Pinning synchronization of time-varying delay coupled complex networks with time-varying delayed dynamical nodes*

Wang Shu-Guo(王树国)[a)b)†] and Yao Hong-Xing(姚洪兴)[b)]

[a)] *Department of Mathematics and Physics, Changzhou Campus, Hohai University, Changzhou* 213022, *China*
[b)] *School of Finance & Economics, Jiangsu University, Zhenjiang* 212013, *China*

(Received 27 July 2011; revised manuscript received 1 November 2011)

This paper deals with the pinning synchronization of nonlinearly coupled complex networks with time-varying coupling delays and time-varying delays in the dynamical nodes. We control a part of the nodes of the complex networks by using adaptive feedback controllers and adjusting the time-varying coupling strengths. Based on the Lyapunov–Krasovskii stability theory for functional differential equations and a linear matrix inequality (LMI), some sufficient conditions for the synchronization are derived. A numerical simulation example is also provided to verify the correctness and the effectiveness of the proposed scheme.

Keywords: complex networks, adaptive feedback controllers, time-varying delays, pinning controllers

PACS: 05.45.Xt, 05.45.Gg **DOI:** 10.1088/1674-1056/21/5/050508

1. Introduction

During the past decade, complex dynamical networks have become a focal research topic and attracted more and more attention in various fields including physics, biology, chemistry, and computer science.[1−5] A complex network is a large set of interconnected nodes, in which each node is a fundamental unit with specific contents. Examples of complex networks include the internet, the world wide web (WWW), neural networks in human brains, food webs, telephone cell graphs, electricity distribution networks, etc. Many of these networks exhibit complexity in the overall topological properties and the dynamical properties of the network nodes and the coupled units. The complex nature of complex networks has led to a series of important studies. In particular, one significant and interesting phenomenon is the synchronization of all the dynamical nodes. So far, many different types of synchronizations have been investigated, such as complete synchronization,[6,7] generalized synchronization,[8] phase synchronization,[9] lag synchronization,[10] and projection synchronization.[11−13] Meanwhile, many approaches to synchronizing complex networks, such as adaptive control,[14] pinning control,[15] impulsive control,[16−21] intermittent control,[22,23] and sliding mode control,[24,25] have been developed.

As is now well known, a real-world complex network normally has a large number of nodes. Therefore, due to the complexity of the dynamical network, it is difficult to add controllers to all nodes to realize the synchronization, just as in the case mentioned in Ref. [26]. A natural way to reduce the number of controllers is to use the pinning control method. Wang and Chen[27] and Li *et al.*[28] proposed an effective approach to pinning a complex network to its equilibrium. Chen *et al.*[29] proposed a pinning control method to achieve the inner synchronization for the complex dynamical network using a single controller. Guo *et al.*[30] investigated the global synchronizations of the complex networks with nondelayed and delayed couplings based on the pinning controllers. Some sufficient conditions for the global synchronization by adding linear and adaptive feedback controllers to some part of the nodes were obtained. Yu *et al.*[31] investigated the pinning synchronization of the delayed dynamical network via periodi-

*Project supported by the National Natural Science Foundation of China (Grant No. 70871056) and the Six Talents Peak Foundation of Jiangsu Province, China (Grant No. 2010-JY70-025).

†Corresponding author. E-mail: wsg97@163.com

© 2012 Chinese Physical Society and IOP Publishing Ltd

cally intermittent control. Wu[32] pointed out that the pinning control should be applied to the roots of the trees in the spanning forest of the interaction graph. To pin an array of linearly coupled neural networks, Zhou *et al.*[33] solved two fundamental problems in the pinning control of complex networks: (i) the number of nodes that need to be pinned in a network with a fixed network structure and fixed coupling strength to reach network synchronization; (ii) the size of the coupling strength required to realize the network synchronization in a network with a fixed network structure and fixed pinning nodes. Lu *et al.*[34] proved that only one controller is required for a strongly connected network, while at least m controllers are needed for an m-reducible coupling matrix. Song and Cao[35] pointed out that the nodes whose out-degrees are bigger than their in-degrees must be chosen as the candidates to be pinned. As far as we know, although some pinning control schemes have been proposed for synchronization, not much work has been done on the synchronization of nonlinearly coupled complex networks with time-varying coupling delays and time-varying delays in the dynamical nodes by using the adaptive feedback controllers to a part of the nodes and adjusting the time-varying coupling strengths.

Motivated by the above discussion, in this paper we investigate the synchronization of nonlinearly coupled complex networks with time-varying coupling delays and time-varying delays in the dynamical nodes. In this study, we design the controllers to ensure that the special networks can reach synchronization. To reduce the number of the controllers, we control the complex networks by adding adaptive feedback controllers to a part of the nodes and adjusting the time-varying coupling strengths. Based on the Lyapunov–Krasovskii stability theory for functional differential equations[36,37] and a linear matrix inequality (LMI), some sufficient criteria are derived from a simple control law to ensure the synchronization of the network. The main contributions of this paper are three-fold: (i) the weight configuration matrix is not necessarily irreducible; (ii) the pinning nodes can be randomly selected; (iii) the synchronization criterion is independent of the time delays. Numerical examples are also provided to demonstrate the effectiveness of the theory.

The rest of this paper is organized as follows. The network model is introduced and some necessary definitions, lemmas, and hypotheses are given in Section 2. The synchronization of the coupled complex networks is discussed in Section 3. An example and its simulation are given in Section 4. Finally, conclusions are drawn in Section 5.

2. Network models and mathematical preliminaries

2.1. Model description

The network with time-varying coupling delays and adaptive coupling strengths can be described by

$$\begin{aligned}\dot{\boldsymbol{x}}_i(t) = {} & \boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) \\ & + \sigma(t)\sum_{j=1}^{N} a_{ij}\boldsymbol{H}_1(\boldsymbol{x}_j(t)) \\ & + c(t)\sum_{j=1}^{N} b_{ij}\boldsymbol{H}_2(\boldsymbol{x}_j(t-\tau_2(t))), \\ & i = 1, 2, \ldots, N, \end{aligned} \tag{1}$$

where $\boldsymbol{x}_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^{\mathrm{T}} \in \mathbf{R}^n$ is the state vector of the i-th dynamical node, $\boldsymbol{f} : \mathbf{R}^+ \times \mathbf{R}^n \to \mathbf{R}^n$ is a vector value function, $\tau_1(t) > 0$ and $\tau_2(t) > 0$ are time-varying delays, $\sigma(t) > 0$ and $c(t) > 0$ are time-varying coupling strengths, $\boldsymbol{H}_1(\cdot)$ and $\boldsymbol{H}_2(\cdot)$ are nonlinear functions, $\boldsymbol{A} = (a_{ij})_{N\times N}$ and $\boldsymbol{B} = (b_{ij})_{N\times N}$ are the weight configuration matrices. If there is a connection between nodes i and $j (j \neq i)$, then couplings $a_{ij} = a_{ji} > 0$, $b_{ij} = b_{ji} > 0$, otherwise, $a_{ij} = a_{ji} = 0$, $b_{ij} = b_{ji} = 0$. The diagonal elements of matrices A and B are defined as $a_{ii} = -\sum\limits_{j=1,j\neq i}^{N} a_{ji}$ and $b_{ii} = -\sum\limits_{j=1,j\neq i}^{N} b_{ji}$, respectively, where $i = 1, 2, \ldots, N$. Particularly, the weight configuration matrix is not necessarily irreducible.

2.2. Mathematical preliminaries

To obtain the main results, the following lemmas and assumptions are needed.

Definition 1 Assume that $\boldsymbol{s}(t) \in \mathbf{R}^n$ are arbitrary smooth dynamics. The controlled complex network (1) is said to be in homogeneous synchronized state $\boldsymbol{s}(t)$ if the solution satisfies $\lim\limits_{t\to\infty} \|\boldsymbol{x}_i(t) - \boldsymbol{s}(t)\| = 0$ for any initial conditions.

Let $C([-\tau, 0], \mathbf{R}^n)$ be the Banach space of continuous functions that map interval $[-\tau, 0]$ into $\mathbf{R}^n$ with norm $\|\boldsymbol{\phi}\| = \sup_{-\tau\leq\theta\leq 0} \|\boldsymbol{\phi}(\theta)\|$. The initial conditions of the functional differential Eq. (1) are given by $\boldsymbol{x}_i(t) = \boldsymbol{\phi}_i(t) \in C([-\tau, 0], \mathbf{R}^n)$, and it is assumed that equation (1) has a unique solution for these initial conditions.

Suppose that $\boldsymbol{s}(t)$ is a solution of the uncoupled system, $\dot{\boldsymbol{s}}(t) = \boldsymbol{f}(\boldsymbol{s}(t))$. In order to pin system (1) into the synchronization manifold $(\boldsymbol{s}(t), \boldsymbol{s}(t), \ldots, \boldsymbol{s}(t))$, we should add controllers $\boldsymbol{u}_i(t) = k_i(\boldsymbol{H}_1(\boldsymbol{x}_i(t)) - \boldsymbol{H}_1(\boldsymbol{s}(t)))$ to a certain selection of the nodes. Without loss of generality, we add the controllers to the first l nodes $(0 \leq l \leq N)$, then we have

$$\begin{cases} \dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) + \sigma(t)\sum\limits_{j=1}^{N} a_{ij}\boldsymbol{H}_1(\boldsymbol{x}_j(t)) \\ \qquad\quad + c(t)\sum\limits_{j=1}^{N} b_{ij}\boldsymbol{H}_2(\boldsymbol{x}_j(t-\tau_2(t))) - \boldsymbol{u}_i(t), \quad i = 1,\ 2,\ \ldots,\ l, \\ \dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) + \sigma(t)\sum\limits_{j=1}^{N} a_{ij}\boldsymbol{H}_1(\boldsymbol{x}_j(t)) \\ \qquad\quad + c(t)\sum\limits_{j=1}^{N} b_{ij}\boldsymbol{H}_2(\boldsymbol{x}_j(t-\tau_2(t))), \quad i = l+1, l+2, \ldots, N. \end{cases} \tag{2}$$

Noting that $\boldsymbol{e}_i(t) = \boldsymbol{x}_i(t) - \boldsymbol{s}(t)$, we have

$$\begin{cases} \dot{\boldsymbol{e}}_i(t) = \tilde{\boldsymbol{f}}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) + \sigma(t)\sum\limits_{j=1}^{N} a_{ij}\boldsymbol{H}_1(\boldsymbol{x}_j(t)) \\ \qquad\quad + c(t)\sum\limits_{j=1}^{N} b_{ij}\boldsymbol{H}_2(\boldsymbol{x}_j(t-\tau_2(t))) - \boldsymbol{u}_i(t), \quad i = 1,\ 2,\ \ldots,\ l, \\ \dot{\boldsymbol{e}}_i(t) = \tilde{\boldsymbol{f}}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) + \sigma(t)\sum\limits_{j=1}^{N} a_{ij}\boldsymbol{H}_1(\boldsymbol{x}_j(t)) \\ \qquad\quad + c(t)\sum\limits_{j=1}^{N} b_{ij}\boldsymbol{H}_2(\boldsymbol{x}_j(t-\tau_2(t))), \quad i = l+1, l+2, \ldots, N, \end{cases} \tag{3}$$

where $\tilde{\boldsymbol{f}}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) = \boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) - f(\boldsymbol{t}, \boldsymbol{s}(t), \boldsymbol{s}(t-\tau_1(t)))$.

Lemma 1[38] For two arbitrary vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, a matrix $\boldsymbol{S} > 0$ with compatible dimensions, one has $2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{y} \leq \boldsymbol{x}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{x} + \boldsymbol{y}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{y}$.

Lemma 2[39] Let $\boldsymbol{Q}$ and $\boldsymbol{R}$ be two symmetric matrices, and matrix $\boldsymbol{S}$ has a suitable dimension, then

$$\begin{pmatrix} \boldsymbol{Q} & \boldsymbol{S} \\ \boldsymbol{S}^{\mathrm{T}} & \boldsymbol{R} \end{pmatrix} < 0$$

if and only if both $\boldsymbol{R} < 0$ and $\boldsymbol{Q} - \boldsymbol{S}\boldsymbol{R}^{-1}\boldsymbol{S}^{\mathrm{T}} < 0$ are satisfied.

Lemma 3[40] For vector-valued function $\boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t)))$, assuming that there exist positive constants $\gamma > 0$ and $\theta > 0$ such that $\boldsymbol{f}$ satisfies the semi-Lipschitz condition, we have

$$\begin{aligned} &(\boldsymbol{x}_i(t) - \boldsymbol{y}_i(t))^{\mathrm{T}}(\boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) \\ &- \boldsymbol{f}(t, \boldsymbol{y}_i(t), \boldsymbol{y}_i(t-\tau_1(t)))) \\ \leq\ &\gamma(\boldsymbol{x}_i(t) - \boldsymbol{y}_i(t))^{\mathrm{T}}(\boldsymbol{x}_i(t) - \boldsymbol{y}_i(t)) + \theta(\boldsymbol{x}_i(t-\tau_1(t))) \\ &- \boldsymbol{y}_i(t-\tau_1(t)))^{\mathrm{T}}(\boldsymbol{x}_i(t-\tau_1(t)) - \boldsymbol{y}_i(t-\tau_1(t))) \end{aligned}$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$ and $t \geq 0$, where $i = 1, 2, \ldots, N$.

Assumption 1[41] (global Lipschitz condition) Suppose that there exist nonnegative constants α, β for all $\forall t \in \mathbf{R}_+$ such that for any time-varying vectors $\boldsymbol{x}(t), \boldsymbol{y}(t) \in \mathbf{R}^n$,

$$\begin{aligned} \|\boldsymbol{H}_1(\boldsymbol{x}) - \boldsymbol{H}_1(\boldsymbol{y})\| &\leq \alpha \|\boldsymbol{x} - \boldsymbol{y}\|, \\ \|\boldsymbol{H}_2(\boldsymbol{x}) - \boldsymbol{H}_2(\boldsymbol{y})\| &\leq \beta \|\boldsymbol{x} - \boldsymbol{y}\|, \end{aligned}$$

where $\|\cdot\|$ denotes the 2-norm throughout the paper.

Assumption 2[41] Time delays $\tau_1(t)$ and $\tau_2(t)$ are differential functions with $0 \leq \dot{\tau}_1(t) \leq \varepsilon \leq 1$ and $0 \leq \dot{\tau}_2(t) \leq \varepsilon \leq 1$. Clearly, this assumption is ensured if delays $\tau_1(t)$ and $\tau_2(t)$ are constants. For convenience, we denote

$$\begin{aligned} \tilde{\boldsymbol{x}}_j(t) &= (x_{1j}(t), x_{2j}(t), \ldots, x_{Nj}(t))^{\mathrm{T}}, \\ \tilde{\boldsymbol{e}}_j(t) &= (e_{1j}(t), e_{2j}(t), \ldots, e_{Nj}(t))^{\mathrm{T}}, \\ \boldsymbol{H}_1(\tilde{\boldsymbol{x}}_j(t)) &= (\boldsymbol{H}_1(x_{1j}(t)), \boldsymbol{H}_1(x_{2j}(t)), \ldots, \\ &\qquad \boldsymbol{H}_1(x_{Nj}(t)))^{\mathrm{T}}, \\ \boldsymbol{H}_2(\tilde{\boldsymbol{x}}_j(t)) &= (\boldsymbol{H}_2(x_{1j}(t)), \boldsymbol{H}_2(x_{2j}(t)), \ldots, \\ &\qquad \boldsymbol{H}_2(x_{Nj}(t)))^{\mathrm{T}}, \\ \tilde{\boldsymbol{e}}_j &= \tilde{\boldsymbol{x}}_j(t) - \tilde{\boldsymbol{s}}_j(t). \end{aligned}$$

3. Main results

In this section, we synchronize system (1) by using adaptive feedback controllers. If parameters k_1,

$k_2, \ldots, k_l$ in coupled system (3) are defined as

$$\dot{k}_i = q_i \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t), \quad i = 1, 2 \ldots l,$$

where q_i $(i = 1, 2, \ldots, l)$ are positive constants and $k_i(0) > 0$, then controller $\boldsymbol{u}_i(t) = k_i(\boldsymbol{H}_1(\boldsymbol{x}_i(t)) - \boldsymbol{H}_1(\boldsymbol{s}(t)))$ is called an adaptive feedback controller. After applying an adaptive feedback controller to system (3), we have

$$\begin{cases} \dot{\boldsymbol{e}}_i(t) = \tilde{\boldsymbol{f}}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) + \sigma(t)\sum\limits_{j=1}^{N} a_{ij}\boldsymbol{H}_1(\boldsymbol{x}_j(t)) \\ \qquad\qquad + c(t)\sum\limits_{j=1}^{N} b_{ij}\boldsymbol{H}_2(\boldsymbol{x}_j(t-\tau_2(t))) - \boldsymbol{u}_i(t), \quad i = 1,\ 2,\ \ldots,\ l, \\ \dot{k}_i = q_i \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t), \quad i = 1, 2, \ldots, l, \\ \dot{\boldsymbol{e}}_i(t) = \tilde{\boldsymbol{f}}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) + \sigma(t)\sum\limits_{j=1}^{N} a_{ij}\boldsymbol{H}_1(\boldsymbol{x}_j(t)) \\ \qquad\qquad + c(t)\sum\limits_{j=1}^{N} b_{ij}\boldsymbol{H}_2(\boldsymbol{x}_j(t-\tau_2(t))), \quad i = l+1, l+2, \ldots, N. \end{cases} \tag{4}$$

Theorem 1 Suppose that Assumptions 1 and 2 hold, the adaptive coupling strengths are obtained as

$$\sigma(t) = \hat{\rho}(t) + \mu_1, \quad c(t) = \hat{\eta}(t) + \mu_2, \tag{5}$$

where $\mu_1 > 0$, $\mu_2 > 0$, $\hat{\rho}(0) > 0$, and $\hat{\eta}(0) > 0$. If $\hat{\rho}(t)$ and $\hat{\eta}(t)$ satisfy the following update laws:

$$\dot{\hat{\rho}}(t) = -\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t),$$

$$\dot{\hat{\eta}}(t) = -\sum_{i=1}^{N} [\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) + \boldsymbol{e}_i^{\mathrm{T}}(t-\tau_2(t))\boldsymbol{e}_i(t-\tau_2(t))],$$

and the following condition holds:

$$\mu_1\alpha\boldsymbol{A} - \alpha k\boldsymbol{\Lambda} + \frac{1}{2(1-\varepsilon)}\boldsymbol{I}_N + \frac{1}{2}\beta^2\mu_2^2\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B} < 0, \tag{6}$$

where k is a positive constant to be determined, then controlled dynamical network (1) is synchronized.

Proof Construct the following Lyapunov–Krasovskii function:

$$\begin{aligned} V(t) = {} & \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) \\ & + \frac{\theta}{(1-\varepsilon)}\sum_{i=1}^{N}\int_{t-\tau_1(t)}^{t} \boldsymbol{e}_i^{\mathrm{T}}(\xi)\boldsymbol{e}_i(\xi)\mathrm{d}\xi \\ & + \frac{1}{2(1-\varepsilon)}\sum_{j=1}^{n}\int_{t-\tau_2(t)}^{t} \tilde{\boldsymbol{e}}_j^{\mathrm{T}}(\xi)\tilde{\boldsymbol{e}}_j(\xi)\mathrm{d}\xi \\ & + \frac{\alpha}{2}\sum_{i=1}^{l}\frac{(k_i-k)^2}{q_i} + \frac{\tilde{\rho}^2}{2\alpha\lambda_1} + \frac{\tilde{\eta}^2}{\beta\lambda_2}, \end{aligned}$$

where $\tilde{\rho}(t) = (\gamma + \frac{\theta}{(1-\varepsilon)}) + \alpha\lambda_1\hat{\rho}(t)$, $\dot{\tilde{\rho}}(t) = \alpha\lambda_1\dot{\hat{\rho}}(t)$, $\tilde{\eta}(t) = \frac{1}{2}\beta\lambda_2\hat{\eta}(t)$, $\dot{\tilde{\eta}}(t) = \frac{1}{2}\beta\lambda_2\dot{\hat{\eta}}(t)$, $\lambda_1 = \max|\lambda(\boldsymbol{A}\otimes\boldsymbol{I})|$, and $\lambda_2 = \max(|\lambda(\boldsymbol{B}\otimes\boldsymbol{I})|)$. By virtue of Assumptions 1 and 2, the time derivative of $V(t)$ along the trajectory of system (4) is derived as

$$\begin{aligned} \dot{V}(t) = {} & \sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)(\tilde{\boldsymbol{f}}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) \\ & + \sigma(t)\sum_{j=1}^{N} a_{ij}\boldsymbol{H}_1(\boldsymbol{x}_j(t)) \\ & + c(t)\sum_{j=1}^{N} b_{ij}\boldsymbol{H}_2(\boldsymbol{x}_j(t-\tau_2(t))) - \boldsymbol{u}_i(t)) \\ & + \frac{\theta}{(1-\varepsilon)}\sum_{i=1}^{N}(\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) - (1-\dot{\tau}_1(t)) \\ & \times \boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1(t))\boldsymbol{e}_i(t-\tau_1(t))) \\ & + \alpha\sum_{i=1}^{l}(k_i-k)\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) \\ & + \frac{1}{2(1-\varepsilon)}\sum_{j=1}^{n}(\tilde{\boldsymbol{e}}_j^{\mathrm{T}}(t)\tilde{\boldsymbol{e}}_j(t) - (1-\dot{\tau}_2(t)) \\ & \times \tilde{\boldsymbol{e}}_j^{\mathrm{T}}(t-\tau_2(t))\tilde{\boldsymbol{e}}_j(t-\tau_2(t))) \\ & + \tilde{\rho}(t)\dot{\hat{\rho}}(t) + \tilde{\eta}(t)\dot{\hat{\eta}}(t). \end{aligned}$$

Using Lemma 3, we obtain

$$\begin{aligned} \dot{V}(t) \le {} & \sum_{i=1}^{N}\left[\gamma\boldsymbol{e}_i(t)^{\mathrm{T}}\boldsymbol{e}_i(t) + \theta\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1(t))\boldsymbol{e}_i(t-\tau_1(t))\right] \\ & + \sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\hat{\rho}(t)\sum_{j=1}^{N} a_{ij}\boldsymbol{H}_1(\boldsymbol{x}_j(t)) \\ & + \mu_1\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\sum_{j=1}^{N} a_{ij}\boldsymbol{H}_1(x_j(t)) \\ & + \sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\hat{\eta}(t)\sum_{j=1}^{N} b_{ij}\boldsymbol{H}_2(\boldsymbol{x}_j(t-\tau_2(t))) \end{aligned}$$

$$
\begin{aligned}
&+\mu_2\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\sum_{j=1}^{N}b_{ij}\boldsymbol{H}_2(\boldsymbol{x}_j(t-\tau_2(t)))\\
&-\alpha\sum_{i=1}^{l}k\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)+\frac{\theta}{(1-\varepsilon)}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)\\
&-\frac{\theta(1-\dot{\tau}_1(t))}{(1-\varepsilon)}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1(t))\boldsymbol{e}_i(t-\tau_1(t))\\
&+\frac{1}{2(1-\varepsilon)}\sum_{j=1}^{n}\tilde{\boldsymbol{e}}_j^{\mathrm{T}}(t)\tilde{\boldsymbol{e}}_j(t)\\
&-\frac{1-\dot{\tau}_2(t)}{2(1-\varepsilon)}\sum_{j=1}^{n}\tilde{\boldsymbol{e}}_j^{\mathrm{T}}(t-\tau_2(t))\tilde{\boldsymbol{e}}_j(t-\tau_2(t))\\
&+\tilde{\rho}(t)\dot{\hat{\rho}}(t)+\tilde{\eta}(t)\dot{\hat{\eta}}(t).
\end{aligned}
$$

Let $\boldsymbol{e}(t)=(\boldsymbol{e}_1^{\mathrm{T}}(t),\boldsymbol{e}_2^{\mathrm{T}}(t),\ldots,\boldsymbol{e}_N^{\mathrm{T}}(t))^{\mathrm{T}}\in\mathbf{R}^{nN}$, by using Assumptions 1 and 2 and Lemma 1, we obtain

$$
\begin{aligned}
\dot{V}(t)\leq&\sum_{i=1}^{N}[\gamma\boldsymbol{e}_i(t)^{\mathrm{T}}\boldsymbol{e}_i(t)\\
&+\theta\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1(t))\boldsymbol{e}_i(t-\tau_1(t))]\\
&+\alpha\lambda_1\hat{\rho}(t)\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)\\
&+\mu_1\alpha\sum_{j=1}^{n}\tilde{\boldsymbol{e}}_j^{\mathrm{T}}(t)\boldsymbol{A}\tilde{\boldsymbol{e}}_j(t)+\frac{1}{2}\beta\lambda_2\hat{\eta}(t)(\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t)\\
&+\boldsymbol{e}^{\mathrm{T}}(t-\tau_2(t))\boldsymbol{e}(t-\tau_2(t)))\\
&+\mu_2\beta\sum_{j=1}^{n}\tilde{\boldsymbol{e}}_j^{\mathrm{T}}(t)\boldsymbol{B}\tilde{\boldsymbol{e}}_j(t-\tau_2(t))\\
&-\alpha\sum_{i=1}^{l}k\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)+\frac{\theta}{(1-\varepsilon)}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)\\
&-\theta\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1(t))\boldsymbol{e}_i(t-\tau_1(t))\\
&+\frac{1}{2(1-\varepsilon)}\sum_{j=1}^{n}\tilde{\boldsymbol{e}}_j^{\mathrm{T}}(t)\tilde{\boldsymbol{e}}_j(t)\\
&-\frac{1}{2}\sum_{j=1}^{n}\tilde{\boldsymbol{e}}_j^{\mathrm{T}}(t-\tau_2(t))\tilde{\boldsymbol{e}}_j(t-\tau_2(t))\\
&+\tilde{\rho}(t)\dot{\hat{\rho}}(t)+\tilde{\eta}(t)\dot{\hat{\eta}}(t).
\end{aligned}
$$

Let $\boldsymbol{\Lambda}=\begin{pmatrix}\boldsymbol{I}_{l\times l}&0\\0&0\end{pmatrix}_{N\times N}$, we obtain

$$
\begin{aligned}
\dot{V}(t)\leq&\sum_{j=1}^{n}\tilde{\boldsymbol{e}}_j^{\mathrm{T}}(t)\Big[\mu_1\alpha\boldsymbol{A}-\alpha k\boldsymbol{\Lambda}+\frac{1}{2(1-\varepsilon)}\boldsymbol{I}\Big]\tilde{\boldsymbol{e}}_j(t)\\
&+\mu_2\beta\sum_{j=1}^{n}\tilde{\boldsymbol{e}}_j^{\mathrm{T}}(t)B\tilde{\boldsymbol{e}}_j(t-\tau_2(t))\\
&-\frac{1}{2}\sum_{j=1}^{n}\tilde{\boldsymbol{e}}_j^{\mathrm{T}}(t-\tau_2(t))\tilde{\boldsymbol{e}}_j(t-\tau_2(t))\\
=&\sum_{j=1}^{n}(\tilde{\boldsymbol{e}}_j^{\mathrm{T}}(t),\tilde{\boldsymbol{e}}_j^{\mathrm{T}}(t-\tau_2(t)))\boldsymbol{Z}_j(\tilde{\boldsymbol{e}}_j(t),\tilde{\boldsymbol{e}}_j(t-\tau_2(t))),
\end{aligned}
$$

where

$$
\boldsymbol{Z}_j=\begin{pmatrix}\mu_1\alpha\boldsymbol{A}-\alpha k\boldsymbol{\Lambda}+\frac{1}{2(1-\varepsilon)}\boldsymbol{I}_N&\frac{1}{2}\beta\mu_2\boldsymbol{B}\\\frac{1}{2}\beta\mu_2\boldsymbol{B}^{\mathrm{T}}&-\frac{1}{2}\boldsymbol{I}_N\end{pmatrix}.
$$

It follows from Lemma 2 that if

$$
\mu_1\alpha\boldsymbol{A}-\alpha k\boldsymbol{\Lambda}+\frac{1}{2(1-\varepsilon)}\boldsymbol{I}_N+\frac{1}{2}\beta^2\mu_2^2\boldsymbol{B}^{\mathrm{T}}\boldsymbol{B}<0,
$$

then $\dot{V}(t)<0$. The proof is completed.

Remark 1 As is now well known, a real-world complex network normally has a large number of nodes, it is usually difficult to control a complex network by adding the controllers to all nodes. To reduce the number of the controllers, the pinning controller is relatively simple and is easy to implement.

Remark 2 By using the Lyapunov–Krasovskii functional method and a linear matrix inequality (LMI), we prove that the dynamical network can be made synchronized with one isolated node by adding the controllers to only a small subset of the nodes. In addition, the pinning nodes can be randomly selected. It indeed provides some new insights into future practical engineering design.

Remark 3 In this paper, the weight configuration matrix is not necessarily irreducible.

Remark 4 The synchronization criteria have been given in Theorem 1. Compared with some similar designs, our pinning adaptive controllers are very simple. In particular, the synchronization criteria are independent of the time delay.

4. Illustrative example

In this section, a numerical example is given to demonstrate the validity of the synchronization criteria obtained in the previous sections. Considering the following network:

$$\begin{cases} \dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) + \sigma(t) \sum\limits_{j=1}^{N} a_{ij} \boldsymbol{H}_1(\boldsymbol{x}_j(t)) \\ \qquad + c(t) \sum\limits_{j=1}^{N} b_{ij} \boldsymbol{H}_2(\boldsymbol{x}_j(t-\tau_2(t))) - \boldsymbol{u}_i(t), \quad i = 1,\ 2,\ \ldots,\ l, \\ \dot{\boldsymbol{x}}_i(t) = \boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) + \sigma(t) \sum\limits_{j=1}^{N} a_{ij} \boldsymbol{H}_1(\boldsymbol{x}_j(t)) \\ \qquad + c(t) \sum\limits_{j=1}^{N} b_{ij} \boldsymbol{H}_2(\boldsymbol{x}_j(t-\tau_2(t))), \quad i = l+1, l+2 \ldots N, \end{cases}$$

where $\boldsymbol{f}(t, \boldsymbol{x}_i(t), \boldsymbol{x}_i(t-\tau_1(t))) = \boldsymbol{D}\boldsymbol{x}_i(t) + \boldsymbol{h}_1(\boldsymbol{x}_i(t)) + \boldsymbol{h}_2(\boldsymbol{x}_i(t-\tau_1(t)))$, and $\boldsymbol{x}_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^{\mathrm{T}}$, with $\tau_1(t) = \frac{2\,\mathrm{e}^t}{(1+\mathrm{e}^t)}$, $\boldsymbol{h}_1(\boldsymbol{x}_i) = (3.247(|x_{i1}+1| - |x_{i1}-1|), 0, 0)^{\mathrm{T}}$, $\boldsymbol{h}_2(\boldsymbol{x}_i) = (0, 0, -3.906\sin(0.5x_{i1}))^{\mathrm{T}}$, $\tau_2(t) = \frac{\mathrm{e}^t}{10(1+\mathrm{e}^t)}$, $N = 4$, $l = 1$, $\boldsymbol{H}_1(x) = \sin \boldsymbol{x} + 2\boldsymbol{x}$, $\boldsymbol{H}_2(x) = \cos \boldsymbol{x} + 3\boldsymbol{x}$,

$$\boldsymbol{D} = \begin{bmatrix} -2.169 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -19.53 & -0.1636 \end{bmatrix},$$

$$\boldsymbol{A} = \boldsymbol{B} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{bmatrix}.$$

The simulation results are given in Figs. 1–4. Figure 1 shows the chaotic behavior of the time-delayed Chua oscillator. Figures 2 and 3 show the time evolutions of the synchronization errors with q_1=1 and q_1=15, respectively. Figure 4 shows the evolution of the feedback strength k_1 with q_1=15. All numerical simulations illustrate the effectiveness of Theorem 1.

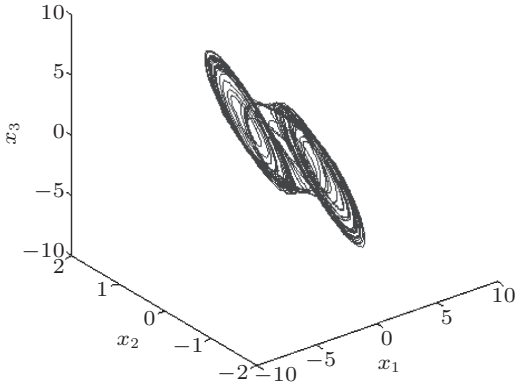


Fig. 1. Chaotic behavior of time-delayed Chua oscillator.

As can be seen, the constant q_i can be chosen properly to adjust the synchronization speed. The larger the adaptive gain q_i, the faster the synchronization of the networks.

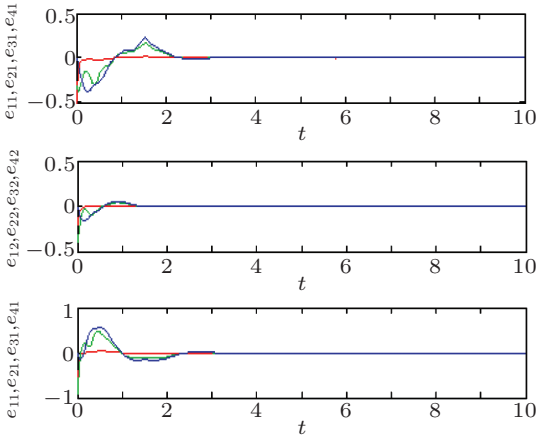


Fig. 2. (colour online) Time evolutions of the synchronization errors with q_1=15.

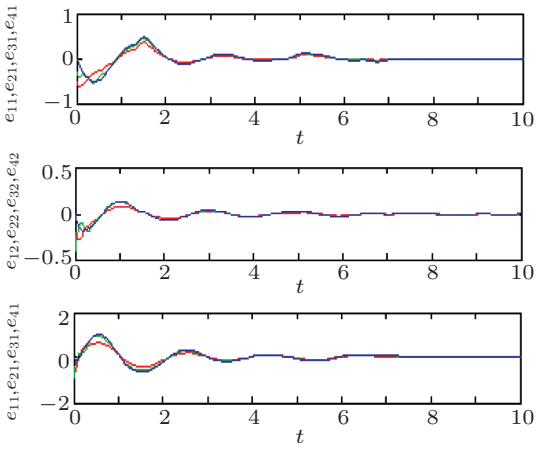


Fig. 3. (colour online) Time evolutions of the synchronization errors with q_1=1.

We have proved in Theorem 1 that system (1) can be synchronized by adding suitable controllers to only a sub-collection of the nodes in the network. The numerical simulation is performed to verify the effectiveness of the theoretical results. Particularly, let $l = 1$, then the results imply that network (1) can be globally synchronized with $(\boldsymbol{s}(t), \boldsymbol{s}(t), \ldots, \boldsymbol{s}(t))$, which also means that we can pin a coupled complex network to a homogenous solution by using only a single controller.

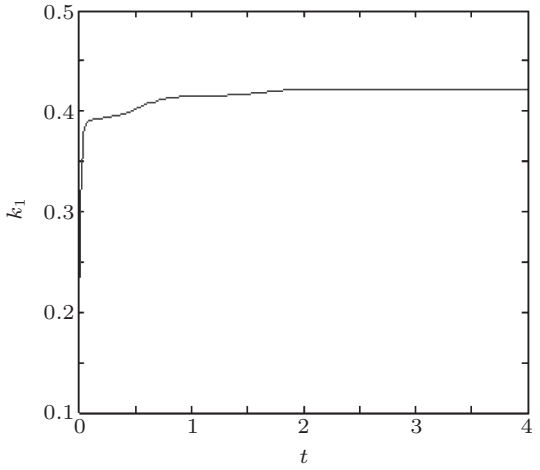


Fig. 4. Evolution of the feedback strength k_1 with q_1=15.

5. Conclusion

In this paper, the synchronization of complex networks with time-varying coupling delays and time-varying delays in the dynamical nodes is investigated based on the pinning controllers. The coupling matrices are not assumed to be irreducible. By using the Lyapunov–Krasovskii functional method and a linear matrix inequality (LMI), we give some criteria for the synchronization. The numerical simulation is performed to verify the effectiveness of the theoretical results. Compared with the existing results, our results reveal that our synchronization is still very useful when the existing methods become invalid. In our future work, we will consider the pinning synchronization problem of directed complex networks.

References

[1] Strogatz S H 2001 *Nature* **410** 268

[2] Barabási A L and Albert R 1999 *Science* **286** 509

[3] Dorogovtesev S N and Mendes J F F 2002 *Adv. Phys.* **51** 1079

[4] Newman M E J 2003 *SIAM Review* **45** 167

[5] Boccaletti S, Latora V, Moreno Y, Chavez M and Hwang D U 2006 *Physics Reports* **424** 175

[6] Lu J, Ho D W C and Liu M 2007 *Phys. Lett.* A **369** 444

[7] Li K, Guan S G , Gong X F and Lai C H 2008 *Phys. Lett.* A **372** 7133

[8] Liu H, Chen J, Lu J and Cao M 2010 *Physica* A **389** 1759

[9] Rosenblum M G, Pikovsky A S and Kurths J 1996 *Phys. Rev. Lett.* **76** 1804

[10] Rosenblum M G, Pikovsky A S and Kurths J 1997 *Phys. Rev. Lett.* **78** 4193

[11] Zhang R, Yang Y Q, Xu Z Y and Hu M F 2010 *Phys. Lett.* A **374** 3025

[12] Lü L, Meng L, Guo L, Zou J R and Yang M 2011 *Acta Phys. Sin.* **60** 124 (in Chinese)

[13] Abdurahman K, Wang X Y and Zhao Y Z 2011 *Acta Phys. Sin.* **60** 81 (in Chinese)

[14] Zhou J, Lu J and Lü J 2006 *IEEE Trans. Automat. Control* **51** 652

[15] Pan L, Zhou W N, Fang J A and Li D Q 2010 *Commun. Nonlinear Sci. Numer. Simulat.* **15** 3754

[16] Guo L X, Hu M F and Xu Z Y 2010 *Chin. Phys.* B **19** 020512

[17] Yang L B and Yang T 2000 *Acta Phys. Sin.* **49** 33 (in Chinese)

[18] Chen J F, Zhang R Y and Peng J H 2003 *Acta Phys. Sin.* **52** 1589 (in Chinese)

[19] Ma T D, Zhang H G and Wang Z L 2007 *Acta Phys. Sin.* **56** 3796 (in Chinese)

[20] Guo L X, Hu M F and Xu Z Y 2010 *Chin. Phys.* B **19** 020512

[21] Wang S G and Yao H X 2011 *Chin. Phys.* B **20** 9 090513

[22] Syntetos A A and Boylan J E 2006 *International Journal of Production Economics* **103** 36

[23] Cai S M, Hao J J, He Q B and Liu Z G 2011 *Phys. Lett.* A **375** 1965

[24] Li H Q, Liao X F and Huang H Y 2011 *Acta Phys. Sin.* **60** 020512 (in Chinese)

[25] Liu F C, Li J Y and Zang X F 2011 *Acta Phys. Sin.* **60** 030504 (in Chinese)

[26] Liu T 2009 *Chaos Solition Fract.* **40** 1506

[27] Wang X and Chen G 2002 *Physica* A **310** 521

[28] Li X, Wang X and Chen G 2004 *IEEE Trans. Circuits Syst.* I **51** 2074

[29] Chen T, Liu X and Lu W 2007 *IEEE Trans. Circuits Syst.* I **54** 1317

[30] Guo W L, Austin F, Chen S H and Sun W 2009 *Phys. Lett.* A **373** 1565

[31] Yu W W, Chen G R and Lue J H 2009 *Automatic* **45** 429

[32] Wu C W 2008 *IEEE Int. Symposium on Circuits and Systems* 2530

[33] Zhou J, Lu J A and Lü J H 2008 *Automatica* **44** 996

[34] Lu J, Ho D W C and Wang Z 2009 *IEEE Trans. Neural Networks* **20** 1617

[35] Song Q and Cao J 2010 *IEEE Trans. Circuits Syst.* I **57** 672

[36] Kolmanovskii V B and Myshkis A D 1999 *Introduction to the Theory and Applications of Functional Differential Equations* (Dordrecht: Kluwer Academic Publishers)

[37] Hale J K and Lunel S M V 1993 *Introduction to Functional Differential Equations* (New York: Springer)

[38] Zuo Z Q, Yang C L and Wang Y J 2010 *Phys. Lett.* A **374** 1989

[39] Zhou J and Chen T P 2006 *IEEE Trans. Circ. Syst.* I **53** 733

[40] Hu C, Yu J, Jiang H J and Teng Z D 2011 *Phys. Lett.* A **375** 873

[41] Liu H, Chen J, Lu J A and Cao M 2010 *Physica* A **389** 1759