Mei conserved quantity directly induced by Lie symmetry in a nonconservative Hamilton system*

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(Received 27 November 2011; revised manuscript received 9 December 2011)

In this paper, we investigate whether the Lie symmetry can induce the Mei conserved quantity directly in a nonconservative Hamilton system and a theorem is presented. The condition under which the Lie symmetry of the system directly induces the Mei conserved quantity is given. Meanwhile, an example is discussed to illustrate the application of the results. The present results have shown that the Lie symmetry of a nonconservative Hamilton system can also induce the Mei conserved quantity directly.

Keywords: Lie symmetry, Mei conserved quantity, nonconservative Hamilton system

PACS: 02.20.Sv, 11.30.–j, 45.20.Jj **DOI:** 10.1088/1674-1056/21/5/050202

1. Introduction

Symmetry is the top-level principle in mechanics and physics. There are intimate relations between symmetries and conserved quantities. If differential equations of motion of a mechanical system cannot be integrated, the existence of some conserved quantities can enable us to obtain some knowledge about the local physical state of the system, and reveal the inner essence and the laws of the system. In 1918, Noether studied the invariance of Hamilton action of a mechanical system under infinitesimal transformations, and found that every continuous symmetry of the action always has a conserved quantity associated with it, first revealed the potential relations between conserved quantities and inner symmetries of the mechanical system, and established the Noether symmetry theory.[1,2] Lutzky applied the Lie group theory to differential equations of motion of mechanical systems in 1979, and began to investigate the Lie symmetry and the conserved quantities of mechanical systems.[3] At the beginning of this century, Mei presented a new symmetry; form invariance[4,5], also known as Mei symmetry.[6,7] At present, the modern methods of finding conserved quantities by using symmetries in analytical dynamics are mainly related with the Noether symmetry, the Lie symmetry, and the Mei symmetry. Those three symmetries can induce the corresponding conserved quantities directly or indirectly, and the conserved quantities induced by them are Noether, Hojman, and Mei conserved quantities.[8−10] The Noether symmetry can induce the Noether conserved quantity directly, and the Hojman and the Mei conserved quantities indirectly; the Lie symmetry can induce the Hojman conserved quantity directly, and the Noether and the Mei conserved quantities indirectly; the Mei symmetry can induce the Mei conserved quantity directly, and the Noether and the Hojman conserved quantities indirectly.[9] In the last ten years, more and more attention has been paid to the research of symmetries and conserved quantities of constrained mechanical systems, and many important results have been obtained.[11−26] In this paper, we investigate whether the Lie symmetry can induce the Mei conserved quantity directly in a nonconservative Hamilton system. A related theorem is presented. We give the condition under which the Lie symmetry of the system can directly induce the Mei conserved quantity. Meanwhile, we discuss an example to illustrate the application of the results.

*Projct supported by the Natural Science Foundation of Shandong Province, China (Grant No. ZR2011AM012) and the Fundamental Research Funds for the Central Universities, China (Grant No. 09CX04018A).

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2. Lie and Mei symmetries of the system

The differential equations of motion of a nonconservative Hamilton system can be written as

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} + Q_s, \quad (s = 1, \ldots, n), \quad (1)$$

where $H = H(t, \boldsymbol{q}, \boldsymbol{p})$ is the Hamiltonian of the system, and Q_s are non-potential generalized forces.

Introduce the infinitesimal transformations

$$\begin{aligned} t^* &= t + \varepsilon\xi_0(t, \boldsymbol{q}, \boldsymbol{p}), \\ q_s^*(t^*) &= q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \boldsymbol{p}), \\ p_s^*(t^*) &= p_s(t) + \varepsilon\eta_s(t, \boldsymbol{q}, \boldsymbol{p}), \end{aligned} \quad (2)$$

where ε is an infinitesimal parameter, ξ_0, ξ_s, and η_s are single-parameter infinitesimal generators.

The Lie symmetry is an invariance of differential equations of motion under the infinitesimal transformations. The determining equations of Lie symmetry of a nonconservative Hamilton system (1) can be expressed as

$$\begin{aligned} &\mathrm{X}^{(1)}\left(\dot{q}_s - \frac{\partial H}{\partial p_s}\right) = 0, \\ &\mathrm{X}^{(1)}\left(\dot{p}_s + \frac{\partial H}{\partial q_s} - Q_s\right) = 0, \end{aligned} \quad (3)$$

where

$$\mathrm{X}^{(1)} = \mathrm{X}^{(0)} + (\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_k} + (\dot{\eta}_k - \dot{p}_k\dot{\xi}_0)\frac{\partial}{\partial \dot{p}_k}, \quad (4)$$

$$\mathrm{X}^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_k\frac{\partial}{\partial q_k} + \eta_k\frac{\partial}{\partial p_k}. \quad (5)$$

The Mei symmetry is an invariance under which the transformed physical quantities still satisfy the original dynamical equations. The determining equations of Mei symmetry of a nonconservative Hamilton system (1) can be expressed as

$$\begin{aligned} &\frac{\partial}{\partial p_s}[\mathrm{X}^{(0)}(H)] = 0, \\ &\frac{\partial}{\partial q_s}[\mathrm{X}^{(0)}(H)] = \mathrm{X}^{(0)}(Q_s). \end{aligned} \quad (6)$$

3. Mei conserved quantity directly induced by Lie symmetry

According to the previous studies, the Mei symmetry of a nonconservative Hamilton system can induce the Mei conserved quantity directly, while the Lie symmetry can only induce the Mei conserved quantity indirectly with the aid of the Mei symmetry.[9] We have the following results.

Theorem 1 For a nonconservative Hamilton system (1), if infinitesimal generators ξ_0, ξ_s, and η_s of the Mei symmetry and a gauge function $G(t, \boldsymbol{q}, \boldsymbol{p})$ satisfy the condition

$$\begin{aligned} &\mathrm{X}^{(0)}(p_s)\dot{\xi}_s + \frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{X}^{(0)}(p_s)]\xi_s - \mathrm{X}^{(0)}[\mathrm{X}^{(0)}(H)] \\ &\quad - \mathrm{X}^{(0)}(H)\dot{\xi}_0 + \mathrm{X}^{(0)}(Q_s)(\xi_s - \dot{q}_s\xi_0) + \dot{G} = 0, \end{aligned} \quad (7)$$

the Mei symmetry of the system can induce Mei conserved quantity

$$I_{\mathrm{M}} = \mathrm{X}^{(0)}(p_s)\xi_s - \mathrm{X}^{(0)}(H)\xi_0 + G = \mathrm{const}. \quad (8)$$

Theorem 2 For a nonconservative Hamilton system (1), if infinitesimal generators ξ_0, ξ_s, and η_s of the Lie symmetry satisfy Eq. (6), and there exists a gauge function $G(t, \boldsymbol{q}, \boldsymbol{p})$ satisfying Eq. (7), the Lie symmetry of the system can induce Mei conserved quantity (8).

Actually, the Lie symmetry of a nonconservative Hamilton system can also induce the Mei conserved quantity directly, which is shown by the following theorem.

Theorem 3 For a nonconservative Hamilton system (1), if infinitesimal generators ξ_0, ξ_s, and η_s of the Lie symmetry and a gauge function $G(t, \boldsymbol{q}, \boldsymbol{p})$ satisfy the condition

$$\begin{aligned} &\mathrm{X}^{(0)}(p_s)\dot{\xi}_s + \frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{X}^{(0)}(p_s)]\xi_s - \mathrm{X}^{(0)}(H)\dot{\xi}_0 \\ &\quad - \xi_0\dot{\xi}_0\frac{\partial H}{\partial t} - \xi_0\mathrm{X}^{(0)}\left(\frac{\partial H}{\partial t}\right) - \xi_0\dot{q}_s\mathrm{X}^{(1)}(Q_s) \\ &\quad - \xi_0\dot{\xi}_s Q_s + \dot{G} = 0, \end{aligned} \quad (9)$$

the Lie symmetry of the system can induce Mei conserved quantity (8) directly.

Proof Taking the derivative of I to t, we have

$$\begin{aligned} \frac{\mathrm{d}I_{\mathrm{M}}}{\mathrm{d}t} &= \mathrm{X}^{(0)}(p_s)\dot{\xi}_s + \frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{X}^{(0)}(p_s)]\xi_s - \mathrm{X}^{(0)}(H)\dot{\xi}_0 \\ &\quad - \frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{X}^{(0)}(H)]\xi_0 + \dot{G}. \end{aligned} \quad (10)$$

Because

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{X}^{(0)}(H)] \\ &= \dot{\xi}_0\frac{\partial H}{\partial t} + \dot{\xi}_k\frac{\partial H}{\partial q_k} + \dot{\eta}_k\frac{\partial H}{\partial p_k} \end{aligned}$$

$$
\begin{aligned}
&+\xi_0\left[\frac{\partial^2 H}{\partial t^2}+\frac{\partial^2 H}{\partial t\partial q_s}\dot{q}_s+\frac{\partial^2 H}{\partial t\partial p_s}\dot{p}_s\right]\\
&+\xi_k\left[\frac{\partial^2 H}{\partial q_k\partial t}+\frac{\partial^2 H}{\partial q_k\partial q_s}\dot{q}_s+\frac{\partial^2 H}{\partial q_k\partial p_s}\dot{p}_s\right]\\
&+\eta_k\left[\frac{\partial^2 H}{\partial p_k\partial t}+\frac{\partial^2 H}{\partial p_k\partial q_s}\dot{q}_s+\frac{\partial^2 H}{\partial p_k\partial p_s}\dot{p}_s\right]\\
=&\ \dot{\xi}_0\frac{\partial H}{\partial t}+\dot{\xi}_k\frac{\partial H}{\partial q_k}+\dot{\eta}_k\frac{\partial H}{\partial p_k}+\mathrm{X}^{(0)}\left(\frac{\partial H}{\partial t}\right)\\
&+\dot{q}_s\mathrm{X}^{(0)}\left(\frac{\partial H}{\partial q_s}\right)+\dot{p}_s\mathrm{X}^{(0)}\left(\frac{\partial H}{\partial p_s}\right),
\end{aligned}
$$

and $\dot{q}_s$ and $\dot{p}_s$ are not embraced in $\partial H/\partial q_s$ and $\partial H/\partial p_s$, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{X}^{(0)}(H)]\\
=&\ \dot{\xi}_0\frac{\partial H}{\partial t}+\dot{\xi}_k\frac{\partial H}{\partial q_k}+\dot{\eta}_k\frac{\partial H}{\partial p_k}\\
&+\mathrm{X}^{(0)}\left(\frac{\partial H}{\partial t}\right)+\dot{q}_s\mathrm{X}^{(1)}\left(\frac{\partial H}{\partial q_s}\right)\\
&+\dot{p}_s\mathrm{X}^{(1)}\left(\frac{\partial H}{\partial p_s}\right)\\
=&\ \dot{\xi}_0\frac{\partial H}{\partial t}+\dot{\xi}_k\frac{\partial H}{\partial q_k}+\dot{\eta}_k\frac{\partial H}{\partial p_k}\\
&+\mathrm{X}^{(0)}\left(\frac{\partial H}{\partial t}\right)+\dot{q}_s\mathrm{X}^{(1)}\left(\frac{\partial H}{\partial q_s}+\dot{p}_s\right)\\
&+\dot{p}_s\mathrm{X}^{(1)}\left(\frac{\partial H}{\partial p_s}-\dot{q}_s\right)-\dot{q}_s\mathrm{X}^{(1)}(\dot{p}_s)\\
&+\dot{p}_s\mathrm{X}^{(1)}(\dot{q}_s)\\
=&\ \dot{\xi}_0\frac{\partial H}{\partial t}+\dot{\xi}_s\frac{\partial H}{\partial q_s}+\dot{\eta}_s\frac{\partial H}{\partial p_s}\\
&+\mathrm{X}^{(0)}\left(\frac{\partial H}{\partial t}\right)+\dot{q}_s\mathrm{X}^{(1)}\left(\frac{\partial H}{\partial q_s}+\dot{p}_s\right)\\
&+\dot{p}_s\mathrm{X}^{(1)}\left(\frac{\partial H}{\partial p_s}-\dot{q}_s\right)-\dot{q}_s(\dot{\eta}_s-\dot{p}_s\dot{\xi}_0)\\
&+\dot{p}_s(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0).
\end{aligned}
$$

Using Eq. (1), we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{X}^{(0)}(H)]\\
=&\ \dot{\xi}_0\frac{\partial H}{\partial t}+\mathrm{X}^{(0)}\left(\frac{\partial H}{\partial t}\right)\\
&+\dot{q}_s\mathrm{X}^{(1)}\left(\frac{\partial H}{\partial q_s}+\dot{p}_s\right)\\
&+\dot{p}_s\mathrm{X}^{(1)}\left(\frac{\partial H}{\partial p_s}-\dot{q}_s\right)+\dot{\xi}_s Q_s. \qquad (11)
\end{aligned}
$$

Substituting Eq. (11) into Eq. (10), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}I_{\mathrm{M}}}{\mathrm{d}t}=&\ \mathrm{X}^{(0)}(p_s)\dot{\xi}_s+\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{X}^{(0)}(p_s)]\xi_s-\mathrm{X}^{(0)}(H)\dot{\xi}_0\\
&-\xi_0\dot{\xi}_0\frac{\partial H}{\partial t}-\xi_0\mathrm{X}^{(0)}\left(\frac{\partial H}{\partial t}\right)\\
&-\xi_0\dot{q}_s\mathrm{X}^{(1)}\left(\frac{\partial H}{\partial q_s}+\dot{p}_s\right)\\
&-\xi_0\dot{p}_s\mathrm{X}^{(1)}\left(\frac{\partial H}{\partial p_s}-\dot{q}_s\right)\\
&-\xi_0\dot{\xi}_s Q_s+\dot{G}. \qquad (12)
\end{aligned}
$$

Substituting the determining equations (3) into Eq. (12) and using Eq. (9), we obtain

$$
\frac{\mathrm{d}I_{\mathrm{M}}}{\mathrm{d}t}=0. \qquad (13)
$$

If the Hamiltonian is time-independent, equation (9) turns into

$$
\begin{aligned}
&\mathrm{X}^{(0)}(p_s)\dot{\xi}_s+\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{X}^{(0)}(p_s)]\xi_s-\mathrm{X}^{(0)}(H)\dot{\xi}_0\\
&-\xi_0\dot{q}_s\mathrm{X}^{(1)}(Q_s)-\xi_0\dot{\xi}_s Q_s+\dot{G}=0. \qquad (14)
\end{aligned}
$$

For the conserved Hamilton system, $Q_s=0$, it is easy to obtain the following deduction from Theorem 3.

Deduction For the conserved Hamilton system, if infinitesimal generators ξ_0, ξ_s, and η_s of the Lie symmetry and a gauge function $G(t,\boldsymbol{q},\boldsymbol{p})$ satisfy the condition

$$
\begin{aligned}
&\mathrm{X}^{(0)}(p_s)\dot{\xi}_s+\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{X}^{(0)}(p_s)]\xi_s-\mathrm{X}^{(0)}(H)\dot{\xi}_0\\
&-\xi_0\dot{\xi}_0\frac{\partial H}{\partial t}-\xi_0\mathrm{X}^{(0)}\left(\frac{\partial H}{\partial t}\right)+\dot{G}=0, \qquad (15)
\end{aligned}
$$

the Lie symmetry of the system can induce Mei conserved quantity (8) directly.

4. Example

The Hamiltonian of a nonconservative Hamilton system is

$$
H=\frac{1}{2}(p_1^2+p_2^2)+q_2+t^2p_1, \qquad (16)
$$

the non-potential generalized forces are

$$
Q_1=0,\quad Q_2=p_2. \qquad (17)
$$

The differential equations of motion (1) give

$$
\dot{q}_1=p_1+t^2,\quad \dot{q}_2=p_2,\quad \dot{p}_1=0,\quad \dot{p}_2=p_2-1. \qquad (18)
$$

We have

$$
\mathrm{X}^{(0)}(H)=\eta_1p_1+\eta_2p_2+\xi_2+2\xi_0tp_1+\eta_1t^2. \qquad (19)
$$

The determining equations (3) of the Lie symmetry give

$$
\dot{\xi}_1-(p_1+t^2)\dot{\xi}_0=\eta_1+2\xi_0t,\quad \dot{\xi}_2-p_2\dot{\xi}_0=\eta_2,
$$

$$\dot{\eta}_1 = 0, \quad \dot{\eta}_2 - (p_2 - 1)\dot{\xi}_0 = \eta_2. \tag{20}$$

Choosing the infinitesimal generators as

$$\begin{aligned} &\xi_0 = 1, \quad \xi_1 = t^2, \quad \xi_2 = p_2, \\ &\eta_1 = 0, \quad \eta_2 = p_2 - 1, \end{aligned} \tag{21}$$

equations (20) are satisfied, so infinitesimal generators (21) are Lie symmetrical. While for infinitesimal generators (21), $\mathrm{X}^{(0)}(H) = 2tp_1 + p_2^2$, the determining equations (6) of the Mei symmetry are not satisfied, so infinitesimal generators (21) are not Mei symmetrical.

According to Theorem 2 and the previous study results, for infinitesimal generators (21), the Lie symmetry of the system cannot induce any Mei conserved quantity without the aid of the Mei symmetry. According to Theorem 3, presented in this paper, for infinitesimal generators (21), the Lie symmetry of the system can induce the Mei conserved quantity directly. For infinitesimal generators (21), equation (9) gives

$$\dot{G} = 2p_1 + p_2 - 1, \tag{22}$$

then we have

$$G = q_1 + p_2 + p_1 t - \frac{1}{3}t^3. \tag{23}$$

According to Theorem 3, from Eq. (8), we have

$$I_{\mathrm{M}} = q_1 - tp_1 - \frac{1}{3}t^3 = \mathrm{const}. \tag{24}$$

If we choose the infinitesimal generators as

$$\xi_0 = 0, \ \xi_1 = t, \ \xi_2 = -p_1, \ \eta_1 = 1, \ \eta_2 = 0, \tag{25}$$

then the determining equations (19) of the Lie symmetry and (6) of the Mei symmetry are both satisfied, the infinitesimal generators (25) are both Lie symmetrical and Mei symmetrical.

For infinitesimal generators (25), from Eqs. (7) and (9), we can obtain the same G, i.e.,

$$G = p_2 - q_2. \tag{26}$$

According to Theorems 2 and 3, we can obtain the following Mei conserved quantity:

$$I_{\mathrm{M}} = p_2 - q_2 + t = \mathrm{const}. \tag{27}$$

5. Conclusion

In previous studies, the Mei conserved quantity induced by the Lie symmetry of a nonconservative Hamilton system can only be obtained indirectly with the aid of the Mei symmetry. The theorem of Mei conserved quantity induced directly by the Lie symmetry of a nonconservative Hamilton system is presented in this paper. The condition under which the Lie symmetry of the system directly induces the Mei conserved quantity is given. The present results show that not only the Mei symmetry but also the Lie symmetry of a nonconservative Hamilton system can induce the Mei conserved quantity directly.

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