Entangled quantum heat engine based on two-qubit Heisenberg XY model*

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Based on a two-qubit isotropic Heisenberg XY model under a constant external magnetic field, we construct a four-level entangled quantum heat engine (QHE). The expressions for the heat transferred, the work, and the efficiency are derived. Moreover, the influence of the entanglement on the thermodynamic quantities is investigated analytically and numerically. Several interesting features of the variations of the heat transferred, the work, and the efficiency with the concurrences of the thermal entanglement of two different thermal equilibrium states in zero and nonzero magnetic fields are obtained.

Keywords: isotropic Heisenberg XY model, thermal entanglement, quantum heat engine, performance characteristics

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1. Introduction

The main objective of quantum thermodynamics is to design and study new thermodynamic processes in the domain where quantum features of matter are relevant. The current active fields in quantum thermodynamics include quantum heat engines,[1−24] work-extraction processes from quantum systems,[25−27] conditions of positive work,[28] and so on. Quantum heat engines (QHEs) produce work using quantum systems, such as the spin system,[2,9−17] harmonic oscillator system,[3−8] two-level or multilevel system,[18,20,22] cavity quantum electrodynamics system[19,21] and coupled two-level system[23] as the working substances. Because of the quantum features of the working substance, many unusual and exotic properties have been found. For example, the QHE can produce finite power at an efficiency close to[25] and even surpassing[21] the Carnot efficiency bound at some conditions.

It is well known that many interesting phenomena in quantum systems are attributable to the existence of the entanglement. Quantum entanglement is studied intensely due to its potential applications in quantum communication and information processing.[29−34] However, most of the QHEs mentioned above only use general quantum systems as the working substance. QHEs working with quantum entangled systems have been rarely investigated. Recently, Zhang *et al.* investigated a four-level entangled quantum heat engine based on a two-qubit isotropic Heisenberg XXX model in a constant external magnetic field.[35] Wang *et al.* studied the performance characteristics of an entangled quantum Otto engine using two two-level identical atoms coupled to a single-mode optical cavity as the working substance.[36] Zhang analyzed entangled QHEs based on two two-spin systems with the Dzyaloshinski–Moriya anisotropic antisymmetric interaction.[37] Some novel results were obtained. To enrich the theory of the entangled quantum heat engines, we construct QHEs with a two-qubit isotropic Heisenberg XY model in a constant external magnetic field. We will show some interesting connections between the entanglement and the basic thermodynamic quantities.

2. Review of two-qubit isotropic Heisenberg XY model

The Hamiltonian of a two-qubit isotropic Heisenberg XY model in a constant external magnetic field B is given by[30]

$$H = \frac{B}{2}(\sigma_1^z + \sigma_2^z) + J(\sigma_1^+\sigma_2^- + \sigma_2^+\sigma_1^-), \qquad (1)$$

where $\sigma^+ = \frac{1}{2}(\sigma^x + \mathrm{i}\,\sigma^y)$ and $\sigma^- = \frac{1}{2}(\sigma^x - \mathrm{i}\,\sigma^y)$ are the raising and the lowering operators, σ^x, σ^y, and σ^z are

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the Pauli operators, and J is the exchange constant, $J > 0$ and $J < 0$ correspond to the antiferromagnetic and the ferromagnetic cases, respectively.

The eigenvalues and the eigenvectors of Hamiltonian H are

$$\begin{aligned} H|00\rangle &= -B|00\rangle, \\ H|11\rangle &= B|11\rangle, \\ H|\psi^{\pm}\rangle &= \pm J|\psi^{\pm}\rangle, \end{aligned} \tag{2}$$

where $|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ are the maximally entangled states.

A state is described by density operator $\rho(T) = \exp(-H/kT)/Z$ when the system is in the equilibrium state at temperature T, where $Z = \mathrm{Tr}[\exp(-H/kT)]$ is the partition function, and k is the Boltzmann constant.

In the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, density matrix $\rho(T)$ is written as

$$\rho(T) = \frac{1}{2(\cosh\frac{J}{T} + \cosh\frac{B}{T})} \times \begin{pmatrix} \mathrm{e}^{-\frac{B}{T}} & 0 & 0 & 0 \\ 0 & \cosh\frac{J}{T} & -\sinh\frac{J}{T} & 0 \\ 0 & -\sinh\frac{J}{T} & \cosh\frac{J}{T} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-\frac{B}{T}} \end{pmatrix}. \tag{3}$$

For convenience, we assume $k = 1$. The concurrence of the thermal entanglement is given by

$$c = \max\left(\frac{\sinh\frac{J}{T} - 1}{\cosh\frac{J}{T} + \cosh\frac{B}{T}}, 0\right). \tag{4}$$

3. Model of a thermal entangled QHE

The Hamiltonian of an arbitrary quantum system with a finite number of energy levels may be expressed as

$$H = \sum_i E_i |i\rangle\langle i|, \tag{5}$$

where $|i\rangle$ is the i-th eigenstate of the system, and E_i is the corresponding eigenenergy. The expectation value of the Hamiltonian is

$$U = \langle H\rangle = \sum_i p_i E_i, \tag{6}$$

where p_i is the occupation probability in the i-th eigenstate. From Eq. (6), we have

$$\mathrm{d}U = \sum_i (E_i \mathrm{d}p_i + p_i \mathrm{d}E_i). \tag{7}$$

Comparing Eq. (7) with the first law of thermodynamics

$$\mathrm{d}U = \bar{\mathrm{d}}Q + \bar{\mathrm{d}}W, \tag{8}$$

we can make the following identifications for infinitesimal heat transferred $\bar{\mathrm{d}}Q$ and work done $\bar{\mathrm{d}}W$:

$$\begin{aligned} \bar{\mathrm{d}}Q &:= \sum_i E_i \mathrm{d}p_i, \\ \bar{\mathrm{d}}W &:= \sum_i p_i \mathrm{d}E_i. \end{aligned} \tag{9}$$

Thus, equation (7) is the first law of thermodynamics in the quantum system. Equation (9) implies that the work done on or by a system can be obtained only through a change in the generalized coordinates of the system, which in turn gives rise to a change in the distribution of the energy levels.[25]

The cycle of the entangled quantum heat engine considered here is composed of two isothermal and two adiabatic processes, and the working substance is described by a two-qubit isotropic Heisenberg XY model. The schematic diagram of the cycle is shown in Fig. 1.

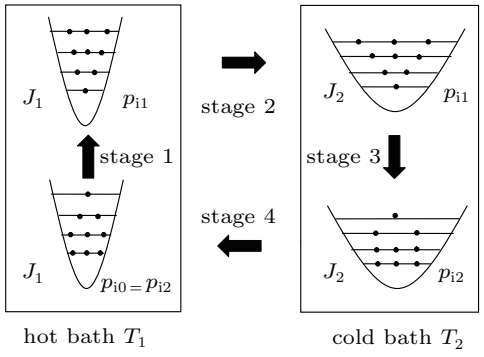


Fig. 1. Schematic diagram of the entangled quantum heat engine.

At stage 1, the working substance is coupled to bath 1 at temperature T_1. At the beginning, the probability of each eigenstate is p_{i0} $(i = 1, 2, 3, 4)$. After some contact time, the probability of the eigenstate changes from p_{i0} to p_{i1}, while the eigenenergy is fixed at E_{i1}, and the exchange constant is maintained at J_1. Thus, only heat is transferred and no work is performed at this stage.

At stage 2, the working substance is decoupled from bath 1 and undergoes a quantum adiabatic expansion to decrease the eigenenergy from E_{i1} to a smaller value E_{i2}, while the probability of each eigenstate is maintained at p_{i1} according to the quantum adiabatic theorem.[38] The exchange constant changes from J_1 to J_2. Thus, some amount of work is performed and no heat is transferred at this stage.

At stage 3, the working substance is coupled to bath 2 at temperature T_2. After some contact time, the probability of each eigenstate changes from p_{i1} to p_{i2}, while the eigenenergy is fixed at E_{i2}, and the exchange constant is maintained at J_2. Thus, only heat is transferred and no work is performed at this stage.

At stage 4, the working substance is decoupled from bath 2 and undergoes a quantum adiabatic compression to increase the eigenenergy from E_{i2} back to the larger value E_{i1}, while the probability of each eigenstate is maintained at p_{i2}, and the exchange constant changes from J_2 back to J_1. Thus, some amount of work is performed and no heat is transferred at this stage. Due to the cyclicity of the heat engine, there is a constraint on the probability, i.e., $p_{i2} = p_{i0}$.

According to Eq. (2), the eigenvalues in baths 1 and 2 are given by

$$\begin{aligned} E_{11} &= -J_1, \\ E_{21} &= -B, \\ E_{31} &= B, \\ E_{41} &= J_1, \end{aligned} \tag{10a}$$

and

$$\begin{aligned} E_{12} &= -J_2, \\ E_{22} &= -B, \\ E_{32} &= B, \\ E_{42} &= J_2, \end{aligned} \tag{10b}$$

respectively. If the system contacts with the two baths sufficiently at stages 1 and 3, they will reach the thermal equilibrium states. Then probabilities p_{i1} and p_{i2} only depend on the temperatures of the baths and the eigenenergy, i.e.,

$$p_{ij} = \frac{\mathrm{e}^{-E_{ij}/kT_j}}{\sum\limits_i \mathrm{e}^{-E_{ij}/kT_j}}, \tag{11}$$

where $i = 1, 2, 3, 4$ and $j = 1, 2$.

According to Eq. (9), the amount of heat absorbed from bath 1 at stage 1 is

$$Q_1 = \sum_i E_{i1}(p_{i1} - p_{i2}), \tag{12}$$

and the amount of heat released to bath 2 at stage 3 is

$$Q_2 = \sum_i E_{i2}(p_{i2} - p_{i1}), \tag{13}$$

where $Q_1 > 0$ and $Q_2 < 0$ must be satisfied. The work performed by the QHE is

$$W = Q_1 + Q_2 = \sum_i (E_{i1} - E_{i2})(p_{i1} - p_{i2}), \tag{14}$$

where $T_1 > T_2$ and $W > 0$. The efficiency can be defined as

$$\eta = \frac{W}{Q_1} = 1 + \frac{Q_2}{Q_1}. \tag{15}$$

4. Basic thermodynamic quantities and thermal entanglement

According to Eq. (4), the concurrences of the thermal entanglement in the two thermal equilibrium states are

$$c_1 = \max\left(\frac{\sinh\frac{J_1}{T_1} - 1}{\cosh\frac{J_1}{T_1} + \cosh\frac{B}{T_1}}, 0\right), \tag{16a}$$

$$c_2 = \max\left(\frac{\sinh\frac{J_2}{T_2} - 1}{\cosh\frac{J_2}{T_2} + \cosh\frac{B}{T_2}}, 0\right). \tag{16b}$$

The entangled state means $c_1 > 0$ and $c_2 > 0$. From Eq. (16), we may obtain

$$J_1 = -B + T_1 \ln\left[\frac{2\,\mathrm{e}^{B/T_1} + c_1 + c_1\,\mathrm{e}^{2B/T_1} + \sqrt{8\,\mathrm{e}^{2B/T_1} + 4c_1\,\mathrm{e}^{3B/T_1} + c_1^2 - 2c_1^2\,\mathrm{e}^{2B/T_1} + c_1^2\,\mathrm{e}^{4B/T_1}}}{2(1 - c_1)}\right], \tag{17a}$$

$$J_2 = -B + T_2 \ln\left[\frac{2\,\mathrm{e}^{B/T_2} + c_2 + c_2\,\mathrm{e}^{2B/T_2} + \sqrt{8\,\mathrm{e}^{2B/T_2} + 4c_2\,\mathrm{e}^{3B/T_2} + c_2^2 - 2c_2^2\,\mathrm{e}^{2B/T_2} + c_2^2\,\mathrm{e}^{4B/T_2}}}{2(1 - c_2)}\right]. \tag{17b}$$

Substituting Eq. (17) into Eqs. (13)–(15), we obtain the expressions for Q_1, Q_2, W, and η as functions of c_1, c_2, T_1, T_2, and B.

We first investigate the simplest case without magnetic field. When $B = 0$, equation (17) becomes

$$J_1 = T_1 \ln\left[\frac{1 + c_1 + \sqrt{2 + c_1}}{(1 - c_1)}\right], \tag{18a}$$

$$J_2 = T_2 \ln\left[\frac{1 + c_2 + \sqrt{2 + c_2}}{(1 - c_2)}\right]. \tag{18b}$$

The amount of heat transferred, the work, and the efficiency may be simplified as

$$Q_1 = J_1\left(\frac{\mathrm{e}^{J_1/T_1} - \mathrm{e}^{-J_1/T_1}}{1 + 1 + \mathrm{e}^{J_1/T_1} + \mathrm{e}^{-J_1/T_1}} - \frac{\mathrm{e}^{J_2/T_2} - \mathrm{e}^{-J_2/T_2}}{1 + 1 + \mathrm{e}^{J_2/T_2} + \mathrm{e}^{-J_2/T_2}}\right), \tag{19a}$$

$$Q_2 = -J_2\left(\frac{\mathrm{e}^{J_1/T_1} - \mathrm{e}^{-J_1/T_1}}{1 + 1 + \mathrm{e}^{J_1/T_1} + \mathrm{e}^{-J_1/T_1}} - \frac{\mathrm{e}^{J_2/T_2} - \mathrm{e}^{-J_2/T_2}}{1 + 1 + \mathrm{e}^{J_2/T_2} + \mathrm{e}^{-J_2/T_2}}\right), \tag{19b}$$

$$W = (J_1 - J_2)\left(\frac{\mathrm{e}^{J_1/T_1} - \mathrm{e}^{-J_1/T_1}}{1 + 1 + \mathrm{e}^{J_1/T_1} + \mathrm{e}^{-J_1/T_1}} - \frac{\mathrm{e}^{J_2/T_2} - \mathrm{e}^{-J_2/T_2}}{1 + 1 + \mathrm{e}^{J_2/T_2} + \mathrm{e}^{-J_2/T_2}}\right), \tag{20}$$

$$\eta_0 = 1 - \frac{J_2}{J_1} = 1 - \frac{T_2 \ln\left[\dfrac{1 + c_2 + \sqrt{2 + c_2}}{(1 - c_2)}\right]}{T_1 \ln\left[\dfrac{1 + c_1 + \sqrt{2 + c_1}}{(1 - c_1)}\right]}, \quad (21)$$

where Q_1 and Q_2 obviously have contrary signs.

In order to analyze how the entanglement affects the basic thermodynamic quantities of the heat engine in detail, we plot an isoline map to show the variation of efficiency with c_1 and c_2, as shown in Fig. 2. It is found from Fig. 2 that all of the isolines are above the diagonal $c_1 = c_2$ line. It means that only when $c_1 < c_2$ can the QHE generate positive work. The Carnot efficiency, defined as $\eta_c = 1 - T_2/T_1$, cannot be achieved. The efficiency increases monotonically with c_1 when c_2 is fixed, and decreases monotonically with c_2 when c_1 is fixed. When $c_1 = c_2$, the efficiency of the QHE is equal to the Carnot efficiency, i.e., $\eta_0 = \eta_c$.

Next we study the complicated case of nonzero magnetic field. In this case, it is too complicated to give analytical expressions for the basic thermodynamic quantities, so we show them graphically. We choose three representative B values to plot the variations of basic thermodynamic quantities with c_1 and c_2, as shown in Figs. 3–5. It is found from Figs. 3–5 that the isolines of the amount of heat transferred are open lines. And $Q_1 > -Q_2 > 0$ is always true. The amount of heat transferred decreases monotonically with c_1 when c_2 is fixed and increases monotonically with c_2 when c_1 is fixed. But the isolines of the work and the efficiency are loops. Thus, they no longer change monotonically with c_1 or c_2. The loop becomes smaller as the magnetic field increases. The magnetic field not only affects the shape of the efficiency but also affects its value. The acceptable ranges of c_1 and c_2 vary with the magnetic field. In a larger magnetic field, the loops also appear when $c_1 > c_2$, whereas in a smaller magnetic field, it seems that only $c_1 < c_2$ is relevant.

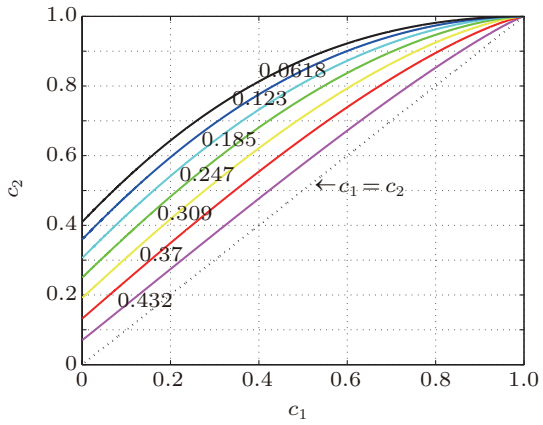


Fig. 2. (colour online) Variation of efficiency η_0 with c_1 and c_2 in an isoline map of efficiency for $kT_1 = 2$, $kT_2 = 1$, and $B = 0$.

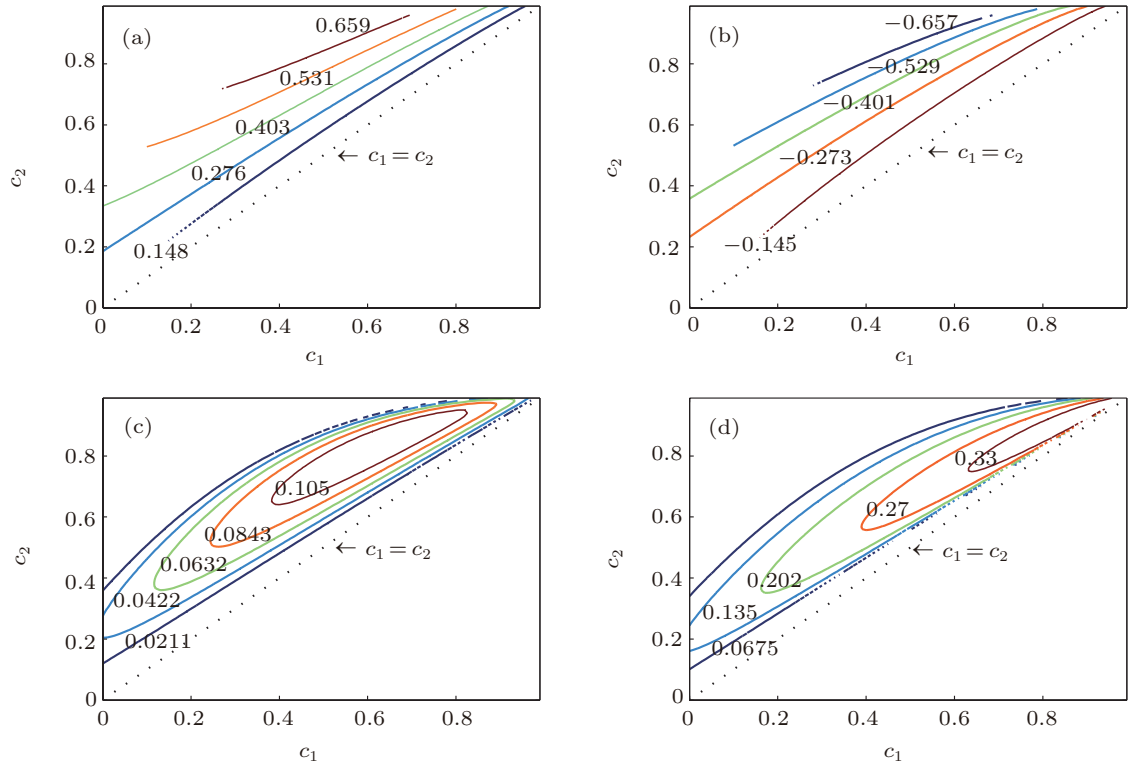


Fig. 3. (colour online) Variations of (a) Q_1, (b) Q_2, (c) W, and (d) η with c_1 and c_2 in isoline maps of efficiency for $kT_1 = 2$, $kT_2 = 1$, and $B = 1$.

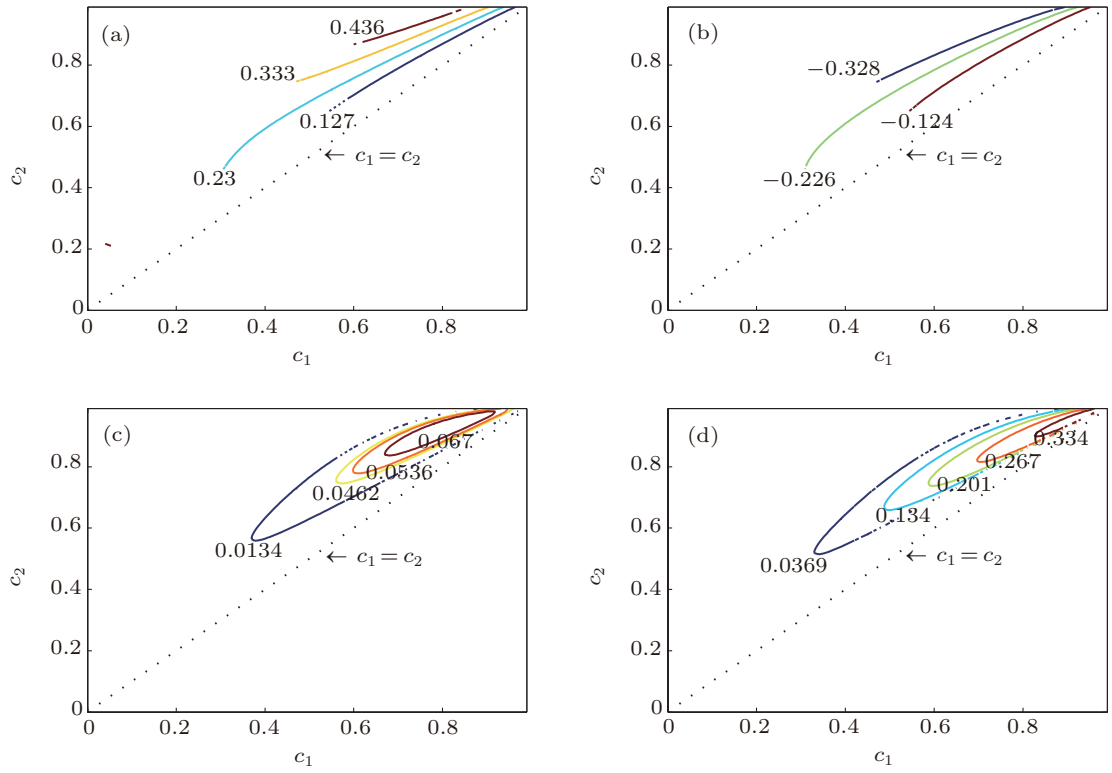


Fig. 4. (colour online) Variations of (a) Q_1, (b) Q_2, (c) W, and (d) η with c_1 and c_2 in isoline maps of efficiency for $kT_1 = 2$, $kT_2 = 1$, and $B = 3$.

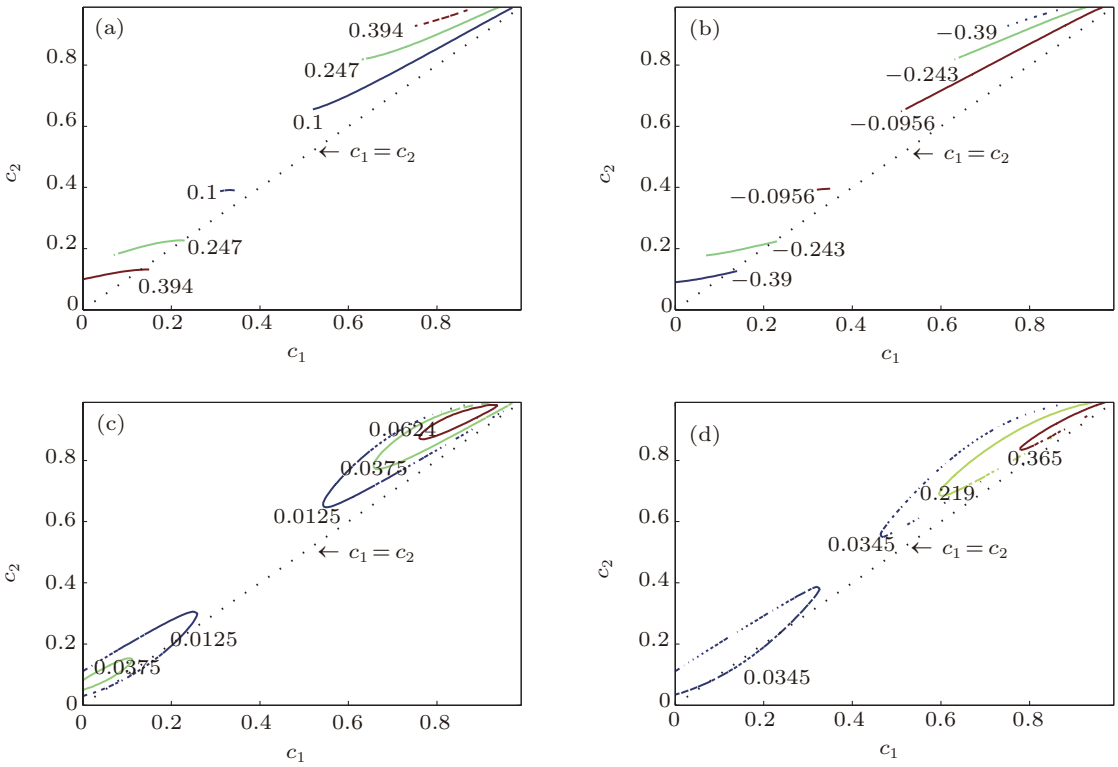


Fig. 5. (colour online) Variations of (a) Q_1, (b) Q_2, (c) W, and (d) η with c_1 and c_2 in isoline maps of efficiency for $kT_1 = 2$, $kT_2 = 1$, and $B = 5$.

5. Conclusion

We have constructed a QHE with the two-qubit isotropic Heisenberg XY model in a constant external magnetic field in this paper. Based on the thermal entanglement and the first law of thermodynamics in the quantum system, the expressions for the heat transferred, the work, and the efficiency have been obtained. It is found that the amount of heat transferred decreases (increases) monotonically with c_1 (c_2) when c_2 (c_1) is fixed. But the isolines of the work and the efficiency are loops. The Carnot efficiency cannot be achieved.

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