frac{2}{5}\sqrt{5E}(tx + m_0 t^3)))}{\cosh(\frac{2}{5}\sqrt{5E}(tx + m_0 t^3))} \right|, \tag{20}$$

which is illustrated in Figs. 1(c) and 1(d).

From Fig. 1, we can find the same pictures as those for the CNLS equation with varying coefficients, the detail can be found in Ref. [16].

Case 2 $m_1(t) = \cos(t)$, $m_2(t) = m_0 \sin(t)$, $\gamma(t) = \gamma_0 \cos(t)^3$, $(\gamma_0 > 0)$. Figures 2(a)–2(d) display the intensity distributions of the following solutions of Eq. (2). The bright soliton intensity (Figs. 2(a) and 2(b)) is given by

$$|\psi_{\mathrm{bs}}(t,x)|^2 = 4\cos(t)^2\gamma_0 \left| \frac{E}{-4\left(\sqrt{\frac{2}{3}}\cosh(2\sqrt{-E}(\cos(t)x + m_0\sin(t))) + 1\right)} \right|, \tag{21}$$

and the dark soliton intensity (Figs. 2(c) and 2(d)) is given by

$$|\psi_{\rm ds}(t,x)|^2=\gamma_0\cos(t)^2\left|\frac{a_1(-1+\cosh(\frac{2}{5}\sqrt{5E}(\cos(t)x+m_0\sin(t))))}{\cosh(\frac{2}{5}\sqrt{5E}(\cos(t)x+m_0\sin(t)))}\right|. \tag{22}$$

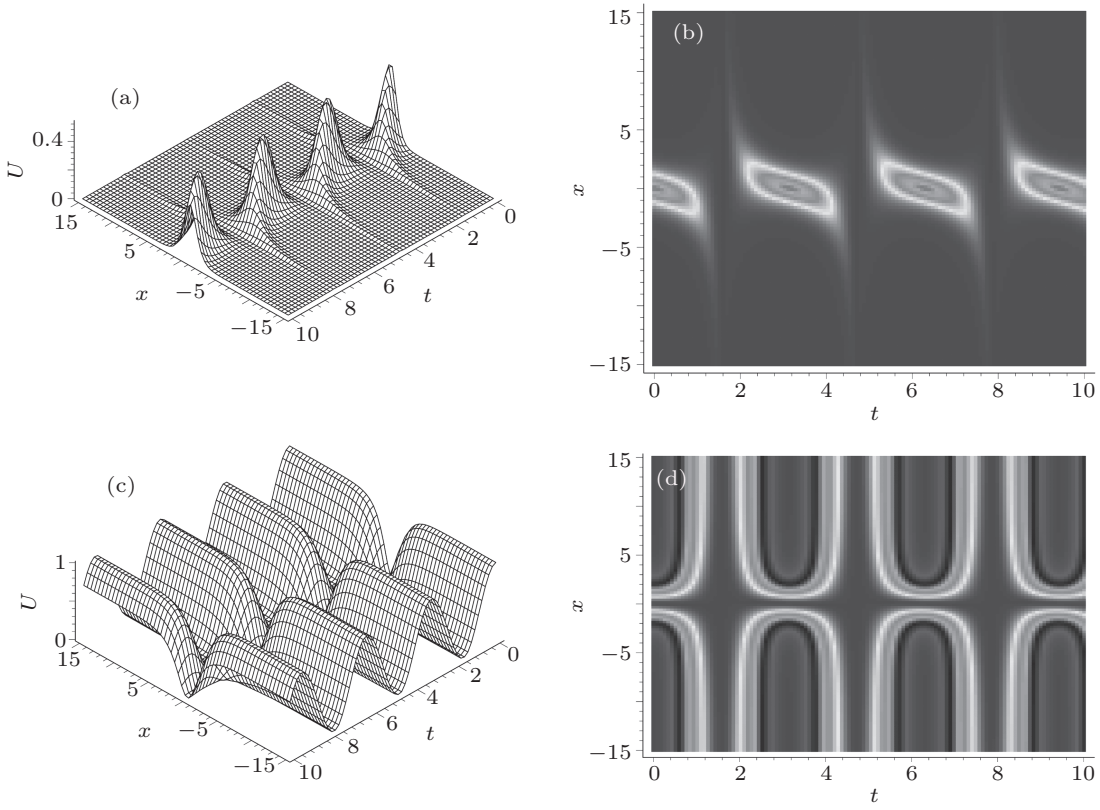


Fig. 2. Intensity $U=|\psi|^2$ distributions of ((a), (b)) the bright soliton given by Eq. (21) with $E=-1.0$, and ((c), (d)) the dark soliton given by Eq. (22) with $E=1.0$. The other parameters are $m_0=0.1$ and $\gamma_0=0.5$.

3.2. $X(t,x)$ being a hyperbolic-sine function of x

Here we choose $X(t,x)$ and $\gamma(t)$ to be hyperbolic-sine functions of x and periodic functions of time. They are given by

$$X(t,x)=\sinh\left[\cos\left(\frac{t}{3}\right)x\right], \tag{23}$$

$$\gamma(t)=\gamma_0\cos(t)^{2n+1},\quad(\gamma_0>0,\ n\geq 0). \tag{24}$$

We obtain

$$\rho(x,t)=\sqrt{\frac{\gamma_0\cos(t)^{2n+1}}{\cosh(\cos(\frac{t}{3})x)\cos(\frac{t}{3})}} \tag{25}$$

from Eq. (11). These functions are substituted into Eqs. (6)–(9) to obtain the other functions for the chosen function ϕ. Thus we can obtain the bright soliton intensity as

$$|\psi_{\rm bs}(t,x)|^2=\frac{\cos(t)^{2n}\gamma_0}{\cosh(\cos(\frac{t}{3})x)\cos(\frac{t}{3})}\left|\frac{-E}{\sqrt{1+\frac{E}{3}}\cosh(2\sqrt{-E}\sinh(\cos(\frac{t}{3})x))+1}\right|, \tag{26}$$

and the dark soliton intensity as

$$|\psi_{\rm ds}(t,x)|^2=\frac{\gamma_0\cos(t)^{2n+1}}{\cosh(\cos(\frac{t}{3})x)\cos(\frac{t}{3})}\left|\frac{a_1(-1+\cosh(\frac{2}{5}\sqrt{5E}\sinh(\cos(\frac{t}{3})x)))}{\cosh(\frac{2}{5}\sqrt{5E}(\sinh(\cos(\frac{t}{3})x)))}\right|. \tag{27}$$

The intensities of these solutions are exhibited in Fig. 3 for different parameters. In particular, n can lead to different wave propagations (see the columns for $n=0$ and $n=0.5$ in Fig. 3).

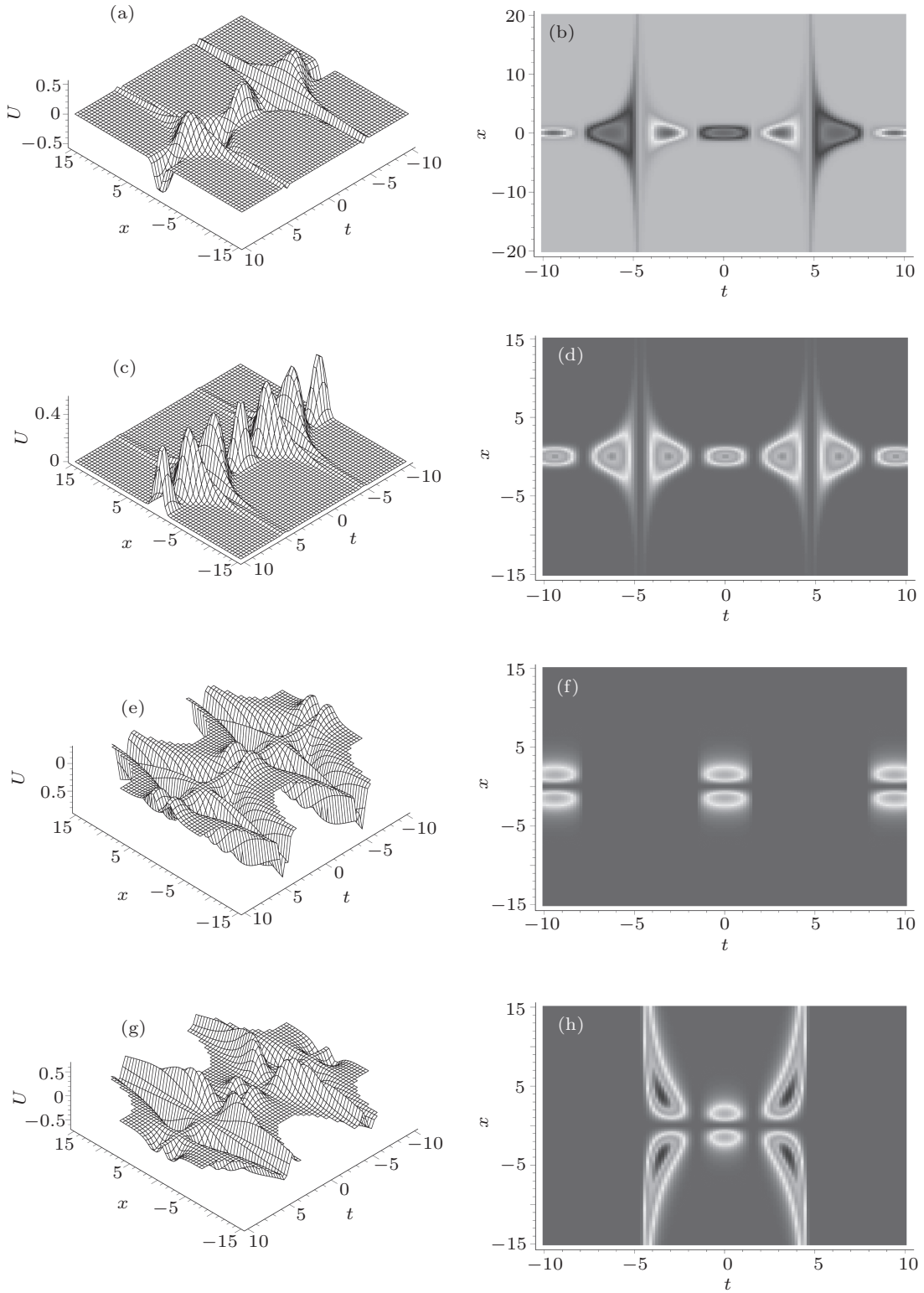


Fig. 3. Intensity $U = |\psi|^2$ distributions of the solutions given by Eq. (25) with $E = -1.0$, ((a), (b)) $n = 0$, ((c), (d)) $n = 0.5$, and the solutions given by Eq. (26) with $E = 1.0$, ((e), (f)) $n = 0$, ((g), (h)) $n = 0.5$. The other parameter is $\gamma_0 = 0.5$.

3.3. $X(t, x)$ being a combination of linear and periodic functions of x

Here we choose $X(t, x)$ and $\gamma(t)$ to be the combination of linear and periodic functions of x and semi-periodic functions of time. They are given by

$$X(t, x) = 2tx + \omega_0 \sin(t^2 x), \tag{28}$$

$$\gamma(t) = \gamma_0 t \cos(t)^2, \quad (\gamma_0 > 0, \quad |\omega_0| < 2). \tag{29}$$

We obtain

$$\rho(x,t)=\sqrt{\frac{\gamma_0\cos(t)^2}{2+\omega_0\cos(t^2x)}} \tag{30}$$

from Eq. (11). These functions are substituted into Eqs. (6)–(9) to obtain the other functions for the chosen function ϕ. Thus we can obtain the bright soliton intensity as

$$|\psi_{\rm bs}(t,x)|^2=\frac{\cos(t)^2\gamma_0}{2+\omega_0\cos(t^2x)t}\left|\frac{-E}{\sqrt{1+\frac{E}{3}}\cosh(2\sqrt{-E}(4tx+2\omega_0\sin(t^2x)))+1}\right|, \tag{31}$$

and the dark soliton intensity as

$$|\psi_{\rm ds}(t,x)|^2=\frac{\gamma_0\cos(t)^2}{2+\omega_0\cos(t^2x)t}\left|\frac{a_1(-1+\cosh(\frac{2}{5}\sqrt{5E}(2tx+\omega_0\sin(t^2x))))}{\cosh(\frac{2}{5}\sqrt{5E}(2tx+\omega_0\sin(t^2x)))}\right|. \tag{32}$$

3.4. $X(t,x)$ being a Jacobi-sn function of x

When we choose $X(t,x)$ and $\gamma(t)$ to be the Jacobi-sn functions of x and the Jacobi-dn functions of time. They are given by

$$X(t,x)=\mathrm{sn}\,(\epsilon(tx+\omega_0t^2),k),\quad \gamma(t)=\gamma_0\,\mathrm{dn}(t,k). \tag{33}$$

We obtain

$$\rho(x,t)=\sqrt{\frac{\gamma_0\,\mathrm{dn}(\epsilon t,k)}{\mathrm{cn}(\epsilon t(\omega_0t+x),k)\,\mathrm{dn}(\epsilon t(\omega_0t+x),k)\epsilon t}} \tag{34}$$

from Eq. (11). These functions are substituted into Eqs. (6)–(9) to obtain the other functions for the chosen function ϕ. Thus we can obtain the bright soliton intensity as

$$|\psi_{\rm bs}(t,x)|^2=\frac{\gamma_0\,\mathrm{dn}(\epsilon t,k)}{\mathrm{cn}(\epsilon t(\omega_0t+x),k)\,\mathrm{dn}(\epsilon t(\omega_0t+x),k)\epsilon t}\left|\frac{-E}{\sqrt{1+\frac{E}{3}}\cosh(2\sqrt{-E}\mathrm{sn}\,(\epsilon(tx+\omega_0t^2),k))+1}\right|, \tag{35}$$

and the dark soliton intensity as

$$|\psi_{\rm ds}(t,x)|^2=\frac{\gamma_0\,\mathrm{dn}(\epsilon t,k)}{\mathrm{cn}(\epsilon t(\omega_0t+x),k)\,\mathrm{dn}(\epsilon t(\omega_0t+x),k)\epsilon t}\left|\frac{a_1(-1+\cosh(\frac{2}{5}\sqrt{5E}\mathrm{sn}\,(\epsilon(tx+\omega_0t^2),k)))}{\cosh(\frac{2}{5}\sqrt{5E}\mathrm{sn}\,(\epsilon(tx+\omega_0t^2),k)}\right|. \tag{36}$$

4. Conclusion

In summary, we have obtained many new types of exact solutions for the generalized nonlinear Schrödinger equation with varying coefficients by means of the combination method of the similarity transformation and the tri-function method. We obtain the bright-like and the dark-like soliton solutions. These obtained solutions contain two free functions of space x and time t and one integration function of time, which provide us with more choices of functions to generate the abundant wave structures of the generalized nonlinear Schödinger equation with varying coefficients. Moreover, we choose four types of free functions to exhibit the wave propagations related to the obtained solutions. These solutions may provide more information to further study the nonlinear physical system.

References

[1] Kivshar Y S and Agrawal G 2003 *Optical Solitons: from Fibers to Photonic Crystals* (New York: Academic Press)

[2] Sulem C and Sulem P 1999 *The Nonlinear Schrödinger Equation*: *Self-focusing and Wave Collapse* (Berlin: Springer-Verlag)

[3] Malomed B A, Mihalache D, Wise F and Torner L 2005 *J. Opt.* B: *Quantum Semiclass. Opt.* **7** 53

[4] Dalfovo F, Giorgini S, Pitaevskii L P and Stringari S 1999 *Rev. Mod. Phys.* **71** 463

[5] Davydov A S 1985 *Solitons in Molecular Systems* (Reidel: Dordrecht)

[6] Vázquez L, Streit L and Pérez-Garcia V M 1997 *Nonlinear Klein-Gordon and Schrödinger Systems: Theory and Applications* (Singapore: World Scientific)

[7] Yan Z Y 2010 *Phys. Lett.* A **374** 4838

[8] Belmonte-Beitia J and Cuevas J 2009 *J. Phys.* A **42** 165201

[9] Ma S H and Zhu J M 2006 *Acta Sinica Quantum Optica* **12** 156

[10] Ma S H and Fang J P 2007 *Acta Optica Sinica* **27** 1090

[11] Ma S H and Fang J P 2006 *Acta Phys. Sin.* **54** 5611 (in Chinese)

[12] Belmonte-Beitia J, Pérez-García V M, Vekslerchik V and Konotop V V 2008 *Phys. Rev. Lett.* **100** 164102

[13] Pérez-García V M, Torres P J and Konotop V V 2006 *Physica* D **221** 31

[14] Serkin V N and Hasegawa A 2000 *Phys. Rev. Lett.* **85** 4502

[15] Yan Z Y 2010 *Phys. Lett.* A **374** 672

[16] Yan Z Y 2010 *Phys. Lett.* A **374** 4838

[17] Yan Z Y and Konotop V V 2009 *Phys. Rev.* E **80** 036607

[18] Mokhtari R, Samadi Toodar A and Chegini N G 2011 *Chin. Phys. Lett.* **28** 020202

[19] Akhmediev N, Ankiewicz A and Taki M 2009 *Phys. Lett.* A **373** 675

[20] Yan Z Y 2009 *Phys. Lett.* A **373** 2432

[21] Akhmediev N, Ankiewicz A and Soto-Crespo J M 2009 *Phys. Rev.* E **80** 026601