Transport enhancement and efficiency optimization in two heat reservoir ratchets*

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We study a Brownian motor moving in a sawtooth potential in the presence of an external driving force and two heat reservoirs. Based on the corresponding Fokker–Planck equation, the analytical expressions of the current and efficiency in the quasi-steady-state limit are obtained. The effects of temperature difference and the amplitude of the external driving force on the current and efficiency are discussed, respectively. The following is our findings. (i) The current increases with both δ and A. In other words, δ and A enhance the transport of the Brownian motor. (ii) The competition between the temperature difference and the amplitude of the external driving force can lead to efficiency optimization. The efficiency is a peaked function of temperature, i.e., $\delta > 0$ and a lower amplitude value of the external driving force is necessary for efficiency optimization. (iii) The efficiency increases with δ, and decreases with A. δ and A play opposite roles with respect to the efficiency, which indicates that δ enhances the efficiency of energy transformation while A weakens it.

Keywords: temperature difference, sawtooth potential, current and efficiency

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1. Introduction

Fluctuation-induced transport has been extensively studied from both experimental and theoretical points of view in the last few years.[1−4] Optical, electrical, and mechanical systems have been shown to have a ratchet property experimentally.[5−9] Theoretically, there have been several extensions of the ratchet effect, for instance, in compound objects and solitons.[10−14] Motivated by this interesting fact, recent studies of thermal ratchet models have shown how work should be extracted from nonequilibrium fluctuations.[15−19] Such models include rocking ratchets,[20] flashing ratchets,[21] diffusion ratchets,[22] correlation ratchets,[23] white-shot-noise ratchets,[24] two-noise ratchets[25] and entropic ratchets.[26,27]

The energetics analysis of these ratchet systems,[28−32] which rectify the zero-mean fluctuation, have undergone thorough investigation in recent years.[33,34] To define the optimal models for such ratchet systems, the maximization of the efficiency of energy transformation is inevitable.[35,36] This discovery has been confirmed by many researchers.[37−41] One important finding among the ratchet models was initiated by the elegant piece of work done by Magnasco,[37] which showed that a Brownian particle subjectedg to external fluctuations can undergo a non-zero drift while moving under the influence of an asymmetric potential. The temperature dependence of the current has been studied and it has been shown that there is a region of the operating regime in which the current can be maximized at finite temperatures, and the existence of thermal fluctuations facilitate the efficiency of energy transformation. An energetic analysis of the same model used by Magnasco was investigated by Kamegawa *et al.*,[38] where they showed that the efficiency of energy transformation cannot be optimized at finite temperatures and that the thermal fluctuations do not facilitate it. The work by Krishnan and co-workers[39] showed that the efficiency can be optimized at finite temperatures in inhomogeneous systems with spatially varying friction coefficient in an adiabatically rocked ratchet. Efficiency optimization in homogeneous nonadiabatical rachet systems was

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presented by Sumithra *et al.*[40] A recent investigation by Ai *et al.*[41] found that the interplay between asymmetry of fluctuation and asymmetry of potential can induce optimized efficiency of energy conversion at finite temperatures. These interesting findings have been interpreted that the existence of thermal fluctuation does not disturb the fluctuation-induced transport, and even facilitates the efficiency of energy transformation. More recently, the transport of a Brownian particle moving in a periodic potential was investigated by Ai *et al.*[42] in the presence of an unbiased fluctuation and two heat reservoirs. Their investigation showed that the competition of the asymmetric parameter of the potential with the temperature difference can lead to phenomena such as current reversal. However, the fluctuation-induced current is not an energetic quantity, and the current is only mimics the energetic efficiency.[38] Therefore, whether the thermal fluctuations actually facilitate the energy transformation in forced homogeneous adiabatical ratchet systems with two heat reservoirs is still an open question and also the subject of the present investigation.

In this paper, the transport of a Brownian motor moving in a sawtooth potential in the presence of an external driving force and two heat reservoirs is investigated. In Section 2, based on the corresponding Fokker–Planck equation, the analytical expressions of the current and efficiency in the quasi-steady-state limit are obtained. Then, effects of the temperature difference and the amplitude of the external driving force on the current and efficiency are discussed. Finally, concluding remarks are given in Section 3.

2. Current and efficiency in a sawtooth potential

Let us consider an overdamped Brownian particle moving in a sawtooth potential with an unbiased external force where the viscous medium is alternately in contact with two heat reservoirs. This model was first proposed to describe a molecular motor in biological systems.[43] The form of the Langevin equation is described by

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{\mathrm{d}V_0(x)}{\mathrm{d}x}+\sqrt{2k_{\mathrm{B}}T(x)}\xi(t)+F(t)-\frac{\mathrm{d}V_{\mathrm{L}}(x)}{\mathrm{d}x}. \quad (1)$$

For the overdamped case, one can use the adiabatic approximation $m\mathrm{d}^2x/\mathrm{d}t^2=0$. Then, Equation (1) can be reduced to

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{\mathrm{d}V_0(x)}{\mathrm{d}x}+\sqrt{2k_{\mathrm{B}}T(x)}\xi(t)+F(t)-\frac{\mathrm{d}V_{\mathrm{L}}(x)}{\mathrm{d}x}, \quad (2)$$

where x represents the state of the ratchet, k_{B} is the Boltzmann constant, and $T(x)$ is the absolute temperature. $\xi(t)$ is a thermal noise satisfying $\langle\xi(t)\xi(t')\rangle=\delta(t-t')$. Here, $\langle\cdots\rangle$ denotes an ensemble average over the distribution of the fluctuating force $\xi(t)$. $F(t)$ is an external driving force with a temporal period of τ ($F(t+\tau)=F(t)$, $\int_0^\tau F(t)dt=0$), as shown in Fig. 1(a). $V_{\mathrm{L}}(x)$ is a potential on which the work is done and $\mathrm{d}V_{\mathrm{L}}(x)\mathrm{d}x=L>0$ is the load force. The geometry of the potential, $V(x)=V_0(x)+V_{\mathrm{L}}(x)$, is displayed in Fig. 1(b). $V_0(x)$ is a piecewise linear and periodic potential, the spatial period of the potential is $\lambda=\lambda_1+\lambda_2$, and $\Delta=\lambda_1-\lambda_2$ is a symmetry breaking amplitude. The potential $V_0(x)$ within the interval $0\le x\le\lambda$ is described by

$$V_0(x)=\begin{cases}\dfrac{Q}{\lambda_1}x, & 0\le x<\lambda_1,\\ \dfrac{Q}{\lambda_2}(\lambda-x), & \lambda_1\le x\le\lambda.\end{cases} \quad (3)$$

The temperature $T(x)$ has the same period as the potential $V_0(x)$ (see Fig. 1(b)). Therefore, $T(x)=T(x+\lambda)$,

$$T(x)=\begin{cases}T+\delta, & 0\le x<\lambda_1,\\ T, & \lambda_1\le x\le\lambda,\end{cases} \quad (4)$$

where δ is the temperature difference.

The Fokker–Planck equation corresponding to Eq. (2) of the ratchet system reads[44]

$$\frac{\partial P(x,t)}{\partial t}=\frac{\partial}{\partial x}\left\{[V_0'(x)-F(t)+L]P(x,t)+\frac{\partial}{\partial x}[k_{\mathrm{B}}T(x)P(x,t)]\right\}=-\frac{\partial j(x,t)}{\partial x}, \quad (5)$$

$$j(x,t)=-[V_0'(x)-F(t)+L]P(x,t)-\frac{\partial}{\partial x}[k_{\mathrm{B}}T(x)P(x,t)], \quad (6)$$

where $j(x,t)$ is the probability current. Equation (5) is its derivative with respect to the space variable x. $P(x,t)$ is the probability density of the particle at

a position of x and at a time of t. It satisfies the normalization condition and the periodicity condition, $P(x,t) = P(x+\lambda,t)$ and $\int_0^\lambda P(x,t)\mathrm{d}x = 1$.

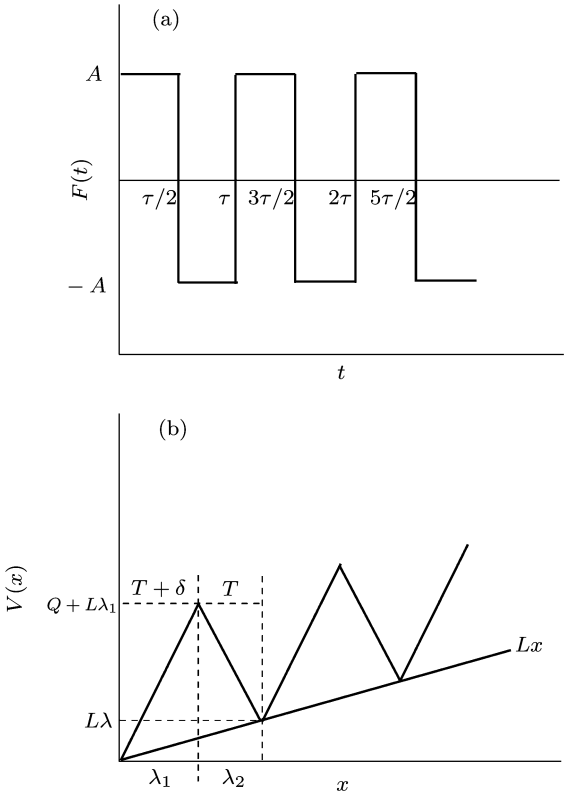


Fig. 1. (a) Temporal evolution of the driving force $F(t)$ which preserves the zero-mean $\langle F(t)\rangle = 0$, A is the amplitude of $F(t)$. (b) Schematic illustration of the potential, $V(x) = V_0(x) + V_{\mathrm{L}}(x)$. The temperature profile is also shown.

If $F(t)$ changes very slowly with respect to t, i.e., its period is longer than any other time scales of the system, there exists a quasi-steady state. In this case, the expression for the current in the adiabatic limit can be obtained[45]

$$j(F(t)) = \frac{\alpha}{\beta_1\beta_2 - \alpha\gamma}, \tag{7}$$

where α, β_1, β_2, and γ read

$$\alpha = 1 - \mathrm{e}^{a-b},$$
$$\beta_1 = \frac{\lambda+\Delta}{2a(T+\delta)}(1-\mathrm{e}^{-a}) + \frac{\lambda-\Delta}{2bT}\mathrm{e}^{-a}(\mathrm{e}^{b}-1),$$
$$\beta_2 = \frac{\lambda+\Delta}{2a}(\mathrm{e}^{a}-1) + \frac{\lambda-\Delta}{2b}\mathrm{e}^{a}(1-\mathrm{e}^{-b}),$$
$$\gamma = P_1 + P_2 + P_3,$$
$$P_1 = \frac{1}{T+\delta}\left(\frac{\lambda+\Delta}{2a}\right)^2(a+\mathrm{e}^{-a}-1),$$
$$P_2 = \frac{\lambda^2-\Delta^2}{4abT}(1-\mathrm{e}^{-a})(\mathrm{e}^{b}-1),$$
$$P_3 = \frac{1}{T}\left(\frac{\lambda-\Delta}{2b}\right)^2(\mathrm{e}^{b}-1-b),$$
$$a = \frac{2Q-(F(t)-L)(\lambda+\Delta)}{2(T+\delta)},$$
$$b = \frac{2Q+(F(t)-L)(\lambda-\Delta)}{2T}.$$

The average current over the fluctuation period is obtained

$$J = \frac{1}{\tau}\int_0^\tau j(F(t))\mathrm{d}t.$$

For such periodic external driving force $F(t)$ of the square wave of amplitude A within the interval of $0 \le t \le \tau$

$$F(t) = \begin{cases} A, & 0 \le t < \tau/2, \\ -A, & \tau/2 \le t \le \tau, \end{cases} \tag{8}$$

the average current is

$$J = \frac{1}{2}[j(A) + j(-A)], \tag{9}$$

where $j(A)$ is a current induced by a constant force $F(t) = A$, and $j(-A)$ is a current induced by a constant force $F(t) = -A$ in Eq. (6).

According to the energetic analysis,[33,34] the input energy R per unit time from the external fluctuation to the ratchet and the useful work W per unit time that the ratchet system extracts from the fluctuation are written, respectively,

$$R = \frac{1}{t_k - t_i}\int_{x=x(t_i)}^{x=x(t_k)} F(t)\mathrm{d}x(t),$$

$$W = \frac{1}{t_k - t_i}\int_{x=x(t_i)}^{x=x(t_k)} \mathrm{d}V[x(t)].$$

For the square wave with amplitude A, we obtain

$$\langle R\rangle = \frac{1}{2}[\langle R(A)\rangle + \langle R(-A)\rangle] = \frac{1}{2}A[j(A) - j(-A)], \tag{10}$$

$$\langle W\rangle = \frac{1}{2}L[j(A) + j(-A)]. \tag{11}$$

An important parameter measuring the operating performance of a Brownian particle is the efficiency of energy transformation, which is generally defined as the ratio of the useful work done by the particle against the load to the input energy.[33,34,46] Therefore, we can obtain the efficiency of energy transformation

$$\eta = \frac{\langle W\rangle}{\langle R\rangle} = \frac{L(j(A)+j(-A))}{A(j(A)-j(-A))}. \tag{12}$$

Because $j(-A)/j(A) < 0$,[37] Equation (12) can be rewritten as

$$\eta = \frac{L}{A}\left(1 - \frac{2\left|\frac{j(-A)}{j(A)}\right|}{1+\left|\frac{j(-A)}{j(A)}\right|}\right). \tag{13}$$

Equation (13) shows that the efficiency η depends on $|j(-A)/j(A)|$. In the limit that $|j(-A)/j(A)| \to 0$, the maximum efficiency of energy transformation for given load L and force amplitude A is realized: $\eta_{\max} = L/A$. We note that the energy transformation from fluctuation to coherent work is impossible in the region where J is negative. Therefore, for a finite work against load, we assume $L \leq A$ throughout this paper.

In Fig. 2, we plot the current J as a function of temperature T for different values of temperature difference δ (see Fig. 2(a)), and for different values of the amplitude A of the external driving force (see Fig. 2(b)), respectively. The current J is certainly maximized at a finite temperature, which indicates that the thermal fluctuations can facilitate the transport of the Brownian motor at finite values. The peak height of J increases with δ and A, respectively. Thus, δ and A play the same role to J. In other words, the temperature difference δ and the amplitude A of the external driving force enhance the transport of the Brownian motor.

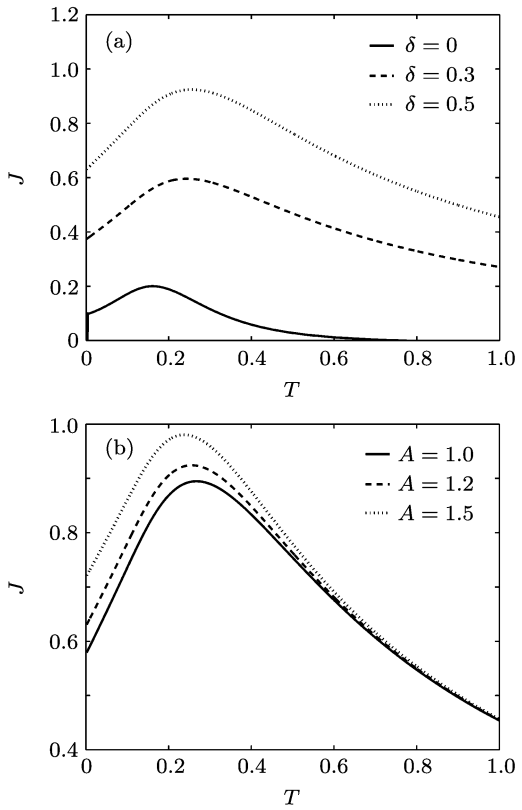


Fig. 2. Plot of the current J as a function of T for different values of (a) temperature difference δ and for different values of (b) the amplitude A of the external driving force. Other parameters: $k_{\rm B} = 1$, $Q = 1.0, \lambda = 1.0, \Delta = 1.0$, and $L = 0.01$. In Fig. 2(a) $A = 1.2$. In Fig. 2(b) $\delta = 0.5$.

In Fig. 3, we study the effect of the temperature difference δ on the efficiency. When the temperature difference $\delta = 0$, the efficiency is a monotonically decreasing function of temperature. When the temperature difference $\delta > 0$ (see $\delta = 0.3$ and 0.5 in Fig. 3), the efficiency is a peaked function of temperature, and thus the efficiency can be optimized with respect to the temperature. For a fixed value of T, the efficiency increases with δ. This implies that the thermal fluctuations may facilitate the energy transformation at finite values for $\delta > 0$, and the temperature difference can enhance the efficiency of the energy transformation.

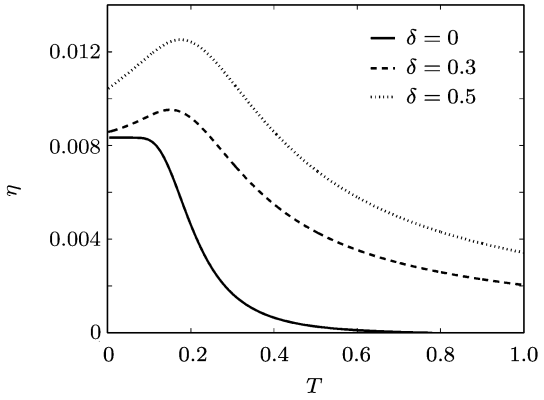


Fig. 3. Plot of the efficiency η as a function of T for different values of temperature difference δ. Other parameters are the same as in Fig. 2.

In Figs. 4(a) and 4(b), we plot the efficiency as a function of temperature for different values of the amplitude A of the external driving force. In Fig. 4(a), when the temperature difference $\delta = 0$, it is shown that the efficiency is a monotonically decreasing function of temperature, and the efficiency decreases with the amplitude A. However, when the temperature difference $\delta > 0$ (i.e., $\delta = 0.5$), as shown in Fig. 4(b), it is found that the efficiency is a peaked function of temperature for smaller amplitude values of the external driving force (see $A = 1.0$, 1.2, and 1.5 in Fig. 4(b)), while for a larger amplitude value of the external driving force (see $A = 6.0$ in the inset of Fig. 4(b)), the efficiency is a decreasing function of temperature.

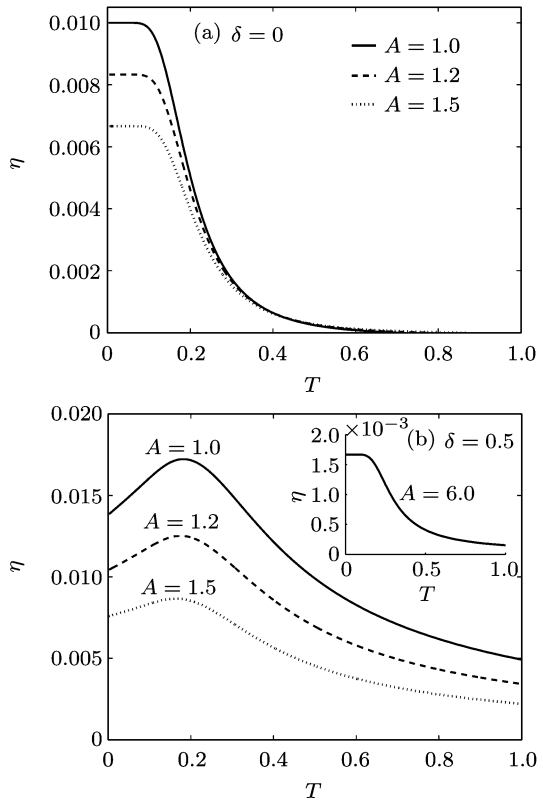


Fig. 4. Plot of the efficiency η as a function of T for different values of the amplitude A of the external driving force. (a) $\delta = 0$; (b)$\delta = 0.5$. Other parameters are the same as in Fig. 2.

It is shown from Figs. 3 and 4 that a Brownian motor is driven by two driving factors. The competition between the temperature difference and the amplitude of the external driving force can lead to the efficiency optimization, i.e., $\delta > 0$ and a smaller amplitude value of the external driving force are necessary for the efficiency optimization. Different from the current case, where δ and A play the same role, in our case δ and A play opposite roles with respect to the efficiency. In other words, δ enhances the efficiency of the energy transformation and A weakens it.

In order to provide an understanding about the aforementioned phenomenon, we plot the input energy $\langle R\rangle$ and useful work $\langle W\rangle$ as a function of temperature in Figs. 5(a)–5(c). If we define the difference between $\langle R\rangle$ and $\langle W\rangle$ as $\Delta U = \langle R\rangle - \langle W\rangle$, then $\eta \propto 1/\Delta U$. For $\delta = 0$ (see Fig. 5(a)), the input energy $\langle R\rangle$ is a monotonically increasing function of temperature, while the useful work $\langle W\rangle$ (scaled up by a factor of 100 for the convenience of comparison) is an initially increasing, and then decreasing, function of temperature. The difference ΔU increases with the temperature, which indicates that the efficiency is a decreasing function of temperature (see Fig. 4(a)). For $\delta > 0$ and a smaller amplitude value of the external driving force, as shown in Fig. 5(b), the difference between $\langle R\rangle$ and $\langle W\rangle$ (scaled up by a factor of 70 for the convenience of comparison) initially decreases and then increases with increasing temperature. Therefore, the efficiency η can be maximized at a finite temperature (see $A = 1.0$, 1.2, and 1.5 in Fig. 4(b)). However, when the amplitude value of the external driving force is larger (see Fig. 5(c)), the difference between $\langle R\rangle$ and $\langle W\rangle$ (scaled up by a factor of 600 for the convenience of comparison) is unchanged for low temperature, while for high temperature the difference between $\langle R\rangle$ and $\langle W\rangle$ increases. Therefore, the efficiency η is unchanged for low temperature, and for high temperature the efficiency decreases (see $A = 6.0$ in the inset of Fig. 4(b)).

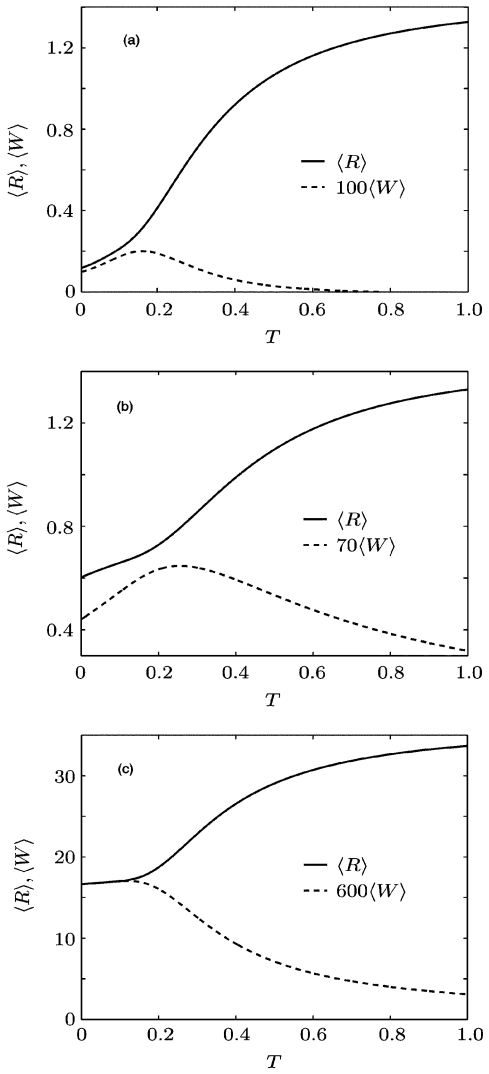


Fig. 5. Plots of the input energy $\langle R\rangle$ and useful work $\langle W\rangle$ as a function of T. (a) $\delta = 0.0$ and $A = 1.2$; (b) $\delta = 0.5$ and $A = 1.2$; (c) $\delta = 0.5$ and $A = 6.0$. Other parameters are the same as in Fig. 2.

3. Concluding remarks

The transport of a Brownian motor moving in a sawtooth potential is investigated in the presence of an external driving force and two heat reservoirs. The viscous medium is alternately in contact with two heat reservoirs. Based on the corresponding Fokker–Planck equation, we obtain the analytical expressions of the current and efficiency in the quasi-steady-state limit. It is found that the current as a function of temperature exhibits a maximum, which indicates that the thermal fluctuations can facilitate the transport of the Brownian motor. The peak height of the current increases with both δ and A. δ and A play the same role with respect to the current, i.e., the temperature difference δ and the amplitude A of the external driving force enhance the transport of the Brownian motor. For $\delta = 0$, the efficiency is a monotonically decreasing function of temperature. For $\delta > 0$ and when the amplitude value of the external driving force is smaller, the efficiency is a peaked function of temperature, where the efficiency can be optimized with respect to the temperature, i.e., $\delta > 0$ and a smaller amplitude value of the external driving force are necessary for efficiency optimization. However, when the amplitude value of the external driving force is larger, the efficiency is a monotonically decreasing function of temperature. The efficiency increases with δ, and decreases with A. δ and A play opposite roles with respect to the efficiency. In other words, δ enhances the efficiency of the energy transformation and A weakens it. In addition, the temperature difference can enhance the efficiency of the energy transformation. This agrees with the diffuse law.

To summarize, it is remarkable that the interplay between two different driving factors generates a rich variety of cooperation effects such as efficiency optimization. We expect that our analysis will facilitate the control over molecular motors and other microscale phenomena.

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