Solving a class of burning disturbed problem with shock layers*

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A class of combustion problem with shock layers is considered. A modified perturbation method is presented. Using this simple and valid technique, we construct the boundary and the shock layers solution to the problem, and the asymptotic behavior of the solution is discussed. The modifying perturbation method is shown to be a valid method.

Keywords: singular perturbation, burning problem, asymptotic solution

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1. Introduction to the burning model

The nonlinear singularly perturbation problem is a very attractive problem in international academic circles.[1,2] Recently, many approximate methods have been refined, including the method of averaging, the boundary layer method, the methods of matched asymptotic expansion and multiple scales. Many scholars, such as Hovhannisyan and Vulanovic,[3] Ramos,[4] D'Aprile and Pistoia,[5] Kellogg and Kopteva,[6] and Faye *et al.*,[7] have done a great deal of work in this area. Using the differential inequalities and other methods, Mo *et al.* considered a class of singularly perturbed nonlinear problems.[8–14] In this paper, we study a class of nonlinear burning singularly perturbation problem using a simple method.

We consider the following burning disturbed equation:[15,16]

$$\varepsilon\,(y'' + a\,y') = (y-t)(y+t), \quad -1 < t < 1, \quad (1)$$

where y is the density of the combustion flame, ε denotes the rate of diffusion effect for the reaction speed, which is a small positive parameter, $\varepsilon\,a$ is the variation rate of density for the combustion flame, and t is the position of combustion. Functions $y-t$ and $y+t$ are the positions of fuel and oxidizer, respectively.

The burning disturbed problem has many applications in theoretical physics, fluid mechanics, aeronautics and cosmonautics, applied chemistry, ecology environment, metallurgical engineering, and so on.

The density of the combustion flame in the burning disturbed problem satisfies the Robin boundary conditions at $t=-1$ and $t=1$

$$y(-1) - k_1\varepsilon^{1/2}\frac{\mathrm{d}y}{\mathrm{d}t}(-1) = A, \quad (2)$$

$$y(1) + k_2\varepsilon^{1/2}\frac{\mathrm{d}y}{\mathrm{d}t}(1) = B, \quad (3)$$

where $k_i > 0$ $(i=1,2)$, A and B are constants.

It is easy to see that the reduced solution Y of the burning disturbed problem (1)–(3) satisfies the equation

$$(Y-t)(Y+t) = 0. \quad (4)$$

From Eq. (4), we can see that problem (1)–(3) has a corner layer behavior solution

$$Y(t) = |\,t\,|, \quad t \in (-1,0)\cup(0,1). \quad (5)$$

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For the reduced solution (5), its derivative is discontinuous at $t = 0$ and may not satisfy the boundary conditions (2) and (3), so we need to construct a shock layer near $t = 0$ and boundary layers near $t = -1$ and $t = 1$ for correction.

2. Shock-layer correction

Now we consider the transitional nonlinear equation near $t = 0$

$$\varepsilon\,\bar{y}'' = \bar{y}^2. \tag{6}$$

Let Eq. (6) have an algebraic type asymptotic solution $\bar{y} = C_0(1 + \alpha_0\,|\,t\,|)^{\beta_0}$, where C_0 is an arbitrary constant. Substituting it into Eq. (6), we have

$$\varepsilon\,C_0\beta_0(\beta_0-1)\alpha_0^2(1+\alpha_0\,|\,t\,|)^{\beta_0-2} = C_0^2(1+\alpha_0\,|\,t\,|)^{2\beta_0}.$$

Then $\alpha_0 = \sqrt{C_0/6\varepsilon}$, $\beta_0 = -2$. The burning model (1)–(3) possesses an interior layer behavior solution $\bar{y}(t) = C_0\left(1 + \sqrt{C_0/6\,\varepsilon}\,|t|\right)^{-2}$ near $t = 0$.

For Eq. (5), $\bar{y}'$ possesses a jump $Y'(+0) - Y'(-0) = 2$ at $t = 0$. Let $\bar{y}'(+0,\varepsilon) = -\bar{y}'(-0,\varepsilon) = (Y'(+0) - Y'(-0))/2 = 1$. We have $C_0 = (3\,\varepsilon/2)^{1/3}$. Therefore, we obtain a shock layer behavior solution near $t = 0$ for the burning model (1)–(3)

$$\bar{y}(t) = \left(\frac{3\,\varepsilon}{2}\right)^{1/3}[1 + (12\varepsilon)^{-1/3}\,|\,t\,|]^{-2}. \tag{7}$$

3. Boundary-layer corrections

Firstly, we consider the transitional nonlinear equation near $t = -1$

$$\varepsilon\,y_{\rm l}'' = y_{\rm l}^2. \tag{8}$$

Let Eq. (8) have an asymptotic solution $y_{\rm l} = C_1(1+\alpha_1(t+1))^{\beta_1}$, where C_1 is an arbitrary constant. Substituting it into Eq. (8), we have

$$\begin{aligned}&\varepsilon\,C_1\beta_1(\beta_1-1)\alpha_1^2(1+\alpha_1\,|\,t\,|)^{\beta_1-2}\\ &= C_1^2(1+\alpha_1\,|\,t\,|)^{2\beta_1}.\end{aligned}$$

Then $\alpha_1 = \sqrt{C_1/6\varepsilon}$, $\beta_1 = -2$. From the theory of boundary layer, the burning model (1)–(3) possesses a boundary layer behavior solution $y_{\rm l}(t) = C_1\left[1 + \sqrt{C_1/6\,\varepsilon}(t+1)\right]^{-2}$ near $t = -1$.

Considering condition (2), we have

$$\begin{aligned}&y_{\rm l}(-1) - k_1\varepsilon^{1/2}y_{\rm l}'\\ &= A - (Y(-1) - k_1\varepsilon^{1/2}Y'(-1))\\ &= (A-1) + k_1\varepsilon^{1/2}.\end{aligned}$$

That is

$$C_1 + \sqrt{\frac{2}{3}}k_1C_1^{3/2} = (A-1) + k_1\varepsilon^{1/2}.$$

It is easy to see that

$$C_1 = (C_{10} + C_{11}\varepsilon^{1/2} + C_{12}\varepsilon + \mathrm{O}(\varepsilon^{3/2}))^2,$$

where C_{10} is a real root of equation

$$\sqrt{\frac{2}{3}}k_1C_{10}^3 + C_{10}^2 = A - 1.$$

That is

$$C_{10} = \frac{1}{\sqrt{6}k_1}(-1 + \sqrt[3]{d_{11}} + \sqrt[3]{d_{12}}), \tag{9}$$

where

$$\begin{aligned}d_{11} =\ & \frac{-1}{\sqrt{6}k_1}[2 + 9\sqrt{6}k_1(1-A)]\\ &+ [9\sqrt{6}k_1(1-A)(4+9\sqrt{6}k_1(1-A)]^{1/2},\end{aligned}$$

$$\begin{aligned}d_{12} =\ & \frac{-1}{\sqrt{6}k_1}[2 + 9\sqrt{6}k_1(1-A)]\\ &- [9\sqrt{6}k_1(1-A)(4+9\sqrt{6}k_1(1-A)]^{1/2}.\end{aligned}$$

And C_{11}, C_{12} are given by

$$\begin{aligned}C_{11} =\ & \frac{k_1}{2C_{10} + \sqrt{6}k_1C_{10}^2},\\ C_{12} =\ & -(C_{11}^2 + 2C_{10}C_{11})/\sqrt{6}k_1C_{10}^2\\ &- C_{10}C_{11}^2.\end{aligned} \tag{10}$$

Then we obtain a boundary layer behavior solution near $t = -1$ for the burning model (1)–(3)

$$y_{\rm l}(t) = \frac{6\varepsilon(C_{10} + C_{11}\varepsilon^{1/2} + C_{12}\varepsilon)^2}{[\sqrt{6\varepsilon} + (C_{10} + C_{11}\varepsilon^{1/2} + C_{12}\varepsilon)(t+1)]^2}, \tag{11}$$

where C_{1i} $(i = 0, 1, 2)$ are given by Eqs. (9) and (10).

Secondly, we consider the transitional nonlinear equation near $t = 1$

$$\varepsilon\,y_{\rm r}'' = y_{\rm r}^2. \tag{12}$$

Analogously, for Eq. (12) the burning model (1)–(3) possesses a boundary layer behavior solution $y_{\rm r}(t) = C_2(1 + \sqrt{C_2/6\,\varepsilon}(1-t))^{-2}$ near $t = 1$, where $C_2 = \left[C_{20} + C_{21}\varepsilon^{1/2} + C_{22}\varepsilon + \mathrm{O}(\varepsilon^{3/2})\right]^2$, and C_{20} is a real root of equation

$$\sqrt{\frac{2}{3}}k_2C_{20}^3 + C_{20}^2 = B - 1.$$

That is

$$C_{20} = \frac{1}{\sqrt{6}k_2}(-1 + \sqrt[3]{d_{21}} + \sqrt[3]{d_{22}}), \tag{13}$$

where

$$\begin{aligned} d_{21} &= \frac{-1}{\sqrt{6}k_2}[2 + 9\sqrt{6}k_2(1-B)] \\ &\quad + [9\sqrt{6}k_2(1-B)(4 + 9\sqrt{6}k_2(1-B))]^{1/2}, \\ d_{22} &= \frac{-1}{\sqrt{6}k_2}[2 + 9\sqrt{6}k_2(1-B)] \\ &\quad - [9\sqrt{6}k_2(1-B)(4 + 9\sqrt{6}k_2(1-B))]^{1/2}. \end{aligned}$$

And $C_{21},\ C_{22}$ are given by

$$\begin{aligned} C_{21} &= \frac{k_2}{2C_{20} + \sqrt{6}k_2C_{20}^2}, \\ C_{22} &= -(C_{21}^2 + 2C_{20}C_{21})/\sqrt{6}k_2C_{20}^2 - C_{20}C_{21}^2. \end{aligned} \tag{14}$$

Then we obtain a boundary layer behavior solution near $t = 1$ for the burning model (1)–(3)

$$y_{\rm r}(t) = \frac{6\varepsilon(C_{20} + C_{21}\varepsilon^{1/2} + C_{22}\varepsilon)^2}{[\sqrt{6\varepsilon} + (C_{20} + C_{21}\varepsilon^{1/2} + C_{22}\varepsilon)(1-t)]^2}, \tag{15}$$

where C_{2i} $(i = 0, 1, 2)$ are given by Eqs. (13) and (14).

4. Asymptotic solution and uniform validity

From Eqs. (5), (7), (11), and (15), we obtain the asymptotic solution for the burning model (1)–(3)

$$\begin{aligned} y(t) &= |\,t\,| + \frac{6\varepsilon(C_{10} + C_{11}\varepsilon^{1/2} + C_{12}\varepsilon)^2}{[\sqrt{6\varepsilon} + (C_{10} + C_{11}\varepsilon^{1/2} + C_{12}\varepsilon)(t+1)]^2} \\ &\quad + \frac{6\varepsilon(C_{20} + C_{21}\varepsilon^{1/2} + C_{22}\varepsilon)^2}{[\sqrt{6\varepsilon} + (C_{20} + C_{21}\varepsilon^{1/2} + C_{22}\varepsilon)(1-t)]^2} \\ &\quad + \left(\frac{3\,\varepsilon}{2}\right)^{1/3}[1 + (12\varepsilon)^{-1/3}\,|\,t\,|]^{-2} + \mathrm{O}(\varepsilon), \\ &\quad -1 \le t \le 1,\ \ 0 < \varepsilon << 1, \end{aligned} \tag{16}$$

where C_{ij} $(i = 1, 2,\ j = 0, 1, 2)$ are given by Eqs. (9), (10), (13), and (14) respectively.

It is easy to see that

$$\begin{aligned} &\varepsilon\,(y'' + a\,y') - (y-1)(y+1) \\ &= -(Y-t)(Y+t) + \varepsilon\,y_{\rm l}'' - y_{\rm l}^2 + \varepsilon\,y_{\rm r}'' \\ &\quad - y_{\rm r}^2 + \varepsilon\,\bar{y}'' - \bar{y}^2 + \mathrm{O}(\varepsilon) \\ &= \mathrm{O}(\varepsilon), \quad -1 < t < 1,\ \ 0 < \varepsilon << 1, \end{aligned} \tag{17}$$

$$\begin{aligned} &y(-1) - k_1y'(-1) - A \\ &= y_{\rm l}(-1) - k_1\varepsilon^{1/2}y_{\rm l}' + (Y(-1) \\ &\quad - k_1\varepsilon^{1/2}Y'(-1)) - A \\ &= \mathrm{O}(\varepsilon), \quad 0 < \varepsilon << 1, \end{aligned} \tag{18}$$

$$y(1) + k_2y'(1) - B = \mathrm{O}(\varepsilon), \quad 0 < \varepsilon << 1. \tag{19}$$

According to the fixed point theorem[1] and with Eqs. (17)–(19), expansion (16) is the uniformly valid asymptotic solution for the burning model (1)–(3) on $t \in [-1, 1]$.

5. Example

We use the following non-dimensional parameters for the combustion flame: $a = 5,\ \ A = B = 50,\ \ k_1 = k_2 = 1,\ \ \varepsilon = 0.1$. The asymptotic solution $y_{\rm asy}(t)$ and the simulated exact solution $y_{\rm exa}(t)$ of the burning model near $t = -1$ and $t = 1$ are shown in Figs. 1 and 2, respectively.

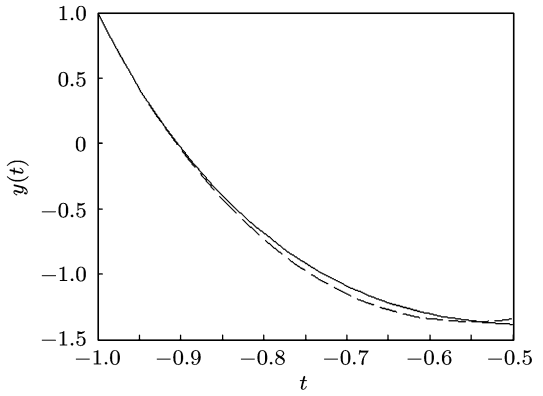


Fig. 1. Shock layer asymptotic solution $y_{\rm asy}(t)$ (dashed line) and simulated exact solution $y_{\rm exa}(t)$ (real line) near $t = 1$.

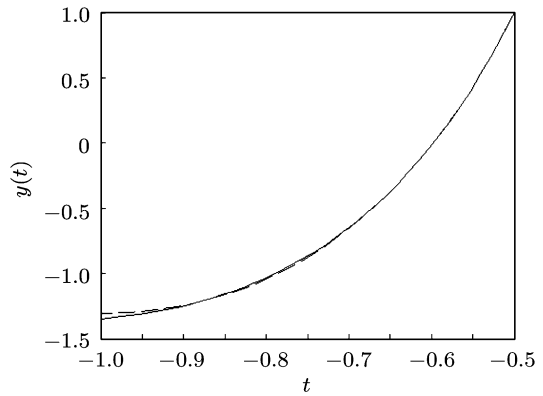


Fig. 2. Shock layer asymptotic solution $y_{\rm asy}(t)$ (dashed line) and simulated exact solution $y_{\rm exa}(t)$ (real line) near $t = -1$.

For different small parameters ε, the results are compared in Figs. 3 and 4.

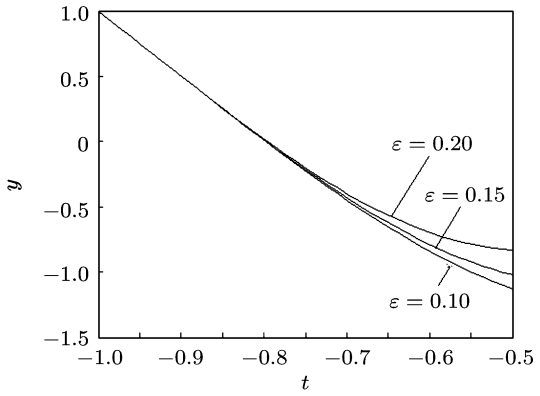


Fig. 3. Density $y(t)$ of the combustion flame near $t = -1$.

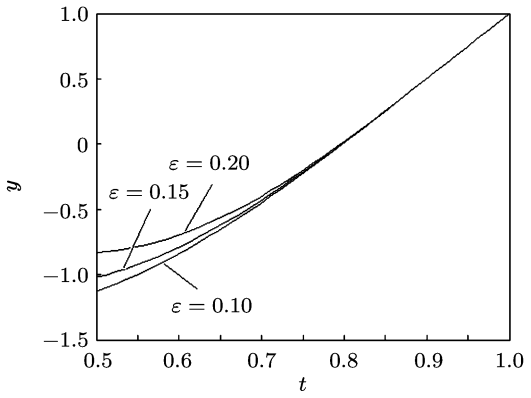


Fig. 4. Density $y(t)$ of the combustion flame near $t = 1$.

We can also obtain $\mathrm{d}y/\mathrm{d}t$ of the combustion flame for the burning model near $t = 0$ with $\varepsilon = 0.020$, $\varepsilon = 0.015$, $\varepsilon = 0.010$ (see Fig. 5).

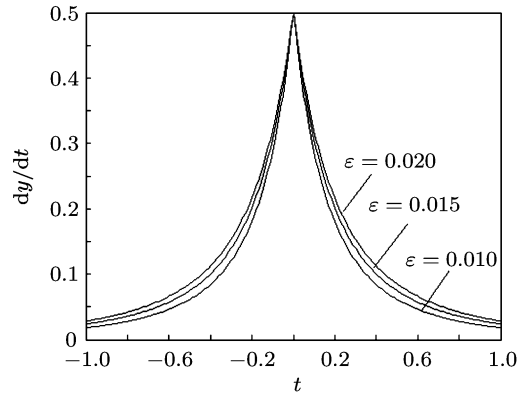


Fig. 5. Shock layer of density $\mathrm{d}y/\mathrm{d}t$ of combustion flame near $t = 0$.

From Figs. (4)–(5), we can see that when the positive parameter ε is smaller, the shock layer is more obvious.

6. Conclusion

The disturbed burning is a complicated natural phenomenon. Hence we need to reduce it to basic models. We solve the models using a modified perturbation method. The modifying perturbation method is an approximate analytic method, which differs from the general numerical method. The modifying perturbation method is shown to be a valid method. Using the modifying perturbation method, we can obtain expansions of the solution.

From the above study of the burning disturbed problem (1)–(3) with shock layers, we find that there are boundary layers near the boundary and the shock of interval. Thus we can control the density of the combustion flame using valid measures.

References

[1] de Jager E M and Jiang F R 1996 *The Theory of Singular Perturbation* (Amsterdam: North-Holland Publishing Co.)

[2] Barbu L and Morosanu G 2007 *Singularly Perturbed Boundary-Value Problems* (Basel: Birkhauserm Verlag AG)

[3] Hovhannisyan G and Vulanovic R 2008 *Nonlinear Stud.* **15** 297

[4] Ramos M 2009 *J. Math. Anal. Appl.* **352** 246

[5] D'Aprile T and Pistoia A 2010 *J. Differ. Equs.* **248** 556

[6] Kellogg R B and Kopteva N A 2010 *J. Differ. Euqs.* **248** 184

[7] Faye L, Frenod E and Seck D 2011 *Discrete Contin. Dyn. Syst.* **29** 1001

[8] Mo J Q 2011 *Acta Phys. Sin.* **60** 020202 (in Chinese)

[9] Mo J Q 2009 *Science in China Ser. G* **52** 1007

[10] Mo J Q 2009 *Chin. Phys. Lett.* **26** 010204

[11] Mo J Q 2009 *Chin. Phys. Lett.* **26** 060202

[12] Mo J Q 2011 *Commun. Theor. Phys.* **55** 387

[13] Mo J Q, Lin Y H and Lin W T 2010 *Chin. Phys.* B **19** 19030202

[14] Mo J Q, Lin W T and Lin Y H 2011 *Chin. Phys.* B **20** 070205

[15] Willams F A 1971 *Ann. Rev. Fluid Mech.* **3** 171

[16] Chang K W and Howes F A 1984 *Nonlinear Singular Perturbation Phenomena: Theory and Applications, Applied Mathemaical Science* (New York: Springer-Verlag)