Density waves in a lattice hydrodynamic traffic flow model with the anticipation effect*

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By introducing the traffic anticipation effect in the real world into the original lattice hydrodynamic model, we present a new anticipation effect lattice hydrodynamic (AELH) model, and obtain the linear stability condition of the model by applying the linear stability theory. Through nonlinear analysis, we derive the Burgers equation and Korteweg-de Vries (KdV) equation, to describe the propagating behaviour of traffic density waves in the stable and the metastable regions, respectively. The good agreement between simulation results and analytical results shows that the stability of traffic flow can be enhanced when the anticipation effect is considered.

Keywords: traffic flow, lattice hydrodynamic model, density waves, anticipation effect

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1. Introduction

To study the dynamics of traffic flow, an increasing number of traffic models have been presented,[1−10] and the most well-known one of them is the lattice hydrodynamic model proposed by Nagatani,[10] which successfully depicts the formation of the density waves and reveals its transition mechanism. Hereafter, inspired by original lattice model, some new lattice hydrodynamic models[11−17] were successively put forward to more realistically describe the traffic nature.

However, the lattice hydrodynamic models mentioned above do not directly describe the complex phenomena caused by anticipation behaviour since this effect is not considered. In fact, with the application of the intelligent transportation system (ITS), drivers can receive information about other vehicles on roads, and then estimate the future traffic situation according to the present traffic status. In this way, a driver can exert advanced control onto vehicles and make the traffic flow more stable. However, few scholars[18] studied the anticipation effect in the lattice hydrodynamic models.

In view of the above reason, in this paper, we propose a new lattice hydrodynamic model by incorporating the anticipation behaviour of traffic flow. In the following section, the new lattice hydrodynamic model is introduced. In Section 3, linear stability analysis is conducted. In Section 4, the effects of anticipation behaviour on the properties of density wave in the stable and the metastable regions in traffic flow are investigated. In Section 5, numerical simulation is carried out to validate the analytic results. In Section 6, we draw some conclusions from the present study.

2. Models

In 1993, Kerner and Konhauser[19] presented a very good high-order continuum traffic flow model (for short, KK model) with the partial differential equations

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{1}$$

$$\partial_t v + v\partial_x v = [V_{\rm e}(\rho) - v]/\tau - (c_0^2/\rho)\partial_x \rho + (\mu/\rho)\partial_x^2 v, \tag{2}$$

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, ρ denotes the traffic density, v represents the space mean velocity, $V_{\rm e}(\rho)$ is the equilibrium speed, c_0 and μ are the anticipation coefficient and viscosity coefficient respectively. The right-hand side of Eq. (2) includes three terms. The first term denotes a relaxation to equilibrium state. The second term is an anticipation term, which re-

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flects that the drivers adaptively adjust his velocity according to the observed traffic density ahead. The final term describes the viscosity factor which represents the presumed tendency to adjust one's speed to that of the surrounding traffic. It is not easy to identify the values of c_0 and μ, and obtain its density wave equation.

For the convenience of analysing the evolution of density wave in traffic flow, in 1998, Nagatani[10] first proposed the simplified version of the hydrodynamic model as follows:

$$\partial_t \rho + \rho_0 \partial_x(\rho v) = 0, \tag{3}$$

$$\partial_t \rho v = a\rho_0 V(\rho(x+\delta)) - a\rho v, \tag{4}$$

where ρ_0 denotes the average density, V is the optimal velocity function, and a is the sensitivity of a driver, $\rho(x+\delta)$ represents the local density at position $x+\delta$ at time t, and δ denotes the average headway, which means $\delta = 1/\rho_0$. Local density is expressed as $\rho(x+\delta) = 1/h(x,t)$, where $h(x,t)$ is the headway. Equation (3) is the continuity equation, corresponding to Eq. (2); equation (4) is the evolution equation, but the anticipation term and viscosity term are omitted for simplicity. In fact, according to actual traffic, the anticipation term in KK model is essential, but this factor is not considered in the lattice hydrodynamic model.

For the above reason, we develop a new model called the anticipation effect lattice hydrodynamic (AELH) model by introducing the anticipation term into the right-hand side of Eq. (4). Thus the evolution equation is changed into

$$\partial_t \rho v = a\rho_0 V(\rho(x+\delta)) - a\rho v + ka\partial_x Q[\rho(x+\delta)], \tag{5}$$

where $\partial_x Q[\rho(x+\delta)]$ is the flux gradient; Q denotes the flux; $\partial_x Q[\rho(x+m\delta)]$ is named anticipation term, which reflects that the driver adaptively adjusts his car according to the observed varying flux on the adjoining neighbour sites ahead; k is a sensitivity coefficient of anticipation term. The ideal of the model is that the variation of traffic flow ρv at position x is determined by the optimal flow $\rho_0 V(\rho(x+\delta))$ at points $x+\delta$, actual flow ρv at position x, and the varying flux $\partial_x Q[\rho(x+\delta)]$. When $k=0$, the extended model reduces into Nagatani's lattice model.[10]

With dimensionless space x (let $\tilde{x} = x/\delta$, and $\tilde{x}$ is indicated as x hereafter), the equation of motion, Eq. (5), with anticipation term is modified into

$$\partial_t \rho v = a\rho_0 V(\rho(x+1)) - a\rho v + k\rho_0 a[Q(\rho(x+1)) - Q(\rho(x))]. \tag{6}$$

The corresponding lattice version of the above system consisting of Eqs. (3) and (6) is rewritten as

$$\partial_t \rho_j + \rho_0(\rho_j v_j - \rho_{j-1} v_{j-1}) = 0, \tag{7}$$

$$\partial_t(\rho_j v_j) = a\rho_0 V(\rho_{j+1}) - a\rho_j v_j + ka\rho_0[Q(\rho_{j+1}) - Q(\rho_j)], \tag{8}$$

where j indicates site j on the one-dimensional lattice.

In this paper, the optimal velocity function $V(\cdot)$ is assumed to be[18]

$$V(\rho_{j+1}) = \tanh\Big(\frac{2}{\rho_0} - \frac{\rho_{j+1}}{\rho_0^2} - \frac{1}{\rho_c}\Big) + \tanh\Big(\frac{1}{\rho_c}\Big). \tag{9}$$

Combining with the fundamental law of traffic flow ($Q = \rho v$), and discretizing the time variable in Eqs. (7) and (8), we obtain

$$\rho_j(t+\tau) - \rho_j(t) + \tau\rho_0[\rho_j(t)v_j(t) - \rho_{j-1}(t)v_{j-1}(t)] = 0, \tag{10}$$

$$\rho_j(t+\tau)v_j(t+\tau) = \rho_0 V(\rho_{j+1}(t)) + k\rho_0[\rho_{j+1}(t)v_{j+1}(t) - \rho_j(t)v_j(t)]. \tag{11}$$

By eliminating the speed v in Eqs. (10) and (11), the density equations are obtained as

$$\rho_j(t+2\tau) - \rho_j(t+\tau) + \tau\rho_0^2[V(\rho_{j+1}) - V(\rho_j)] - k\rho_0[\Delta\rho_j(t+\tau) - \Delta\rho_j(t)] = 0, \tag{12}$$

where $\Delta\rho_j(t) = \rho_{j+1}(t) - \rho_j(t)$.

3. Linear stability analysis

For all types of lattice hydrodynamic models, the performance of traffic system stability, without a doubt, is an important issue. In the lattice hydrodynamic model, the traffic system stability is defined as an uniform distribution state for all vehicles with constant density ρ_0 and optimal velocity $V(\rho_0)$. In order to investigate the influence of the new consideration on traffic flow, we first give the solution of the steady state for the AELH model as

$$\rho_j(t) = \rho_0 \quad \text{and} \quad v_j(t) = V(\rho_0). \tag{13}$$

Supposing that $y_j(t)$ is a small deviation from the steady state flow, and expressed as

$$\rho_j(t) = \rho_0 + y_j(t). \tag{14}$$

Substituting Eq. (14) into Eq. (12) and linearizing it, then

$$y_j(t+2\tau) - y_j(t+\tau) + \tau\rho_0^2 V' \Delta y_j(t) - k\rho_0[\Delta y_j(t+\tau) - \Delta y_j(t))] = 0. \tag{15}$$

where $\Delta y_j(t) = y_{j+1}(t) - y_j(t)$ and $V' = [\mathrm{d}V(\rho_j)/\mathrm{d}\rho_j]|_{\rho_j=\rho_0}$.

Expanding $y_j(t) = \exp(\mathrm{i}kj + zt)$, in the Fourier model, where $z = z_1\mathrm{i}k + z_2(\mathrm{i}k)^2 + \cdots$, and inserting them into Eq. (15) give the first- and the second-order

terms of coefficients in the expression of z, respectively, as follows:

$$z_1 = -\rho_0^2 V', \tag{16}$$

$$z_2 = -\frac{3}{2}\tau(\rho_0^2 V')^2 - \frac{1}{2}\rho_0^2 V' - k\rho_0^3 V'. \tag{17}$$

If z_2 is a negative value, the uniform steady-state flow becomes unstable for long-wavelength mode, while when z_2 is a positive value, the uniform flow is stable. Thus the neutral stable criterion for this steady state is given by

$$\tau = -\frac{1+2k\rho_0}{3\rho_0^2 V'}. \tag{18}$$

For small disturbances with long wavelengths, the homogeneous traffic flow is stable when

$$\tau < -\frac{1+2k\rho_0}{3\rho_0^2 V'}. \tag{19}$$

As $k = 0$, the stable condition of extended model is the same as that of the Nagatani's model,[10] i.e.,

$$\tau < -\frac{1}{3\rho_0^2 V'}. \tag{20}$$

In Fig. 1, the neutral stability lines are plotted in the parameter space (ρ, a). There is a critical point (ρ_c, a_c), which is the apex of each curve. Specially, when $k = 0$, in the AELH model the neutral stability line is consistent with that of the original Nagatani's model. In Fig. 1, the region above the neutral stability lines represents the stability region where the traffic jam does not appear. Below the neutral stability lines, accordingly, is the unstable region where traffic flow is unstable and density waves appear as the disturbances are added into it.

Furthermore, in Fig. 1, one can find out that the stability region in the AELH model is larger than that in the Nagatani's model. Specially, the bigger the value k, the larger the stable region is. That means that the traffic flow is stabilized further when the anticipation effect is considered.

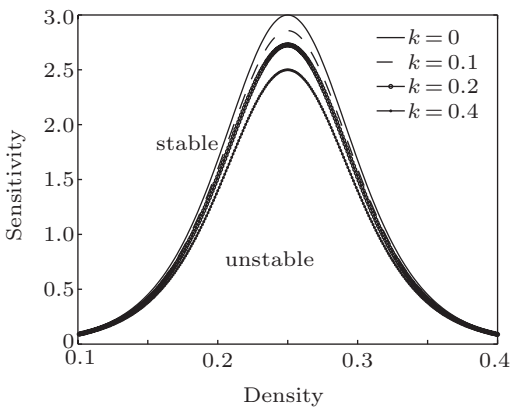


Fig. 1. Phase diagram in the density-sensitivity (ρ, a) space.

4. Nonlinear analysis

Vehicle density can fluctuate to form traffic congestion due to various reasons, and it behaves as density wave propagation. Different nonlinear wave equations have been deduced to describe the properties of density waves in distinct regions from those in a car-following model. In this section, we also examine this nature of our extended lattice hydrodynamic model by using a nonlinear analysis approach, and investigate the slowly varying behaviour in the stable and the metastable regions by employing a small positive scaling parameter ε.

4.1. Burgers equation in the stable flow region

To investigate the evolution characteristics of a traffic jam in the stable region, we introduce slow scales for space variable j and time variable t, and define the slow variables X and T as

$$X = \varepsilon(j + bt) \quad \text{and} \quad T = \varepsilon^2 t,\ 0 < \varepsilon \le 1, \tag{21}$$

where b is a constant to be determined. Let ρ_j satisfy the following equation:

$$\rho_j(t) = \rho_c + \varepsilon R(X, T), \tag{22}$$

where ρ_c denotes the safety critical density, and $0 < \varepsilon \le 1$ is a small positive scaling parameter.

Substituting Eqs. (21) and (22) into Eq. (12) and expanding the resulting equation to the third order of ε, one obtains the following partial differential equation:

$$\begin{aligned}&\varepsilon^2[b + \rho_c^2 V']\partial_X R \\ &+ \varepsilon^3\Big[\partial_T R + \frac{3b^2\tau + \rho_c^2 V' - 2k\rho_c b}{2}\partial_X^2 R + \rho_c^2 V'' R\partial_X R\Big] \\ &= 0.\end{aligned} \tag{23}$$

Let $b = -\rho_c^2 V'$ and eliminate the second order of Eq. (23), then the resulting equation will be

$$\begin{aligned}&\partial_T R + \rho_c^2 V'' R\partial_X R \\ &= -\Big[\frac{1+2k\rho_c}{2} + \frac{3(\rho_c^2 V')\tau}{2}\Big]\rho_c^2 V'\partial_X^2 R.\end{aligned} \tag{24}$$

In accordance with criterion Eq. (19), the coefficient $-[(1+2k\rho_c)/2 + 3(\rho_c^2 V')\tau/2]$ of the second derivatives is positive in the stable region. Thus, in the stable region, Eq. (24) is just the Burgers equation, of which the solution is a train of N-shock wave

and given by

$$R(X,T)=\frac{1}{|\rho_c^2V''|T}\Big[X-\frac{1}{2}(\eta_j+\eta_{j+1})\Big] - \frac{1}{2|\rho_c^2V''|T}(\eta_{j+1}-\eta_j) \times\tanh\Big[\frac{C_1}{4|\rho_c^2V''|T}(\eta_{j+1}-\eta_j)(X-\xi_j)\Big], \tag{25}$$

where $C_1=-[(1+2k\rho_c)/2+3(\rho_c^2V')\tau/2]$, ξ_j is the coordinate of the shock front and η_j is the coordinate of the intersection of slope with the x axis $(n=1,2,\ldots,N)$, $R(X,T)\to 0$ when $t\to\infty$, which means that any density wave expressed by Eq. (25) in stable traffic flow region evolves into a uniform flow in the course of time.

4.2. KdV equation in the metastable flow region

Focusing on the system behaviour near the neutral stability line, we introduce slow scales for space variable j and time variable t, and define the slow variables X and T as

$$X=\varepsilon(j+bt) \quad \text{and} \quad T=\varepsilon^3t,\ 0<\varepsilon\le 1, \tag{26}$$

where b is a constant to be determined. Let ρ_j satisfy the following equation:

$$\rho_j(t)=\rho_c+\varepsilon^2R(X,T). \tag{27}$$

Substituting Eqs. (26) and (27) into Eq. (12) and making the Taylor expansions to the sixth order of ε, then the following expression is obtained:

$$\varepsilon^3[b+\rho_c^2V']\partial_XR+\varepsilon^4\Big[\frac{3b^2\tau}{2}+\frac{\rho_c^2V'}{2}-k\rho_0b\Big]\partial_X^2R+\varepsilon^5\Big[\partial_TR+\frac{7b^3\tau^2+\rho_c^2V'-3bk\rho_0(1+b\tau)}{6}\partial_X^3R+\rho_c^2V''R\partial_XR\Big] +\varepsilon^6\Big[(3b\tau-k\rho_0)\partial_T\partial_XR+\frac{15b^4\tau^3+\rho_c^2V'-k\rho_0(4b^3\tau^2+6b^2\tau+4b)}{24}\partial_X^4R+\frac{\rho_c^2V''}{4}\partial_X^2R^2\Big]=0, \tag{28}$$

where $V'=\dfrac{\mathrm{d}V(\rho_j)}{\mathrm{d}\rho_j}\Big|\rho_j=\rho_c$, $V''=\dfrac{\mathrm{d}^2V(\rho_j)}{d\rho_j^2}\Big|\rho_j=\rho_c$. Near the neutral stability line $\tau=(1-\varepsilon^2)\tau_c$, taking $b=-\rho_c^2V'$, and eliminating the third- and the fourth-order terms of ε from Eq. (28), the simplified equation is obtained as

$$\varepsilon^5[\partial_TR-m_1\partial_X^3R-m_2R\partial_XR]+\varepsilon^6[-m_3\partial_X^2R+m_4\partial_X^4R+m_5\partial_X^2R^2]=0, \tag{29}$$

where

$$m_1=-\frac{7b^3\tau_c^2+\rho_c^2V'-3bk\rho_c(1+b\tau_c)}{6},\quad m_2=-\rho_c^2V'',\quad m_3=\frac{3b^2\tau_c}{2},$$
$$m_4=\frac{15b^4\tau_c^3+\rho_c^2V'-k\rho_0(4b^3\tau_c^2+6b^2\tau_c+4b)-(12b\tau_c-3k\rho_0)[7b^3\tau_c^2+\rho_c^2V'-3bk\rho_0(1+b\tau)]}{24},$$
$$m_5=\frac{1-6b\tau_c+2k\rho_c}{4}.$$

In order to obtain the standard KdV equation, we make the following transformation for Eq. (29):

$$T=\sqrt{m_1}T',\quad X=-\sqrt{m_1}X',\quad R=\frac{1}{m_2}R'. \tag{30}$$

Hence, we can obtain the regularized KdV equation

$$\partial_{T'}R'+\partial_{X'}^3R'+R'\partial_{X'}R'+\varepsilon M[R']=0, \tag{31}$$

where the $M[R']$ is the correction term with the high order of an infinitesimal for ε and expressed as

$$M[R']=\sqrt{\frac{1}{m_1}}\Big(-m_3\partial_{X'}^2R'+\frac{m_4}{m_1}\partial_{X'}^4R'+\frac{m_5}{m_2}\partial_{X'}^2R'^2\Big). \tag{32}$$

Ignoring the $O(\varepsilon)$ term, we obtain the KdV equation with the soliton

$$R_0'(X',T')=A\,\mathrm{sech}^2\Big[\sqrt{\frac{A}{12}}(X'-\frac{A}{3}T')\Big]. \tag{33}$$

With the method described in Ref. [20], we obtain the selected amplitude A as

$$A=\frac{21m_1m_2m_3}{24m_1m_5-5m_2m_4}. \tag{34}$$

Hence, we obtain the KdV soliton as

$$\rho_j(t)=\rho_c+\frac{A}{m_2}\Big(1-\frac{\tau}{\tau_s}\Big) \times\mathrm{sech}^2\Big[\sqrt{\frac{A}{12m_1}\Big(1-\frac{\tau}{\tau_s}\Big)}\Big] \times\Big\{j+\Big(b-\varepsilon^2\frac{A}{3}\Big)t\Big\}. \tag{35}$$

5. Numerical simulation

To validate the above theoretical results, numerical simulation is carried out for the extended model described by Eq. (12), in which the periodic boundary condition is adopted. The considered road is 20 km

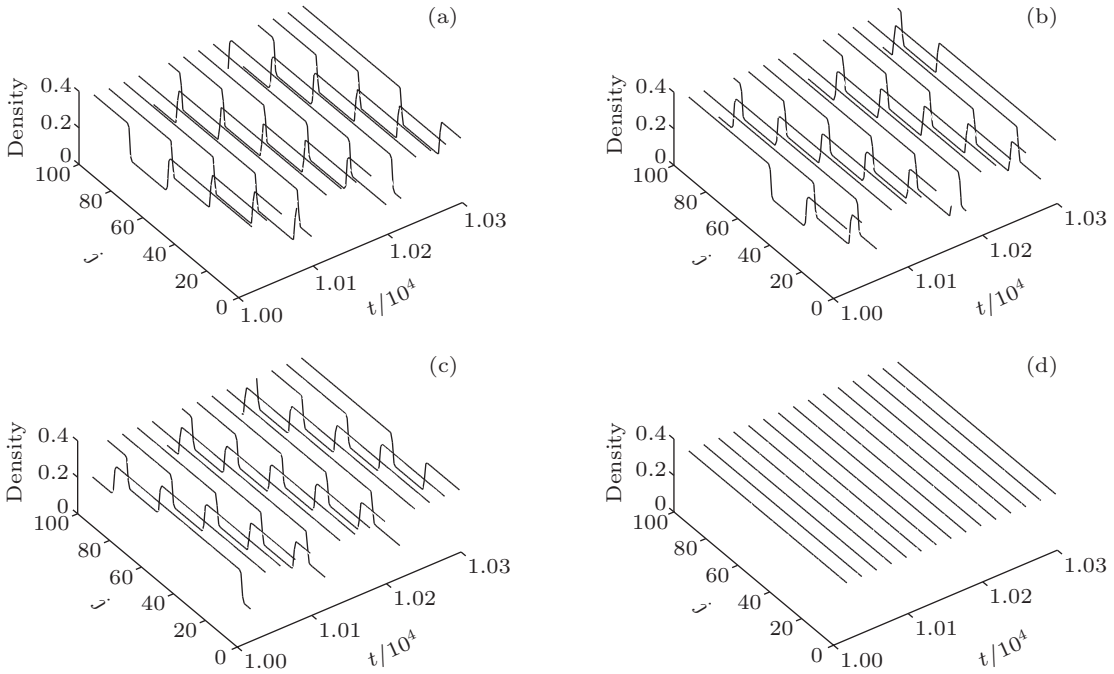


Fig. 2. Space–time evolutions of the density after $t = 10^4$ in AELH model for different k values: $k = 0$ (a), 0.1 (b), 0.2 (c), and 0.4 (d).

long, which is divided into 100 lattices equally. Parameter values are set as follows: $\rho_j(0) = \rho_0 = 0.3$, $\rho_j(1) = \rho_j(0) = 0.3$, for $j \neq 49, 50$; $\rho_j(1) = 0.3 - 0.05$ for $j = 49$; $\rho_j(1) = 0.3 + 0.05$ for $j = 50$; $\rho_c = 0.25$, and delay time $\tau = 0.39$. The simulation inputs are shown in Fig. 2.

Figure 2 shows the space–time evolutions of the density for the AELH model after $t = 10^4$ time steps. The pattern (a) with $k = 0$ exhibits the result obtained according to the original Nagatani model. The patterns (b), (c), and (d), respectively, for $k = 0.1$, 0.2, and 0.4 correspond to three cases of AELH model. In the patterns (a), (b), and (c), because stability condition (19) cannot be satisfied for $k = 0$, 0.1, and 0.2 under the condition of $\tau = 0.39$, the uniform traffic flow becomes unstable and the traffic jam appears when uniform traffic flow encounters stochastic small disturbances. Due to the fact that the anticipation effect is considered, the amplitudes of density waves in pattern (b) are less than those in pattern (a) under the same sensitivity, which means that the anticipation effect should not be omitted simply in the lattice hydrodynamic traffic flow model. Besides, with the increase of the value k, from patterns (a) to (c), the amplitudes of the inhomogeneous traffic density waves are weakened gradually. Especially, as we set $k = 0.4$, stable condition (19) holds true, thus the traffic jams in pattern (d) disappear and the inhomogeneous traffic flow recovers the uniform state. Therefore the simulation outcomes are in agreement with analytical results and show that the effect of the anticipation effect can stabilize the traffic flow.

6. Conclusion

In recent years, wireless communication technology has been widely applied to ITS, and thus much more information is available for drivers under ITS environments than ever before. With the information, drivers can anticipate the traffic situation ahead. However, few studies of the existing lattice hydrodynamic models directly deal with the anticipation factor. In this paper, we present a new lattice hydrodynamic model (AELH model) by considering the effect of anticipation in single-lane freeway traffic. Due to the fact that a new consideration is introduced, the stability of the uniform flow is strengthened efficiently. In addition, the density waves for traffic flow are investigated in the AELH model with nonlinear analysis method in the stable and the metastable regions. We find that the developing process and the evolution forms of density waves in the stable and metastable regions can be described by the Burgers equation and the KdV equation respectively. Numerical simulation

results show that the performance of the new model is better than that of the original lattice hydrodynamic model by introducing the new consideration. The analytical results are in good agreement with the simulation results.

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