Mei symmetry and Mei conserved quantity of the Appell equation in a dynamical system of relative motion with non-Chetaev nonholonomic constraints*

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The Mei symmetry and the Mei conserved quantity of Appell equations in a dynamical system of relative motion with non-Chetaev nonholonomic constraints are studied. The differential equations of motion of the Appell equation for the system, the definition and the criterion of the Mei symmetry, and the expression of the Mei conserved quantity deduced directly from the Mei symmetry for the system are obtained. An example is given to illustrate the application of the results.

Keywords: non-Chetaev nonholonomic constrained system, dynamics of relative motion, Appell equation, Mei conserved quantity

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1. Introduction

At present, the symmetries in constrained mechanical systems mainly include the Noether symmetry,[1−8] the Lie symmetry,[9−17] and the Mei symmetry.[18−27] The Mei symmetry is a new symmetry advanced by Mei.[28] Symmetries and conserved quantities in constrained mechanical systems are major research directions in modern mathematics, mechanics, and physics. The Mei symmetry refers to an invariance by which the dynamic functions in the equations of motion also satisfy the original equations after the infinitesimal transformation. In 1899, Appell constructed the first-order linear Appell equations in a nonholonomic constrained system. The Appell equations become important in analytical mechanics. In recent decades, Chinese scholars have obtained important results and made a series of contributions in studying the Appell systems and their applications.[29−38] But so far, the Mei symmetry and the Mei conserved quantity of the Appell equations in a dynamical system of relative motion with non-Chetaev nonholonomic constraints have not been studied. This paper studies the definition and the criterion of the Mei symmetry for the Appell equations in a dynamical system of relative motion with non-Chetaev nonholonomic constraints. The condition and the expression for the Mei conserved quantity deduced from the Mei symmetry have been obtained. An example is given to illustrate the application of the results.

2. Differential equations of the system

Suppose that the configuration of the mechanical system is determined by N generalized coordinates q_s $(s = 1, 2, \ldots, N)$, the g two-sided ideal nonholonomic constrained equations of Chetaev type subjected by the system are

$$f_\beta\left(t, \boldsymbol{q}, \dot{\boldsymbol{q}}\right) = 0, \quad (\beta = 1, 2, \ldots, g). \tag{1}$$

The Chetaev conditions of constrained equations (1) putting on virtual displacement δq_s are

$$f_{\beta s}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\delta q_s = 0. \tag{2}$$

Generally speaking, there is no contract between $f_{\beta s}$ and $\partial f_\beta / \partial \dot{q}_s$. If

$$f_{\beta s} = \frac{\partial f_\beta}{\partial \dot{q}_s}, \tag{3}$$

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then non-Chetaev constraints become of the Chetaev type.

We assume that the speed of the origin of the moving coordinate system fixed on the reference materials is $\boldsymbol{\nu}_0$, and the angular velocity of the reference materials is $\boldsymbol{\omega}$, both are known functions of time t. We adopt Einstein's summation convention. Then the Appell function for the mechanical system and the Appell equations in the dynamical system of relative motion with non-Chetaev nonholonomic constraints are

$$S_{\mathrm{r}} = S_{\mathrm{r}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \frac{1}{2} m_i \ddot{\boldsymbol{r}}'_i \cdot \ddot{\boldsymbol{r}}'_i, \quad (i = 1, 2, \ldots, N), \tag{4}$$

$$\frac{\partial S_{\mathrm{r}}}{\partial \ddot{q}_s} = Q_s - \frac{\partial}{\partial q_s}(V^0 + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s + \lambda_\beta f_{\beta s}, \quad (s = 1, 2, \ldots, N), \tag{5}$$

where Q_s are the generalized forces, $V^0 = M(\dot{\boldsymbol{\nu}}_0 + \boldsymbol{\omega} \times \boldsymbol{\nu}_0) \cdot \boldsymbol{r}'_{\mathrm{c}}$ is the potential energy in a uniform field, $V^\omega = -\frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\theta}^0 \cdot \boldsymbol{\omega}$ is the potential energy of the centrifugal force,

$$Q_s^{\dot{\omega}} = -(\dot{\boldsymbol{\omega}} \times m_i \boldsymbol{r}'_i) \cdot \frac{\partial \boldsymbol{r}'_i}{\partial q_s}$$

is the generalized Coriolis inertial force,

$$\Gamma_s = 2\boldsymbol{\omega} \cdot \left(m_i \frac{\partial \boldsymbol{r}'_i}{\partial q_s} \times \frac{\partial \boldsymbol{r}'_i}{\partial q_k} \right) \dot{q}_k$$

is the generalized gyroscopic force, and λ_β is the constraint multiplier.

Suppose the system is non-singular, namely,

$$\det\left(\frac{\partial^2 L_{\mathrm{r}}}{\partial \dot{q}_s \partial \dot{q}_k} \right) \neq 0, \tag{6}$$

then from Eqs. (1) and (5), we can obtain each λ_β as a function of $(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$. Hence, equation (5) can be written as

$$\frac{\partial S_{\mathrm{r}}}{\partial \ddot{q}_s} = Q_s - \frac{\partial}{\partial q_s}(V^0 + V^\omega) + Q_s^{\dot{\omega}} + \Gamma_s + \Lambda'_s, \quad (s = 1, 2, \ldots N), \tag{7}$$

where

$$\Lambda'_s = \lambda_\beta f_{\beta s} = \Lambda'_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{8}$$

Equation (7) is the equation of relative motion for the holonomic system corresponding to Eqs. (1) and (5) of relative motion in a dynamical system with non-Chetaev nonholonomic constraints. The equation of relative motion in a dynamical system with non-Chetaev nonholonomic constraints can be found in solutions of the equations of relative motion in a holonomic system. If the initial conditions of motion satisfy the constraint equations of relative motion in a dynamical system with non-Chetaev nonholonomic constraints, then the solution of Eq. (7) of relative motion in a holonomic system gives the motions of Eqs. (1) and (5) of relative motion in a dynamical system with non-Chetaev nonholonomic constraints. From Eq. (7), we can solve all generalized accelerations as

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, 2, \ldots, N). \tag{9}$$

3. Definition of Mei symmetry

Introduce the infinitesimal transformations of groups of time and quasi-coordinates as

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),\ q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (s = 1, 2, \ldots, N), \tag{10}$$

where ε is an infinitesimal parameter, ξ_0 and ξ_s are infinitesimal generators. Form Eq. (10), we have

$$\frac{\mathrm{d}q_s^*}{\mathrm{d}t^*} = \frac{\mathrm{d}q_s + \varepsilon \mathrm{d}\xi_s}{\mathrm{d}t + \varepsilon \mathrm{d}\xi_0} = \dot{q}_s + \varepsilon(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) + \mathrm{O}(\varepsilon^2),$$
$$\frac{\mathrm{d}^2 q_s^*}{\mathrm{d}t^{*2}} = \ddot{q}_s + \varepsilon\left[(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)^{\cdot} - \ddot{q}_s\dot{\xi}_0\right] + \mathrm{O}(\varepsilon^2). \tag{11}$$

By transformation (10), dynamic functions S_{r}, Q_s, V^0, V^ω, $Q_s^{\dot{\omega}}$, Γ_s, Λ'_s, and f_β are changed to S_{r}^*, Q_s^*, V^{0*}, $V^{\omega*}$, $Q_s^{\dot{\omega}*}$, Γ_s^*, $\Lambda_s'^*$, and f_β^*, respectively. Taking the Taylor expansions of S_{r}^*, Q_s^*, V^{0*}, $V^{\omega*}$, $Q_s^{\dot{\omega}*}$, Γ_s^*, $\Lambda_s'^*$, and f_β^* at $(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ along the trajectory of motion of the system, we have

$$\begin{aligned} S_{\mathrm{r}}^* = {} & S_{\mathrm{r}}\left(t^*, q^*, \frac{\mathrm{d}q^*}{\mathrm{d}t^*}, \frac{\mathrm{d}^2 q^*}{\mathrm{d}t^{*2}}\right) = S_{\mathrm{r}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) \\ & + \varepsilon\left\{ \frac{\partial S_{\mathrm{r}}}{\partial t}\xi_0 + \frac{\partial S_{\mathrm{r}}}{\partial q_s}\xi_s + \frac{\partial S_{\mathrm{r}}}{\partial \dot{q}_s}\left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right) \right. \\ & \left. + \frac{\partial S_{\mathrm{r}}}{\partial \ddot{q}_s}\left[\left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right)^{\cdot} - \ddot{q}_s\dot{\xi}_0\right] \right\} + \mathrm{O}\left(\varepsilon^2\right), \end{aligned} \tag{12}$$

that is

$$S_{\mathrm{r}}^* = S_{\mathrm{r}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) + \varepsilon\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}}) + \mathrm{O}\left(\varepsilon^2\right), \tag{13}$$
$$Q_s^* = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{\mathrm{X}}^{(1)}(Q_s) + \mathrm{O}\left(\varepsilon^2\right), \tag{14}$$
$$V^{0*} = V^0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{\mathrm{X}}^{(1)}\left(V^0\right) + \mathrm{O}\left(\varepsilon^2\right), \tag{15}$$
$$V^{\omega*} = V^\omega(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{\mathrm{X}}^{(1)}(V^\omega) + \mathrm{O}\left(\varepsilon^2\right), \tag{16}$$
$$Q_s^{\dot{\omega}*} = Q_s^{\dot{\omega}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{\mathrm{X}}^{(1)}\left(Q_s^{\dot{\omega}}\right) + \mathrm{O}\left(\varepsilon^2\right), \tag{17}$$
$$\Gamma_s^* = \Gamma_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{\mathrm{X}}^{(1)}(\Gamma_s) + \mathrm{O}\left(\varepsilon^2\right), \tag{18}$$
$$\Lambda_s'^* = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{\mathrm{X}}^{(1)}(P_s) + \mathrm{O}\left(\varepsilon^2\right), \tag{19}$$
$$f_\beta^* = f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + \varepsilon\tilde{\mathrm{X}}^{(1)}(f_\beta) + O(\varepsilon^2), \tag{20}$$

where $s = 1, 2, \ldots, N$,

$$\tilde{\mathrm{X}}^{(2)} = \tilde{\mathrm{X}}^{(1)} + \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - \dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) - \ddot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right]\frac{\partial}{\partial\ddot{q}_s}, \quad (21)$$

$$\tilde{\mathrm{X}}^{(1)} = \mathrm{X}^{(0)} + \left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - \dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\frac{\partial}{\partial\dot{q}_s}, \quad (22)$$

$$\mathrm{X}^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s}, \quad (23)$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial\dot{q}_s} + \dot{\alpha}_s\frac{\partial}{\partial\ddot{q}_s}. \quad (24)$$

Definition If equation (7) in a dynamical system of relative motion with non-Chetaev nonholonomic constraints keeps its form invariant when dynamic functions S_{r}, Q_s, V^0, V^ω, $Q_s^{\dot{\omega}}$, Γ_s, Λ'_s, and f_β are substituted by functions S_{r}^*, Q_s^*, V^{0*}, $V^{\omega*}$, $Q_s^{\dot{\omega}*}$, Γ_s^*, $\Lambda_s'^*$, and f_β^* obtained through infinitesimal transformation (10), that is

$$\frac{\partial S_{\mathrm{r}}^*}{\partial\ddot{q}_s} = Q_s^* - \frac{\partial}{\partial q_s}\left(V^{0*} + V^{\omega*}\right) + Q_s^{\dot{\omega}*} + \Gamma_s^* + \Lambda_s'^*, \quad (s = 1, 2, \ldots, N), \quad (25)$$

then the invariance is called the Mei symmetry of the holonomic system corresponding to Eqs. (1) and (5) of relative motion in the dynamical system with non-Chetaev nonholonomic constraints.

4. Criterion of Mei symmetry

Substituting S_{r}^*, Q_s^*, V^{0*}, $V^{\omega*}$, $Q_s^{\dot{\omega}*}$, Γ_s^*, $\Lambda_s'^*$, and f_β^* into Eq. (25) and neglecting the terms that are of ε^2 order or higher, by means of Eq. (7), we obtain

$$\frac{\partial}{\partial\ddot{q}_s}\left[\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})\right] - \tilde{\mathrm{X}}^{(1)}\left[Q_s - \frac{\partial}{\partial q_s}\left(V^0 + V^\omega\right) + Q_s^{\dot{\omega}} + \Gamma_s + \Lambda'_s\right] = 0, \quad (s = 1, 2, \ldots, N). \quad (26)$$

Equation (26) is called the criterion equation of the Mei symmetry of the holonomic system corresponding to Eqs. (1) and (5) of relative motion in the dynamical system with non-Chetaev nonholonomic constraints.

Criterion For Eq. (7) of relative motion in the holonomic system corresponding to Eqs. (1) and (5) of relative motion in a dynamical system with non-Chetaev nonholonomic constraints, if infinitesimal generators ξ_0 and ξ_s make the criterion Eq. (26) tenable, then the symmetry is called the Mei symmetry of the system.

5. Mei conserved quantity deduced from Mei symmetry

Proposition If infinitesimal generators ξ_0 and ξ_s ξ_0, ξ_s and gauge function $G_{\mathrm{M}} = G_{\mathrm{M}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ of the Appel equation (Eqs. (1) and (5)] in a dynamical system of relative motion with non-Chetaev nonholonomic constraints make the following structural equations tenable

$$\begin{aligned}&\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \tilde{\mathrm{X}}^{(1)}\left[\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})\right] \\ &\quad + (\xi_s - \dot{q}_s\xi_0)\,\tilde{E}_s\left[\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})\right] \\ &\quad + \xi_0\Big\{\tilde{\mathrm{X}}^{(1)}\Big[Q_s - \frac{\partial}{\partial q_s}\left(V^0 + V^\omega\right) \\ &\quad + Q_s^{\dot{\omega}} + \Gamma_s + \Lambda'_s\Big]\Big\}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t} + \frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t} = 0, \end{aligned} \quad (27)$$

then the Mei conserved quantities deduced from the Mei symmetry for the system can be expressed as

$$\begin{aligned}I_{\mathrm{M}} &= \xi_0\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}}) + \frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial\dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\mathrm{M}} \\ &= \text{const.}\end{aligned} \quad (28)$$

Proof Using Eq. (24), we have

$$\begin{aligned}\frac{\bar{\mathrm{d}}I_{\mathrm{M}}}{\mathrm{d}t} &= \Big[\frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial t} + \dot{q}_s\frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial q_s} \\ &\quad + \alpha_s\frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial\dot{q}_s} + \frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial\ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t}\Big]\xi_0 \\ &\quad + \tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial\dot{q}_s}\right](\xi_s - \dot{q}_s\xi_0) \\ &\quad + \frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial\dot{q}_s}\left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - \alpha_s\xi_0 - \dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) \\ &\quad + \frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t}. \end{aligned} \quad (29)$$

We know

$$\begin{aligned}\tilde{\mathrm{X}}^{(1)}\left[\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})\right] &= \xi_0\frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial t} + \xi_s\frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial q_s} \\ &\quad + \left(\frac{\bar{\mathrm{d}}\xi_s}{\mathrm{d}t} - \dot{q}_s\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)\frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial\dot{q}_s}, \end{aligned} \quad (30)$$

so equation (29) can be written as

$$\begin{aligned}\frac{\bar{\mathrm{d}}I_{\mathrm{M}}}{\mathrm{d}t} &= \tilde{\mathrm{X}}^{(1)}\left[\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})\right] - \xi_s\frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial q_s} \\ &\quad + \xi_0\dot{q}_s\frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial q_s} + \xi_0\frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial\ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t} \\ &= \tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial\dot{q}_s}\right] \\ &\quad \times (\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t}\end{aligned}$$

$$
\begin{aligned}
&= \tilde{\mathrm{X}}^{(1)}\left[\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})\right] - (\xi_s - \dot{q}_s\xi_0)\frac{\partial \tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial q_s} \\
&\quad + \xi_0 \frac{\partial \tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial \ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t} \\
&= \tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \left[\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial \dot{q}_s}\right] \\
&\quad \times (\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t}. \qquad (31)
\end{aligned}
$$

Noting the structural Eq. (27) and the criterion Eq. (26), we have

$$
\begin{aligned}
\frac{\bar{\mathrm{d}}I_{\mathrm{M}}}{\mathrm{d}t} &= \tilde{\mathrm{X}}^{(1)}\left[\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})\right] + \xi_0\frac{\partial \tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial \ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t} \\
&\quad + (\xi_s - \dot{q}_s\xi_0)\,\tilde{E}_s\left[\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})\right] \\
&\quad + \xi_0\frac{\partial \tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial \ddot{q}_s}\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t} + \tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} + \frac{\bar{\mathrm{d}}G_{\mathrm{M}}}{\mathrm{d}t} \\
&= \xi_0\frac{\bar{\mathrm{d}}\alpha_s}{\mathrm{d}t}\left\{\frac{\partial \tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}})}{\partial \ddot{q}_s} - \tilde{\mathrm{X}}^{(1)}\left[Q_s - \frac{\partial}{\partial q_s}(V^0 + V^\omega)\right.\right. \\
&\quad \left.\left. + Q_s^{\dot{\omega}} + \Gamma_s + \Lambda_s'\right]\right\} = 0. \qquad (32)
\end{aligned}
$$

6. The two-degree-of-freedom dynamical system of relative motion with non-Chetaev nonholonomic constraints

The Appell function for the dynamical system of relative motion with non-Chetaev nonholonomic constraints is

$$
S_{\mathrm{r}} = \frac{1}{2}m\left(\ddot{q}_1^2 + \ddot{q}_2^2\right), \qquad (33)
$$

the constraint equation is

$$
f = \dot{q}_1 + bt\dot{q}_2 - bq_2 + t = 0, \qquad (34)
$$

the virtual displacement equation is

$$
\delta q_1 + bt\delta q_2 = 0, \qquad (35)
$$

and

$$
\begin{aligned}
&Q_1 = \frac{2\omega q_1 - \omega\dot{q}_1}{1+t^2}, \quad Q_2 = \frac{2\omega q_2 - \omega\dot{q}_2}{1+t^2}, \\
&\Gamma_1 = \frac{\omega\dot{q}_1}{1+t^2}, \quad \Gamma_2 = \frac{\omega\dot{q}_2}{1+t^2}, \\
&V^\omega = \omega\frac{q_1^2 + q_2^2}{1+t^2}, \quad V^0 = Q_1^{\dot{\omega}} = Q_2^{\dot{\omega}} = 0, \qquad (36)
\end{aligned}
$$

where b, m, and ω are constants. We study the Mei symmetry and the Mei conserved quantity deduced directly from the Mei symmetry in this two-degree-of-freedom dynamical system of relative motion with non-Chetaev nonholonomic constraints.

From Eq. (5), and noting Eq. (33) and the criterion Eq. (25), we can obtain

$$
m\ddot{q}_1 = \Lambda_1' = \lambda, \quad m\ddot{q}_2 = \Lambda_2' = \lambda bt. \qquad (37)
$$

Combined Eq. (37) with Eq. (34), we obtain

$$
\lambda = -\frac{m}{1+b^2t^2}. \qquad (38)
$$

Noting Eq. (8), and substituting Eq. (38) into Eq. (37), we have

$$
\begin{aligned}
\ddot{q}_1 = \alpha_1 = \frac{\Lambda_1'}{m} = -\frac{1}{1+b^2t^2}, \\
\ddot{q}_2 = \alpha_2 = \frac{\Lambda_2'}{m} = -\frac{bt}{1+b^2t^2}. \qquad (39)
\end{aligned}
$$

Using Eqs. (21), (22), and (23), we have

$$
\begin{aligned}
\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}}) &= m\ddot{q}_1\frac{\bar{\mathrm{d}}^2\xi_1}{\mathrm{d}t^2} + m\ddot{q}_2\frac{\bar{\mathrm{d}}^2\xi_2}{\mathrm{d}t^2} \\
&\quad - 2m\left(\ddot{q}_1^2 + \ddot{q}_2^2\right)\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} \\
&\quad - m\left(\dot{q}_1\ddot{q}_1 + \dot{q}_2\ddot{q}_2\right)\frac{\bar{\mathrm{d}}^2\xi_0}{\mathrm{d}t^2}, \qquad (40)
\end{aligned}
$$

$$
\frac{\partial}{\partial \ddot{q}_1}\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}}) = m\frac{\bar{\mathrm{d}}^2\xi_1}{\mathrm{d}t^2} - 4m\ddot{q}_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} - m\dot{q}_1\frac{\bar{\mathrm{d}}^2\xi_0}{\mathrm{d}t^2}, \qquad (41)
$$

$$
\frac{\partial}{\partial \ddot{q}_2}\tilde{\mathrm{X}}^{(2)}(S_{\mathrm{r}}) = m\frac{\bar{\mathrm{d}}^2\xi_2}{\mathrm{d}t^2} - 4m\ddot{q}_2\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} - m\dot{q}_2\frac{\bar{\mathrm{d}}^2\xi_0}{\mathrm{d}t^2}, \qquad (42)
$$

$$
\begin{aligned}
&\tilde{\mathrm{X}}^{(1)}\left[Q_1 - \frac{\partial}{\partial q_1}(V^o + V^\omega) + Q_1^{\dot{\omega}} + \Gamma_1 + \Lambda_1'\right] \\
&= -\frac{\partial}{\partial t}\left(\frac{m}{1+b^2t^2}\right)\xi_0, \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\mathrm{X}}^{(1)}\left[Q_2 - \frac{\partial}{\partial q_2}(V^o + V^\omega) + Q_2^{\dot{\omega}} + \Gamma_2 + \Lambda_2'\right] \\
&= -\frac{\partial}{\partial t}\left(\frac{mbt}{1+b^2t^2}\right)\xi_0. \qquad (44)
\end{aligned}
$$

Then from Eq. (26), we obtain

$$
\begin{aligned}
&m\frac{\bar{\mathrm{d}}^2\xi_1}{\mathrm{d}t^2} - 4m\ddot{q}_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} - m\dot{q}_1\frac{\bar{\mathrm{d}}^2\xi_0}{\mathrm{d}t^2} \\
&\quad + \frac{\partial}{\partial t}\left(\frac{m}{1+b^2t^2}\right)\xi_0 = 0, \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
&m\frac{\bar{\mathrm{d}}^2\xi_2}{\mathrm{d}t^2} - 4m\ddot{q}_2\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} - m\dot{q}_2\frac{\bar{\mathrm{d}}^2\xi_0}{\mathrm{d}t^2} \\
&\quad + \frac{\partial}{\partial t}\left(\frac{mbt}{1+b^2t^2}\right)\xi_0 = 0. \qquad (46)
\end{aligned}
$$

Taking the following infinitesimal generators:

$$
\xi_0 = 0, \quad \xi_1 = \dot{q}_2 - bt\dot{q}_1 + bq_1 + bt, \quad \xi_2 = 1, \qquad (47)
$$

obviously, they all satisfy Eq. (26). From the structural equation (27), we have

$$G_{\rm M} = -b^2 t. \tag{48}$$

By means of Eq. (28), we obtain the Mei conserved quantity deduced directly from the Mei symmetry of the Apell equation in this dynamical system of relative motion with non-Chetaev nonholonomic constraints

$$I_{\rm M} = b\left(\dot{q}_2 - bt\dot{q}_1 + bq_1\right) = \text{const.} \tag{49}$$

7. Conclusion

In this paper, we study the definition and the criterion for the Mei symmetry of the Appell equations in a dynamical system of relative motion with non-Chetaev nonholonomic constraints. The condition and the expression for the Mei conserved quantity deduced from the Mei symmetry are obtained. The main results are criterion (26) of the Mei symmetry, structural equation (27), and expression (28) of the Mei conserved quantity. The results in this paper can be applied to the Appell equations in dynamical systems of relative motion with nonholonomic constraints of the Chetaev type, the Appell equations in dynamical systems of relative motion with the holonomic constraint, the Appell equations in mechanical systems with non-Chetaev nonholonomic constraints, the Appell equations in mechanical systems with nonholonomic constraints of the Chetaev type, the Appell equations in mechanical systems with holonomic constraints, and so on. Therefore, the conclusion of this paper is universal and has practical significance.

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