Synchronization of impulsively coupled complex networks*

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We investigate the synchronization of complex networks, which are impulsively coupled only at discrete instants. Based on the comparison theory of impulsive differential systems, a distributed impulsive control scheme is proposed for complex dynamical networks to achieve synchronization. The proposed scheme not only takes into account the influence of all nodes to network synchronization, which depends on the weight of each node in the network, but also provides us with a flexible method to select the synchronized state of the network. In addition, it is unnecessary for the impulsive coupling matrix to be symmetrical. Finally, the proposed control scheme is applied to a chaotic Lorenz network and Chua's circuit network. Numerical simulations are used to illustrate the validity of this control scheme.

Keywords: complex network, impulsive control scheme, synchronization

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1. Introduction

Complex networks are representative of the intricate connections between elements in diverse systems such as the Internet, metabolic networks, neural networks, food webs, communication networks, transport networks, and social networks.[1,2] One of the interesting and important subjects that has received much attention is the study of individual systems behaving in unison. Such collective behaviour of networks, called collective synchronization, has become a potential candidate for engineering applications.[3,4]

How to design a coupling scheme to generate collective synchronization, and how to determine the stability of synchronous motion with regard to coupling strength are questions that have been intensively investigated in chaotic oscillator networks.[5−8]

There are many control methods used to achieve network synchronization, such as linear state feedback control,[9,10] pinning control,[11] adaptive control,[12−16] and impulsive control.[17−29] Recently, the concept of control topology[24] was introduced to describe the control of information flow transmitted between nodes and it is used to investigate the problem of distributed impulsive synchronization of complex dynamical networks (CDNs) with system delay and multiple coupling delay. This method, which combines continuous coupling control and impulsive control, can be easily carried out in practice.

Based on the idea of impulsive control proposed in Refs. [17–29], in this paper, we give an analytical systematic method to explore the collective synchronization of a whole complex network with only discontinuous couplings. It is proved rigorously via the comparison theory of impulsive differential systems that the whole complex network can reach an agreement by using a simple distributed impulsive control scheme. Due to the utilization of information from neighboring nodes at discrete instants through impulsive connections, collective synchronization can be achieved at a low control cost. In addition, the proposed distributed impulsive control scheme provides with us a flexible method to select the synchronized state of the network, as compared with the scheme in Ref. [26]. Controller implementation difficulty can also be avoided. Moreover, the impulsive coupling matrix does not have to be symmetrical.

This paper is organized as follows: the model and preliminaries are given in Section 2; the sufficient condition of synchronization of an impulsively coupled complex network is obtained in Section 3; numerical

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simulations for verifying theoretical results are presented in Section 4; the conclusion can be found in Section 5.

2. Model and preliminaries

We begin by considering a complex network consisting of N identical nodes in the form of

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}(\boldsymbol{x}_i(t)) + \boldsymbol{u}_i(t), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $\boldsymbol{x}_i(t) = (x_{i1}, x_{i2}, \ldots, x_{in})^{\mathrm{T}}$ is the state vector of the ith node, $\boldsymbol{f}(\cdot) \in R^n$ is a continuous map satisfying

$$\begin{aligned} H_1 \ : \ & (\boldsymbol{x}_i - \boldsymbol{x}_j)^{\mathrm{T}}(\boldsymbol{f}(\boldsymbol{x}_i) - \boldsymbol{f}(\boldsymbol{x}_j)) \\ & \leq \alpha(\boldsymbol{x}_i - \boldsymbol{x}_j)^{\mathrm{T}}(\boldsymbol{x}_i - \boldsymbol{x}_j), \end{aligned}$$

where $\boldsymbol{x}_i,\ \boldsymbol{y}_j \in R^n$, and $\boldsymbol{u}_i(t)$ $(i = 1, 2, \ldots, N)$ is a distributed impulsive control input which utilizes the information from neighboring nodes, i.e.,

$$\boldsymbol{u}_i(t) = \sum_{k=1}^{\infty} \boldsymbol{B}_k[\boldsymbol{x}_i(t_k) - \sum_{j=1}^{N} \theta_{ij}\boldsymbol{x}_j(t_k)]\delta(t - t_k), \tag{2}$$

or equivalently

$$\boldsymbol{u}_i(t) = \sum_{k=1}^{\infty} \boldsymbol{B}_k \sum_{j=1}^{N} \phi_{ij}\boldsymbol{x}_j(t_k)\delta(t - t_k),$$

where

$$\boldsymbol{\phi} = (\phi_{ij})_{N\times N} = \begin{bmatrix} 1-\theta_{11} & -\theta_{12} & \ldots & -\theta_{1N} \\ -\theta_{21} & 1-\theta_{22} & \ldots & -\theta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -\theta_{N1} & -\theta_{N2} & \ldots & 1-\theta_{NN} \end{bmatrix}.$$

The discrete set $\{t_k\}$ satisfies

$$0 \leq t_0 < t_1 < \cdots < t_k \to \infty \ \ (t \to \infty).$$

In addition, $\boldsymbol{B}_k$ is the control gain, $\delta(\cdot)$ is the Dirac impulsive function, $\theta_{ij} \geq 0$ denotes the influence weight of the state of the j-th node on that of the ith node, and $\sum_{j=1}^{N} \theta_{ij} = 1$. θ_{ij} is designed as follows: if the j-th node can obtain the information of the ith node, $\theta_{ij} > 0$; otherwise, $\theta_{ij} = 0$. It must be pointed out that the impulsive coupling matrix $\boldsymbol{\Theta} = (\theta_{ij})$ does not need to be symmetrical.

The network (1) with distributed impulsive control (2) can then be described as

$$\begin{aligned} \dot{\boldsymbol{x}}_i(t) \ &= \ \boldsymbol{f}(\boldsymbol{x}_i(t)), \quad i = 1, 2, \ldots, N, \quad t \neq t_k, \\ \Delta\boldsymbol{x}_i \ &= \ \boldsymbol{B}_k \sum_{j=1}^{N} \phi_{ij}\boldsymbol{x}_j(t_k), \quad t = t_k, \end{aligned} \tag{3}$$

where

$$\boldsymbol{x}_i(t_k^+) = \lim_{t\to t_k^+} \boldsymbol{x}_i(t)$$

and

$$\boldsymbol{x}_i(t_k^-) = \lim_{t\to t_k^-} \boldsymbol{x}_i(t).$$

Moreover, any solution of Eq. (3) is left continuous at each $t = t_k$, i.e., $\boldsymbol{x}_i(t_k^-) = \boldsymbol{x}_i(t_k)$.

The impulsively coupled network (3) is considered to be globally synchronized if

$$\lim_{t\to\infty} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| = 0, \quad i,\ j = 1, 2, \ldots, N,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\boldsymbol{R}^n$.

Rewriting the network (3) with the Kronecker product

$$\begin{aligned} \dot{\boldsymbol{x}}(t) \ &= \ \boldsymbol{F}(\boldsymbol{x}(t)), \quad t \neq t_k, \\ \Delta\boldsymbol{x} \ &= \ \boldsymbol{B}_k^N \boldsymbol{\Phi}\boldsymbol{x}(t_k), \end{aligned} \tag{4}$$

where

$$\boldsymbol{x}(t) = (\boldsymbol{x}_1^{\mathrm{T}}, \boldsymbol{x}_2^{\mathrm{T}}, \ldots, \boldsymbol{x}_N^{\mathrm{T}})^{\mathrm{T}},$$

$\boldsymbol{B}_k^N = I_N \otimes \boldsymbol{B}_k$, $\boldsymbol{\Phi} = \boldsymbol{\phi} \otimes \boldsymbol{I}_n$, and $\boldsymbol{F}(\boldsymbol{x}(t)) = (\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}_1(t)), \ldots, \boldsymbol{f}^{\mathrm{T}}(\boldsymbol{x}_N(t)))^{\mathrm{T}}$.

In the following, we introduce a necessary lemma. Supposing that $\boldsymbol{T}(\epsilon)$ denotes the matrix set such that the sum of elements in each row is equal to the real number ϵ. The set $\boldsymbol{M}_1$ is defined as follows: if $\boldsymbol{M} = (M_{ij})_{(N-1)\times N} \in \boldsymbol{M}_1$, each row of $\boldsymbol{M}$ contains exactly one element 1 and one element -1, and all other elements are zero. $j_{i1}(j_{i2})$ denotes the column index of the first (second) nonzero element in the i-th row. The set H is defined as $H = \{\{j_{11}, j_{12}\}, \{j_{21}, j_{22}\}, \ldots, \{j_{P1}, j_{P2}\}\}$. The set $\boldsymbol{M}_2$ is defined as follows: $\boldsymbol{M}_2 \subset \boldsymbol{M}_1$; if $\boldsymbol{M} = (M_{ij})_{(N-1)\times N} \in \boldsymbol{M}_2$, for any pair of column indexes j_s and j_t, there exists indexes $j_1,\ j_2, \ldots,\ j_l$, with $j_1 = j_s$ and $j_t = j_l$ so that $j_m,\ j_{m+1} \in H$ for $m = 1,\ 2, \ldots,\ l - 1$.

Lemma 1[9] Supposing $\boldsymbol{M} = (M_{ij}) \in \boldsymbol{M}_2$ is an $(N-1) \times N$ matrix and $\boldsymbol{A} \in \boldsymbol{T}(\epsilon)$ is an $N \times N$ matrix, there exists an $N \times (N-1)$ matrix $\boldsymbol{G}_{\boldsymbol{M}}$ such that $\boldsymbol{M}\boldsymbol{A} = \hat{\boldsymbol{A}}\boldsymbol{M}$, where $\hat{\boldsymbol{A}} = \boldsymbol{M}\boldsymbol{A}\boldsymbol{G}_{\boldsymbol{M}}$, $\boldsymbol{M}\boldsymbol{G}_{\boldsymbol{M}} = \boldsymbol{I}_{N-1}$. Moreover, let $\boldsymbol{\Gamma}$ be an $n \times n$ constant matrix and $\boldsymbol{A}_{\boldsymbol{\Gamma}} = \boldsymbol{A} \otimes \boldsymbol{\Gamma}$, then $\boldsymbol{M}_{\boldsymbol{I}}\boldsymbol{A}_{\boldsymbol{\Gamma}} = \hat{\boldsymbol{A}}_{\boldsymbol{\Gamma}}\boldsymbol{M}$, where $\hat{\boldsymbol{A}}_{\boldsymbol{\Gamma}} = \hat{\boldsymbol{A}} \otimes \boldsymbol{\Gamma}$ and $\boldsymbol{M}_{\boldsymbol{I}} = \boldsymbol{M} \otimes \boldsymbol{I}_n$. The function, which maps $\boldsymbol{A}$ to $\hat{\boldsymbol{A}}$ by $\boldsymbol{S}_{\boldsymbol{M}}$, is defined as

$$\hat{\boldsymbol{A}} = \boldsymbol{S}_{\boldsymbol{M}}(\boldsymbol{A}) = \boldsymbol{M}\boldsymbol{A}\boldsymbol{G}_{\boldsymbol{M}}.$$

3. Impulsive synchronization with nonlinear coupling

For $\boldsymbol{M} \in \boldsymbol{M}_2$, $\boldsymbol{M_I} = \boldsymbol{M} \otimes \boldsymbol{I}_n$, then

$$\boldsymbol{y}(t) = \boldsymbol{M_I}\boldsymbol{x}(t) = (\boldsymbol{y}_1^{\mathrm{T}}(t), \boldsymbol{y}_2^{\mathrm{T}}(t), \ldots, \boldsymbol{y}_{N-1}^{\mathrm{T}}(t))^{\mathrm{T}},$$
$$\boldsymbol{y}_i(t) = (y_{i1}, y_{i2}, \ldots, y_{in})^{\mathrm{T}}, \quad i = 1, \ldots, N-1.$$

Given that j_{i1} (j_{i2}) denotes the column index of the first (second) nonzero element in the i-th row, it is obvious that $\boldsymbol{y}_i(t) = \boldsymbol{x}_{j_{i1}}(t) - \boldsymbol{x}_{j_{i2}}(t)$.

In view of the assumption that $\boldsymbol{M} \in \boldsymbol{M}_2$, the crucial property of $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M_I^{\mathrm{T}}}\boldsymbol{M_I}\boldsymbol{x}$ is that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M_I^{\mathrm{T}}}\boldsymbol{M_I}\boldsymbol{x} \to 0$ if and only if $\|\boldsymbol{x}_i - \boldsymbol{x}_j\| \to 0$, for all $i, j = 1, 2, \ldots, N$.

Constructing the Lyapunov function in the following form:

$$V(t) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}_{\boldsymbol{I}}^{\mathrm{T}}\boldsymbol{M_I}\boldsymbol{x} = \sum_{i=1}^{N-1} \boldsymbol{y}_i^{\mathrm{T}}(t)\boldsymbol{y}_i(t).$$

Then, for any $t \in (t_{k-1}, t_k)$, taking the derivative of the trajectories of Eq. (4), we have

$$\begin{aligned}
\dot{V}(t) &= 2\boldsymbol{y}^{\mathrm{T}}\boldsymbol{M_I^{\mathrm{T}}}\boldsymbol{M_I}\boldsymbol{F}(\boldsymbol{x}) \\
&= 2\sum_{i=1}^{N-1} \boldsymbol{y}_i^{\mathrm{T}}(t)[\boldsymbol{f}(\boldsymbol{x}_{j_{i1}}(t)) - \boldsymbol{f}(\boldsymbol{x}_{j_{i2}}(t))] \\
&= 2\sum_{i=1}^{N-1} (\boldsymbol{x}_{j_{i1}}(t) - \boldsymbol{x}_{j_{i2}}(t))^{\mathrm{T}} \\
&\quad \times [\boldsymbol{f}(\boldsymbol{x}_{j_{i1}}(t)) - \boldsymbol{f}(\boldsymbol{x}_{j_{i2}}(t))] \\
&\leq 2\alpha \sum_{i=1}^{N-1} (\boldsymbol{x}_{j_{i1}}(t) - \boldsymbol{x}_{j_{i2}}(t))^{\mathrm{T}}(\boldsymbol{x}_{j_{i1}}(t) - \boldsymbol{x}_{j_{i2}}(t)) \\
&\leq 2\alpha \sum_{i=1}^{N-1} \boldsymbol{y}_i^{\mathrm{T}}(t)\boldsymbol{y}_i(t) \\
&= 2\alpha V((t)). \qquad (5)
\end{aligned}$$

In addition, if $t = t_k$, we have

$$\begin{aligned}
V(t_k^+) &= \boldsymbol{x}^{\mathrm{T}}(t_k)(\boldsymbol{I}_{Nn} + \boldsymbol{B}_k^N\Phi)^{\mathrm{T}}\boldsymbol{M_I^{\mathrm{T}}}\boldsymbol{M_I} \\
&\quad \times (\boldsymbol{I}_{Nn} + \boldsymbol{B}_k^N\boldsymbol{\Phi})\boldsymbol{x}(t_k) \\
&= \boldsymbol{x}^{\mathrm{T}}(t_k)\boldsymbol{M}^{\mathrm{T}}(\boldsymbol{I}_{(N-1)n} \\
&\quad + (\boldsymbol{I}_{N-1} \otimes \boldsymbol{B}_k)(\hat{\boldsymbol{\phi}} \otimes \boldsymbol{I}_n))^{\mathrm{T}} \\
&\quad \times (\boldsymbol{I}_{(N-1)n} + (\boldsymbol{I}_{N-1} \otimes \boldsymbol{B}_k)(\hat{\boldsymbol{\phi}} \otimes \boldsymbol{I}_n))\boldsymbol{M}\boldsymbol{x}(t_k) \\
&\leq \sum_{i=1}^{N-1} \boldsymbol{y}_i^{\mathrm{T}}(t_k)(\boldsymbol{I}_{(N-1)n} \\
&\quad + (\boldsymbol{I}_{N-1} \otimes \boldsymbol{B}_k)(\hat{\boldsymbol{\phi}} \otimes \boldsymbol{I}_n))^{\mathrm{T}} \\
&\quad \times (\boldsymbol{I}_{(N-1)n} + (\boldsymbol{I}_{N-1} \otimes \boldsymbol{B}_k)(\hat{\boldsymbol{\phi}} \otimes \boldsymbol{I}_n))\boldsymbol{y}_i(t_k) \\
&\leq \lambda_{\max}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}) \sum_{i=1}^{N-1} \boldsymbol{y}_i^{\mathrm{T}}(t_k)\boldsymbol{y}_i(t_k) \\
&\leq \beta_k V(t_k), \qquad (6)
\end{aligned}$$

where $\boldsymbol{c} = (\boldsymbol{I}_{(N-1)n} + (\boldsymbol{I}_{N-1} \otimes \boldsymbol{B}_k)(\hat{\boldsymbol{\phi}} \otimes \boldsymbol{I}_n))$, $\beta_k = \lambda_{\max}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C})$. When $k = 1$ in Eq. (5), we have

$$V(t) \leq V(t_0^+)\,\mathrm{e}^{2\alpha(t-t_0)}$$

for all $t \in (t_0, t_1]$, which leads to $V(t_1) \leq V(t_0^+)\,\mathrm{e}^{2\alpha(t_1-t_0)}$. Similarly, based on Eq. (6), we have

$$V(t_1^+) \leq \beta_1 V(t_1) \leq \beta_1 V(t_0^+)\,\mathrm{e}^{2\alpha(t_1-t_0)},$$

and therefore we have, in general,

$$V(t) \leq V(t_0^+)\beta_1\beta_2 \ldots \beta_{k-1}\,\mathrm{e}^{2\alpha(t-t_0)}$$

for $t \in (t_{k-1}, t_k]$, $(k = 1, 2, \ldots)$. If there exists a constant $\xi > 1$ such that $\ln(\xi\beta_k) + 2\alpha\Delta_k \leq 0$ for any k and $t \in (t_{k-1}, t_k]$, then we have

$$\begin{aligned}
V(t) &\leq \beta_1\beta_2 \ldots \beta_{k-1}V(t_0^+)\,\mathrm{e}^{2\alpha(t_k-t_0)} \\
&\leq V(t_0^+)[\beta_1\,\mathrm{e}^{2\alpha(t_1-t_0)}] \ldots [\beta_{k-1}\,\mathrm{e}^{2\alpha(t_{k-1}-t_{k-2})}] \\
&\quad \times\ \mathrm{e}^{2\alpha(t_k-t_{k-1})} \\
&\leq \frac{1}{\xi^{k-1}} V(e(t_0^+))\,\mathrm{e}^{2\alpha(t_k-t_{k-1})}.
\end{aligned}$$

Thus, based on the Lyapunov function and comparison principle, the synchronization of the impulsively coupled network (3) is achieved. In practice, the gain matrix $\boldsymbol{B}_k$ and impulsive distance Δ_k are usually chosen to be constants for convenience, i.e., $\Delta_k = \Delta$, $\boldsymbol{B}_k = \boldsymbol{B'}$ $(k = 1, 2, \ldots)$. Then, the impulsively coupled networks (3) is globally and asymptotically synchronized if there exists a constant $\xi > 1$ such that

$$\ln(\xi\beta) + 2\alpha\Delta \leq 0.$$

4. Numerical results

In this section, numerical simulations are presented to verify the theoretical results obtained in previous sections. We take two networks composed of 10 nodes as examples. The first example is the Lorenz network, in which each node is described by

$$\begin{aligned}
\dot{x}_{i1} &= a(x_{i2} - x_{i1}), \\
\dot{x}_{i2} &= cx_{i1} - x_{i1}x_{i3} - x_{i2}, \\
\dot{x}_{i3} &= x_{i1}x_{i2} - bx_{i3}, \qquad (7)
\end{aligned}$$

where $a = 10$, $b = 8/3$, and $c = 28$.

The total synchronization error is expressed as follows:

$$E_{\mathrm{s}}(t) = \sum_{i=1}^{N} \sqrt{\sum_{k=1}^{3}(x_{ik}(t) - \sum_{j=1}^{N} \theta_{ij}x_{jk}(t))^2}.$$

Figure 1 shows the evolution process of the total error and the states of the Lorenz network with $\boldsymbol{u}_i(t) = 0$. It is easy to see that the network is not synchronized without an impulsive control. The impulses are equidistant and separated by an interval of $\Delta = 0.02$, and the control gain $\boldsymbol{B'} = \mathrm{diag}\{-0.6, -0.6, -0.6\}$. The initial values of these systems are chosen randomly in a real number interval $[-20, 25]$, respectively. For simplicity, we assume that all controllers share the same state information from common nodes, i.e., $\theta_{ij} = \theta_j$.

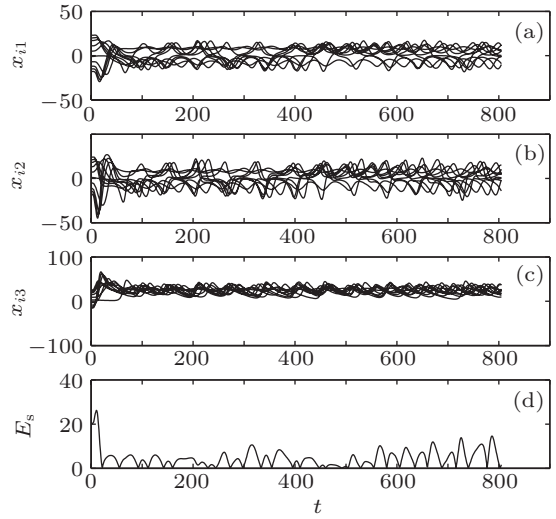


Fig. 1. The evolution of states x_{i1}, x_{i2}, x_{i3}, and the total error E_s of the Lorenz network without impulsive couplings.

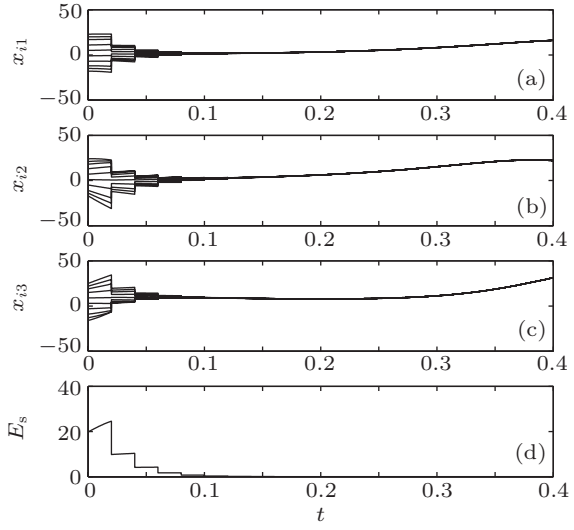


Fig. 2. The evolution of states x_{i1}, x_{i2}, x_{i3}, and the total error E_s of the impulsively coupled Lorenz network with $\Delta = 0.02$, $t_0 = 0$, and $\theta_i = 0.1$ ($i = 1,\ 2, \ldots,\ 10$).

Let $\theta_j = 0.1$ ($j = 1,\ 2,\ \ldots,\ 10$), one can see from Fig. 2 that all the states tend to synchronize asymptotically as time evolves, which implies that the synchronization of the impulsively coupled Lorenz network is achieved.

Figure 3 shows the evolution of states x_{i1}, x_{i2}, x_{i3}, and E_s of the continuous linearly coupled Lorenz network

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}_1(\boldsymbol{x}_i(t)) + c\sum_{j=1}^{N}\theta_{ij}\boldsymbol{x}_j(t),$$
$$i = 1, 2, \ldots, 10, \tag{8}$$

where $\boldsymbol{f}_1(\boldsymbol{x}_i(t))$ is the dynamic Lorenz system, $c = 100$, and $\theta_{ij} = \theta_j = 0.1$ ($j = 1,\ 2,\ \ldots,\ 10$).

It can be seen from Figs. 2 and 3 that although the impulsively coupled Lorenz network has a low control cost, its synchronization performance is superior to that of the continuous linearly coupled Lorenz network at given initial values and coupled matrix $\boldsymbol{\phi}$.

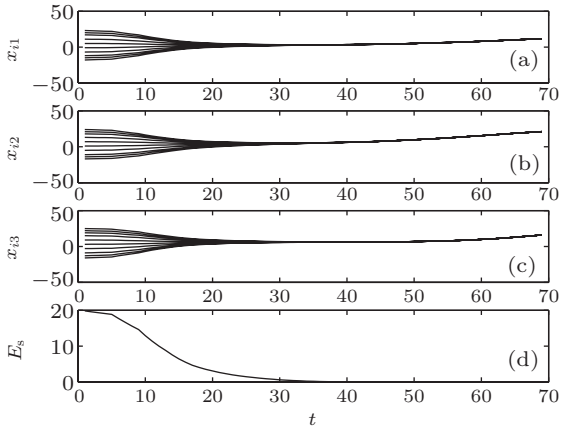


Fig. 3. The evolution of states x_{i1}, x_{i2}, x_{i3}, and the total error E_s of the continuous linearly coupled Lorenz network with $c = 100$, where initial values and coupled matrix $\boldsymbol{\phi}$ are the same as in Fig. 2.

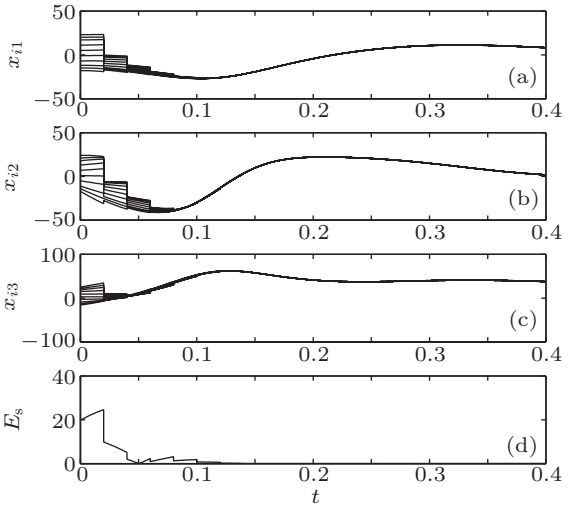


Fig. 4. The evolution of states x_{i1}, x_{i2}, x_{i3}, and the total error E_s of the impulsively coupled Lorenz network with $\Delta = 0.02$, $t_0 = 0$, $\theta_1 = \theta_2 = \theta_3 = 1/3$, and $\theta_i = 0$ ($i = 4,\ 5, \ldots,\ 10$).

When $\theta_1 = \theta_2 = \theta_3 = 1/3$, and $\theta_i = 0$ ($i = 4, 5, \ldots, 10$), the synchronization of the impulsively coupled Lorenz network can also be achieved, which is shown in Fig. 4.

Let $\theta_1 = 1$, $\theta_j = 0$ ($j = 2, 3, \ldots, 10$), one can see from Fig. 5 that all the states converge to the first node asymptotically as time evolves, which implies that the synchronization of the impulsively coupled Lorenz network is achieved.

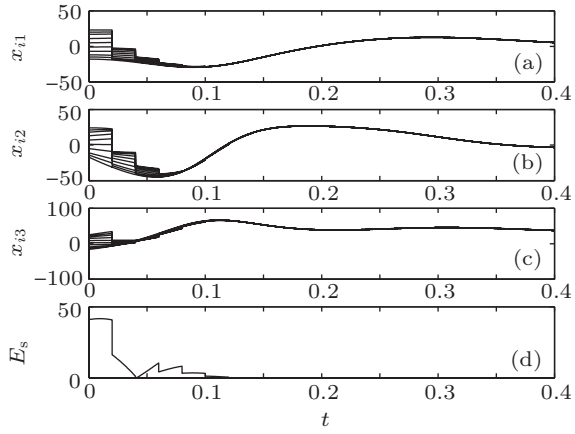


Fig. 5. The evolution of states x_{i1}, x_{i2}, x_{i3}, and the total error E_s of the impulsively coupled Lorenz network with $\Delta = 0.02$, $t_0 = 0$, $\theta_1 = 1$, and $\theta_i = 0$ ($i = 2, 3, \ldots, 10$).

Let $\theta_j = 0.1$ ($j = 1, 2, \ldots, 10$), and $\Delta = 2.2$, Fig. 6 shows the evolution process of the total error and states of the impulsively coupled Lorenz network. It can be seen that the synchronization of the impulsively coupled Lorenz network cannot be achieved if the impulse interval is greater than a critical value.

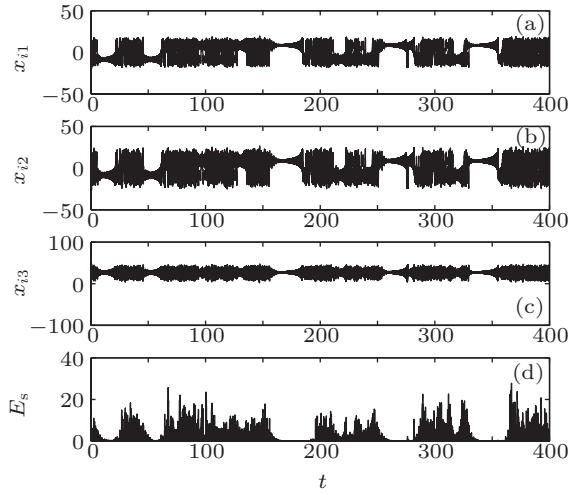


Fig. 6. The evolution of states x_{i1}, x_{i2}, x_{i3}, and the total error E_s of the impulsively coupled Lorenz network with $\Delta = 2.0$, $t_0 = 0$, and $\theta_i = 0.1$ ($i = 1, 2, \ldots, 10$).

Chua's circuit is taken as a second example of a network node system, which is described by

$$\begin{aligned} \dot{x}_{i1} &= m(x_{i2} - x_{i1} - g(x_{i1})), \\ \dot{x}_{i2} &= x_{i1} - x_{i2} + x_{i3}, \\ \dot{x}_{i3} &= -l x_{i2}, \end{aligned} \tag{9}$$

where $m = 9$, $l = 100/7$, $g(x) = 2x/7 - 3(|x+1| - |x-1|)/14$.

Figure 7 shows the evolution process of the total error and states of a Chua circuit network with $\boldsymbol{u}_i(t) = 0$. It is clearly seen that Chua's circuit network is not synchronized without impulsive controls.

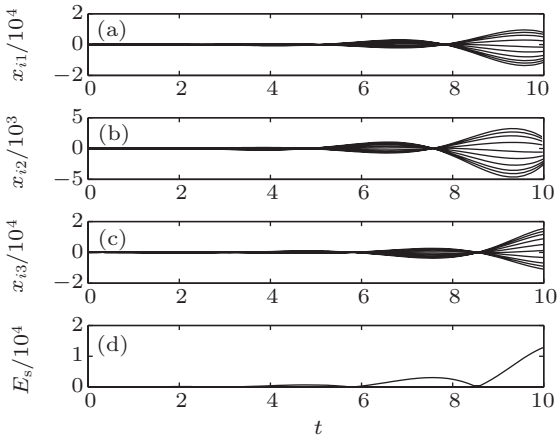


Fig. 7. The evolution of states x_{i1}, x_{i2}, x_{i3}, and the total error E_s of a Chua circuit network without impulsively couplings.

Values for the impulse Δ, the control gain $\boldsymbol{B'}$, and the initial values of these systems are taken as the same as in the first example. In Fig. 8, let $\theta_j = 0.1$ ($j = 1, 2, \ldots, 10$), one can see that all the states tend to synchronizize asymptotically as time evolves, which implies that the synchronization of the impulsively coupled Chua circuit network is achieved.

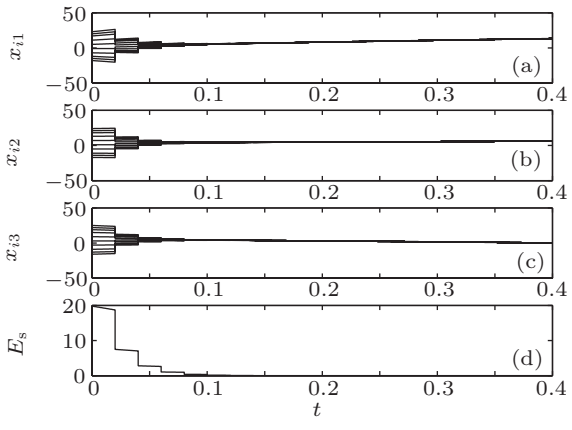


Fig. 8. The evolution of states x_{i1}, x_{i2}, x_{i3}, and the total error E_s of a Chua circuit network with $\Delta = 0.02$, $t_0 = 0$, and $\theta_i = 0.1$ ($i = 1, 2, \ldots, 10$).

Figure 9 shows the evolution of states x_{i1}, x_{i2}, x_{i3} and the total error $E_{\rm s}$ of the continuous linearly coupled Chua circuit network

$$\dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}_2(\boldsymbol{x}_i(t)) + c\sum_{j=1}^{N}\phi_{ij}\boldsymbol{x}_j(t),$$
$$i = 1,\ 2,\ \ldots,\ 10, \tag{10}$$

where $\dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}_2(\boldsymbol{x}_i(t))$ is the dynamic Chua circuit, $c = 100$, and $\theta_{ij} = \theta_j = 0.1$ ($j = 1,\ 2,\ \ldots,\ 10$). One can see that the synchronization performance of the impulsively coupled Chua circuit network is better than that of the continuous linearly coupled Chua circuit network at the given initial values and coupled matrix $\boldsymbol{\phi}$.

When $\theta_1 = \theta_2 = \theta_3 = 1/3$, and $\theta_i = 0$ ($i = 4,\ 5,\ \ldots,\ 10$), the synchronization of the impulsively coupled Chua circuit network is also achieved, as shown in Fig. 10.

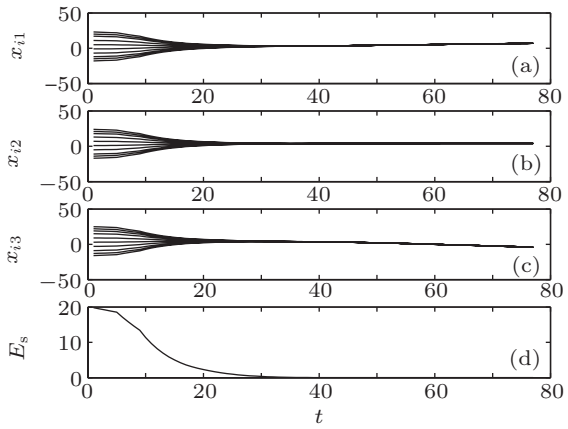


Fig. 9. The evolution of states x_{i1}, x_{i2}, x_{i3}, and the total error $E_{\rm s}$ of the continuous linearly coupled Chua circuit network with $c = 100$, where initial values and coupled matrix $\boldsymbol{\phi}$ are the same as in Fig. 8.

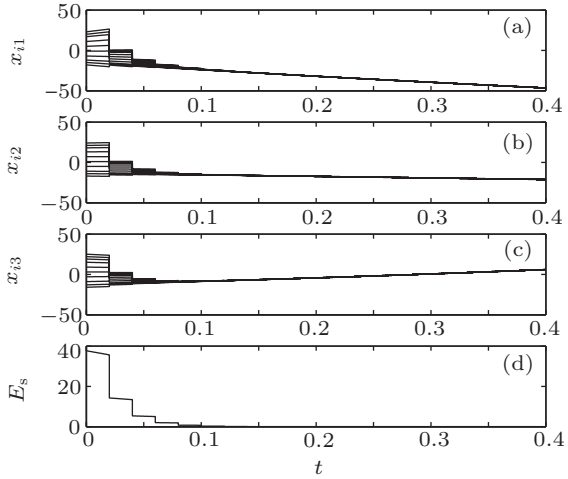


Fig. 10. The evolution of states x_{i1}, x_{i2}, x_{i3}, and the total error $E_{\rm s}$ of the impulsively coupled Chua circuit network with $\Delta = 0.02$, $t_0 = 0$, $\theta_1 = \theta_2 = \theta_3 = 1/3$, and $\theta_i = 0$ ($i = 4,\ 5,\ \ldots,\ 10$).

Let $\theta_1 = 1$, $\theta_j = 0$ ($j = 2,\ 3,\ \ldots,\ 10$) such that the first node's state information is shared by all other nodes, one can see from Fig. 11 that all the states converge to the first node asymptotically as time evolves, which implies that synchronization of the impulsively coupled Chua circuit network is achieved.

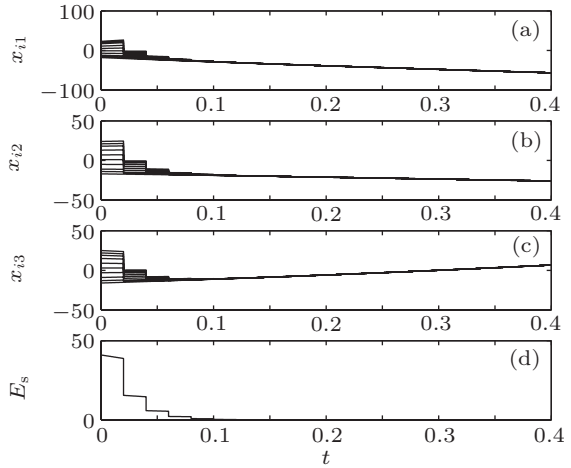


Fig. 11. The evolution of states x_{i1}, x_{i2}, x_{i3}, and the total error $E_{\rm s}$ of the impulsively coupled Chua circuit network with $\Delta = 0.02$, $t_0 = 0$, $\theta_1 = 1$, and $\theta_i = 0$ ($i = 2,\ 3,\ \ldots,\ 10$).

Let $\theta_j = 0.1$ ($j = 1,\ 2,\ \ldots,\ 10$) and $\Delta = 2.2$, Fig. 12 shows the evolution process of the total error and states of the impulsively coupled Chua circuit network, from which one can see that synchronization of an impulsively coupled Chua circuit network cannot be achieved when an impulse interval is greater than a critical value.

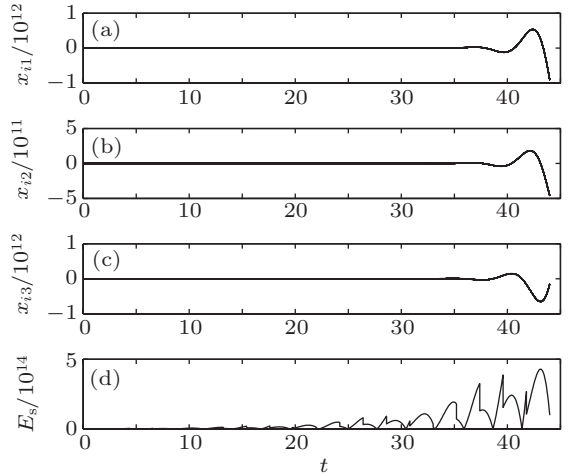


Fig. 12. The evolution of states x_{i1}, x_{i2}, x_{i3}, and the total error $E_{\rm s}$ of the impulsively coupled Chua circuit network with $\Delta = 2.2$, $t_0 = 0$, $\theta_1 = 1$, and $\theta_i = 0$ ($i = 2,\ 3,\ \ldots,\ 10$).

In all, the above two examples offer a valid demonstration of the effectiveness of the developed distributed impulsive control scheme.

5. Conclusion

To summarize, we have proposed a rigorous, systematic distributed impulsive control scheme to achieve the synchronization of complex dynamical networks with discontinuous couplings only, which is a significant extension of current studies, i.e., an extension of the continuously coupled network to a much more general and realistic situation: a non-continuously coupled network. We demonstrate the validity of the method with the well-known Lorenz network and Chua's circuit network. In comparison with previous methods for continuous coupling, the proposed distributed impulsive control scheme is very effective, low-cost, systematic, and easy to implement in practice, and provides us with a flexible method to select the synchronized state of a network. The implementation difficulty of the controller may also be avoided. Therefore, this method will be of great practical value.

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