Tortuosity for streamlines in porous media*

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(Received 22 March 2011; revised manuscript received 24 November 2011)

An analysis of tortuosity for streamlines in porous media is presented by coupling the circle and square models. It is assumed that some particles in porous media do not overlap and that fluid in porous media is incompressible. The relationship between tortuosity and porosity is attained with different configurations by using a statistical method. In addition, the tortuosity fractal dimension is expressed as a function of porosity. Those correlations do not include any empirical constant. The percolation threshold and tortuosity fractal dimension threshold of porous media are also presented as: $\phi_c = 0.32$, $D_{T_c} = 1.07$. The predicted correlations of the tortuosity and the porosity agree well with the existing experimental and simulated results.

Keywords: tortuosity, tortuosity fractal dimension, porous media

PACS: 47.55.Mh, 03.40.Kf, 02.70 **DOI:** 10.1088/1674-1056/21/4/044701

1. Introduction

Material transport is one of the most ubiquitous phenomena in porous media, such as fluid flow and gas diffusion. The actual trajectory is microscopic and complicated because of tortuousity in the transported material. In recent years, the behavior of tortuous flow in porous media has attracted considerable attention.[1−22] It is critical for calculating gas diffusion and electrical conductivity, etc., because these transport parameters are usually related to the tortuosity of porous media.[23−29] In order to describe the quantitative transport properties of porous material, the tortuosity must first be determined. However, there is little information available on this aspect, especially the analysis results for tortuosity.[4,8,11]

Tortuosity is usually determined by experiments and numerical methods, where those results include one or more empirical constants.[1,5−7,9,12] Recently, two simple geometric models of tortuosity were obtained by Yu *et al.*[4] and Yuan *et al.*[8] by analyzing flow path in porous media comprising square and circular particles. Similar to Yu's work, Yuan *et al.*[11] extended the model from two dimensions to three dimensions, and then proposed a model with plate-like particles. Their models are only applicable to porous media with single-type particles, such as squares, spheres or plate-like particles in two-dimensional or three-dimensional space. In reality, most porous media are not simplex, so a tortuosity model for mixed media is important.

The tortuosity is defined as[2,3,27]

$$\tau = \frac{L_e}{L}, \tag{1}$$

or

$$\tau = L_{\min}^{1-D_T} L^{D_T-1} = \left(\frac{L}{L_{\min}}\right)^{D_T-1}, \tag{2}$$

where $L_{\min}$ is the smallest scale of measurement, which is of the order of the pore size, L_e and L are the actual and the straight lengths of fluid flow through porous media, respectively. Equation (1) will be used in this paper.

A correlation between the average flow path tortuosity and the porosity was given by the experiments based on flow through beds packed with cubic and circular particles[7]

$$\tau = 1 + p\ln(1/\phi), \tag{3}$$

where ϕ is the porosity. The p is the empirical constant and can be obtained by using the experimental

*Project supported by the National Natural Science Foundation of China (Grant No. 10932010, 10972199, 11005093, 11072220, and 11079029), the Natural Science Foundation of Zhejiang Province of China (Grant No. Z6090556 and Y6100384), and the Research Project for the Higher Educational Institutions of Inner Mongolia Autonomous Region (Grant No. NJZZ11284).

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data, $p = 0.63$ and $p = 0.41$ in porous media with square and circular particles, respectively.

Koponen *et al.*[5] applied the LG method (lattice-gas cellular automaton method) to study a creeping flow of Newtonian and incompressible fluid in a two-dimensional porous medium, which is constructed by randomly placing square particles with the condition of unrestricted overlap. They obtained a correlation between the average flow path tortuosity and the porosity ϕ as

$$\tau = 1 + 0.8(1 - \phi). \tag{4}$$

In their later work, by using the same method and model, the percolation threshold (ϕ_c) was considered and another correlation between the average flow path tortuosity and the porosity was given as

$$\tau = 1 + a\frac{(1-\phi)}{(\phi - \phi_c)^m}. \tag{5}$$

The values of the fitting model being $a = 0.65$, $m = 0.19$, and respect to the porosity range is 0.33 to 1.0.

In this paper, we study the tortuosity in a mixed porous medium packed with circular and square particles. The tortuosity in porous media with several different arrangements of circular and square particles are discussed. The function of the proposed tortuosity has no empirical constant. The correlation between the tortuosity fractal dimension and the porosity is also discussed. The results are compared with existing experimental and simulated results.

2. The model of tortuosity in porous media

Figures 1(a)–1(c) display three types of possible configurations of porous media composed of circular and square particles. The dashed rectangle is chosen to be a unit cell for investigating the tortuosity, in which the distances between three center points within the circles and square are equal (viz. $L_{O_1,O_2} = L_{O_2,O_3} = L_{O_1,O_3}$), d is the length of the gap between the square particles, R is the radius of the circular particles, while the length of a square particle's side is $2R$ as shown in Figs. 1(a)–1(c). Assume that the fluid in porous media is incompressible, and the thickness of the fluid layer on the surface of particles is quite thin. All particles in the porous media are not allowed to be overlapped. In Figs. 1(a)–1(c), a series of ideal flow tracks are presented, and the real flow track might be averaged by them, so the tortuosity can be expressed as

$$\tau = \frac{1}{N}\sum \tau_i. \tag{6}$$

The length of the unit cell is $(\sqrt{3}/2)(2R+d)$, the width of the unit cell is $2R+d$, as shown in Figs. 1–3, so the total area of the unit cell V_t is

$$V_t = \frac{\sqrt{3}}{2}(2R+d)^2, \tag{7}$$

and the pore area in the unit cell is

$$V_p = \frac{\sqrt{3}}{2}(2R+d)^2 - 2R^2 - \frac{1}{2}\pi R^2. \tag{8}$$

Based on Eqs. (7) and (8), the unit cell porosity is

$$\phi = \frac{V_p}{V_t} = 1 - \frac{4R^2 + \pi R^2}{\sqrt{3}(2R+d)^2}. \tag{9}$$

We define $k = d/R$, then Eq. (8) is transformed into

$$k = \sqrt{\frac{4+\pi}{\sqrt{3}(1-\phi)}} - 2. \tag{10}$$

In the case of Fig. 1(a), the actual length of the flow path is $L_e = L_{AB} + L_{\widehat{BC}}$. Through a geometrical algorithm, $L_{\widehat{BC}} = (1/2)\pi R$, so the actual length of the flow path is $L_e = (\sqrt{3}/2)(2R+d) - R + (1/2)\pi R$, the straight length is $L = (\sqrt{3}/2)(2R+d)$. From Eq. (1), the tortuosity for Fig. 1(a) is

$$\tau_{1a} = 1 - \frac{\pi - 2}{\sqrt{3}(2+k)}. \tag{11}$$

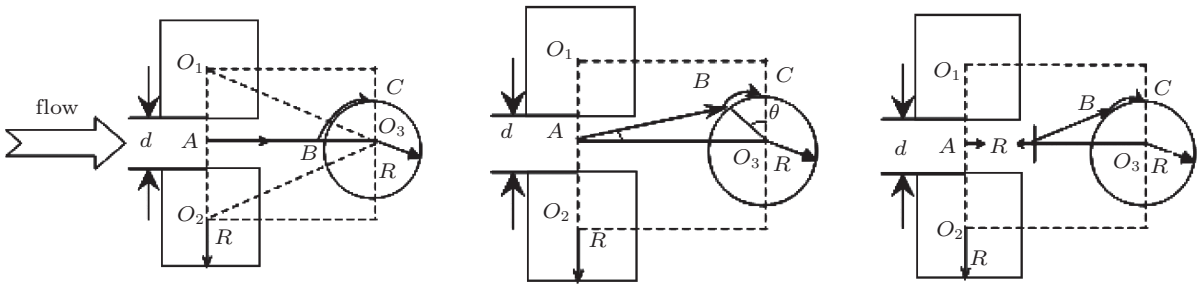


Fig. 1. Fluid flow from square particles to spherical particles in a square area, panels (a), (b), and (c) are possible cases, and the labels A, B and C represent their position, respectively.

For the case of Fig. 1(b), line AB is perpendicular to line BO_3, the actual length of the flow path is $L_{\rm e} = L_{AB} + L_{\widehat{BC}}$,

$$\begin{aligned} L_{AB} &= \sqrt{L_{AO_3}^2 - L_{O_3B}^2} = \sqrt{(3/4)(2R+d)^2 - R^2}, \\ L_{\widehat{BC}} &\simeq \theta R \\ &= R\arcsin(L_{O_3B}/L_{AO_3}) \\ &\quad \times R\arcsin(2R/\sqrt{3}(2R+d)), \end{aligned}$$

the straight length is $L = (\sqrt{3}/2)(2R+d)$.

According to the definition of Eq. (1), the obtained tortuosity for Fig. 1(b) is

$$\tau_{1\rm b} = \frac{\sqrt{\dfrac{3}{4}(2+k)^2} + \arcsin\left(\dfrac{2}{\sqrt{3}(2+k)}\right)}{\dfrac{\sqrt{3}}{2}(2+k)}. \tag{12}$$

Particles in actual porous media randomly distribute, which means that some particles may overlap each other. Therefore, $\widehat{BC}$ in Fig. 1(b) might be sheltered, leading to some streamlines skipping up the $\widehat{BC}$.[8] The actual length of the flow path is L_{AB}. Figure 1(c) is of a similar flow path to Fig. 1(b). They are transformed by using a geometry algorithm and Eq. (1), we could obtain $\tau_{1\rm c^1}$ and $\tau_{1\rm c^2}$

$$\tau_{1\rm c^1} = \frac{1}{\sqrt{1 - \left(\dfrac{2}{\sqrt{3}(2+k)}\right)^2}}, \tag{13}$$

$$\tau_{1\rm c^2} = \frac{1}{\sqrt{1 - \left(\dfrac{2}{\sqrt{3}(2+k) - 2}\right)^2}}. \tag{14}$$

According to Eqs. (11)–(14) and Eq. (6), the averaged tortuosity is

$$\tau_1 = \frac{\tau_{1\rm a} + \tau_{1\rm b} + \tau_{1\rm c^1} + \tau_{1\rm c^2}}{4}. \tag{15}$$

In the case of Fig. 2(a), the actual length of the flow path is $L_{\rm e} = L_{AD} + L_{DB} + L_{BC} = (\sqrt{3}+1)R + (\sqrt{3}/2)d$. The straight length is $L = (\sqrt{3}/2)(2R+d)$. From Eq. (1), the tortuosity for Fig. 2(a) is

$$\tau_{2\rm a} = \frac{2(\sqrt{3}+1)R + \sqrt{3}k}{\sqrt{3}(2+k)}. \tag{16}$$

In the case of Fig. 2(b), the actual length of the flow path is

$$\begin{aligned} L_{\rm e} &= L_{AB} + L_{BC} = \sqrt{L_{AD}^2 + L_{DB}^2} + R \\ &= \sqrt{\left[\frac{\sqrt{3}}{2}(R+d) - R\right]^2 + R^2} + R, \end{aligned}$$

the straight length is

$$L = L_{AO} = \frac{\sqrt{3}}{2}(2R+d).$$

Due to Eq. (1), the tortuosity for Fig. 2(b) is

$$\tau_{2\rm b} = \frac{2\sqrt{\left[\frac{\sqrt{3}}{2}(1+k) - 1\right]^2 + 1} + 1}{\sqrt{3}(1+k)}. \tag{17}$$

Figure 2(c) is another similar flow path situation. The length of L_{BC} is possibly sheltered, so the actual length of the flow path is

$$L_{\rm e} = \sqrt{L_{A_1D}^2 + L_{DB}^2} = \sqrt{\left[\frac{\sqrt{3}}{2}(2R+d) - 2R\right]^2 + R^2},$$

the straight length is

$$L = L_{A_1O} = \frac{\sqrt{3}}{2}(2R+d) - R.$$

According to Eq. (1), we have

$$\tau_{2\rm c} = \frac{\sqrt{\left[\frac{\sqrt{3}}{2}(2R+d) - 2R\right]^2 + R^2}}{\frac{\sqrt{3}}{2}(2R+d) - 2R}. \tag{18}$$

According to Eqs. (16)–(18) and Eq. (6), in this case, the averaged tortuosity is

$$\tau_2 = \frac{\tau_{2\rm a} + \tau_{2\rm b} + \tau_{2\rm c}}{3}. \tag{19}$$

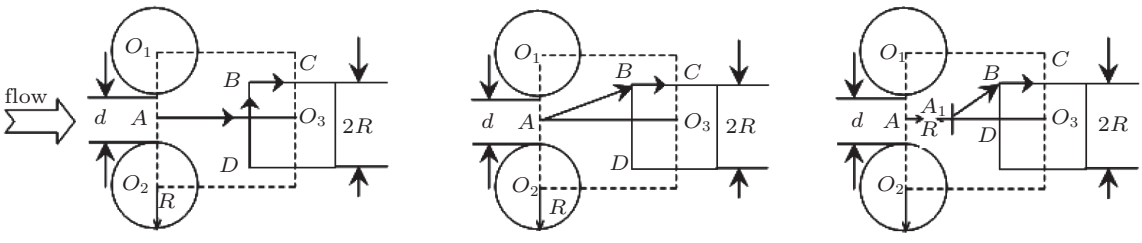


Fig. 2. Fluid flow from spherical particles to square particles in a square form, panels (a), (b), and (c) are possible cases, respectively.

Since particles in actual porous media distribute irregularly, we consider another possible case that circles and squares are arranged alternately, as shown in Fig. 3.

For streamline 1 in Fig. 3, the actual length of the flow path is $L_e = d + 4R$, the straight length is $L = 2R + d$. According to Eq. (1), we obtain

$$\tau_{31} = 1 + \frac{2}{2+k}. \tag{20}$$

For streamline 2 in Fig. 3, the actual length of the flow path is $L_e = d + \pi R$, the straight length is $L = 2R + d$. So its tortuosity is

$$\tau_{32} = \frac{k+\pi}{2+k}. \tag{21}$$

For streamline 3 in Fig. 3, the actual length is equal to the straight length of the flow path in the unit cell, so its tortuosity is 1

$$\tau_{33} = 1. \tag{22}$$

The tortuosity of the weighted average is expressed as

$$\tau_3 = \frac{2R}{\sqrt{3}(2R+d)}\tau_{31} + \frac{2R}{\sqrt{3}(2R+d)}\tau_{32} + \frac{\sqrt{3}(2R+d)-4R}{\sqrt{3}(2R+d)}\tau_{33}, \tag{23}$$

where $2R/\sqrt{3}(2R+d)$ and

$$[\sqrt{3}(2R+d)-4R]/\sqrt{3}(2R+d)$$

are defined as the weighted factor.

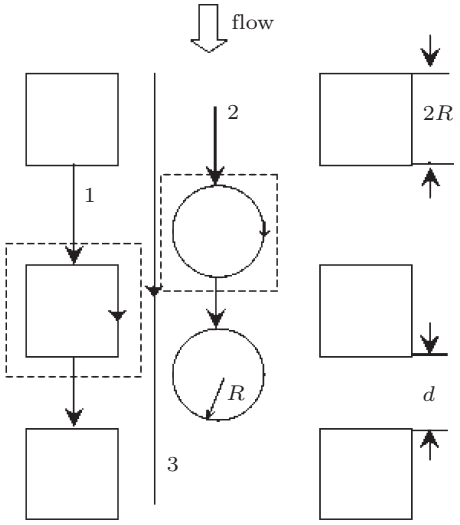


Fig. 3. A possible case, fluid flow through media where spheres and squares are side by side.

Finally, we can obtain the averaged tortuosity of the flow path in two-dimensional mixed porous media, where the circular and square particles are arranged with different configurations by averaging Eqs. (15), (19), (23), and Eq. (6).

$$\tau = \frac{\tau_1 + \tau_2 + \tau_3}{3}. \tag{24}$$

It is evident that the tortuosity (Eq. (24)) is a function of porosity for a simple and approximate flow path model in this paper, and there is no empirical constant in this model.

In addition, in Eq. (2), the tortuosity is related to the tortuosity fractal dimension, so substituting Eq. (24) into Eq. (2) gives

$$D_T = \frac{\ln \tau}{\ln\left(\dfrac{L}{L_{\min}}\right)}. \tag{25}$$

We define the ratio of the maximum pore size to the minimum pore size $\alpha = L/L_{\min}$. It is seen that equation (25) is a function of porosity ϕ and ratio α.

3. Results and discussion

In Fig. 4, the predicted results of tortuosity by using the present model predictions τ are compared with the available experimental and simulation data (viz. Eqs. (3)–(5)). Equation (3) is attained from experiment by using a circular model ($p = 0.41$), rectangle model ($p = 0.63$).[7] Equations (4) and (5) are attained from LG simulation by using a random rectangle model.[5,6] From Fig. 4, it can be seen that the curve of the present model lies between the curves of the rectangle model and the circular model. It is not surprising that the present model is obtained in mixed porous media packed with circles and squares, which is closer to the real situation. Furthermore, in an ideal model composed only of circles or squares, it can be shown that the tortuosity of the circular model is always smaller than that of the square model. It can be explained that the square model is a longer flow path than the circular model, the increased speed of the actual flow path length of L_e is quicker than that in the circular model. In addition, when the porosity $\phi \to 1$, the tortuosity tends to one. This is consistent with the physical situation.[2,3,27] It is noted that when porosity $\phi < 0.46$, the tortuosity of our model is higher than that in circular and square models. Therefore,

the present model is in agreement with the existing results when the porosity ranges from 0.46 to 1.

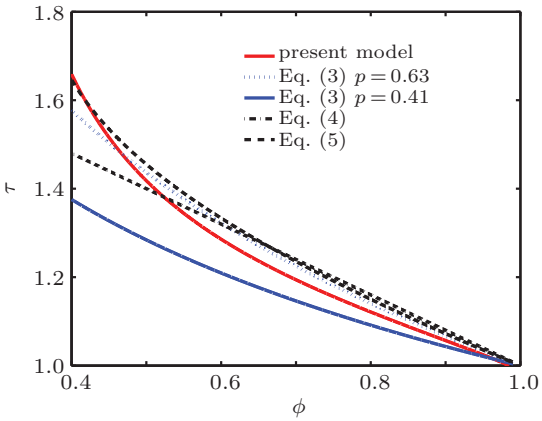


Fig. 4. (colour online) A comparison between the present model and existing validity experiment[7] and simulation,[5,6] the tortuosity versus porosity ϕ.

We compare the predicted result of the present model with the result from Eq. (5), which is obtained by the LG simulation.[6] It is seen from Fig. 5 that the percolation threshold in the present model is close to $\phi_c = 0.33$. The reason is that flow paths are blocked when the porosity is lower than a certain value. It has been found that the percolation threshold at the core region in packed beds is usually between $0.3 \sim 0.4$.[30,31]

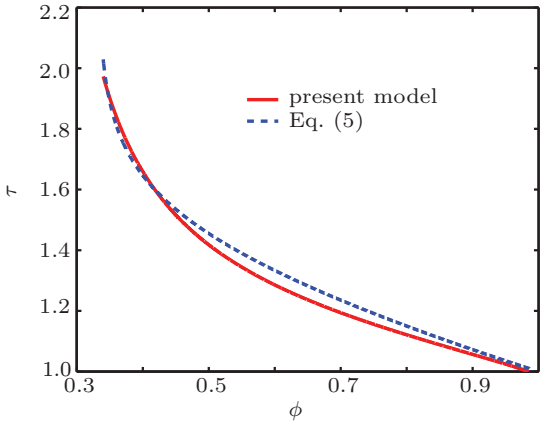


Fig. 5. (colour online) A comparison between the present model and an existing model[6] of the tortuosity versus porosity ϕ in the percolation threshold.

The proposed model is expressed as a function of porosity and there is no empirical constant. Although experimental correlations look simple, they have one or more empirical constants which have no physical meaning. It is also worth while pointing out that the correlation (Eq. (3)) allows the porosity to be zero, which is inconsistent with the practical situation. Therefore, the experimental correlation Eq. (3),[7] may present unreasonable results at low porosity. In all, a good agreement is found between the present model and the existing experimental and simulated results.

Figure 6 presents the tortuosity fractal dimension D_T versus porosity ϕ with different ratios α. It is found from Fig. 6 that the fractal dimension of tortuous flow paths decreases with the increase of the ratio α. This implies that larger pores dominate the creeping flow behavior in porous media, and usually a capillary with a larger diameter has a less tortuous flow path. Thus, we have lower values of the fractal dimension for tortuous flow paths at a higher ratio α.[32] The range ($10^3 \sim 10^4$) of the ratio α of the maximum pore size to the minimum pore size was obtained by Katz and Thompson[33] by using scanning electron microscopy. When the ratio is 10^4, the tortuosity fractal dimension $D_c = 1.07 \pm 0.009$. This is very close to the tortuosity fractal dimension $D_T = 1.081$,[27] which was obtained by Wheatctaft and Tylers by simulating fluid flow through porous media based on the Monte Carlo method. In addition, the tortuosity fractal dimension obtained by Yu *et al.*[34] is 1.10 as measured by the box-counting method,[35] and they also recognized that the ratio is greater than 10^2. It is seen from Fig. 6 that the ratio is 10^3, which is in good agreement with existing results. From Fig. 6, it is seen that the porosity is close to 1, $D_T = 1.0$. This is expected because the streamline in the medium is straight when porosity is 1, so the tortuosity fractal dimension $D_T = 1.0$.[29–31] A limited case of $D_T = 2.0$ has a highly tortuous streamline that fills a plane according to the tortuous theory.[27,28] From Fig. 6, the value of the tortuosity fractal dimension is close to 2 at the low ratio $\alpha = 2$. In the present model, because the percolation threshold is a constant $\phi_c = 0.320$, the porosity

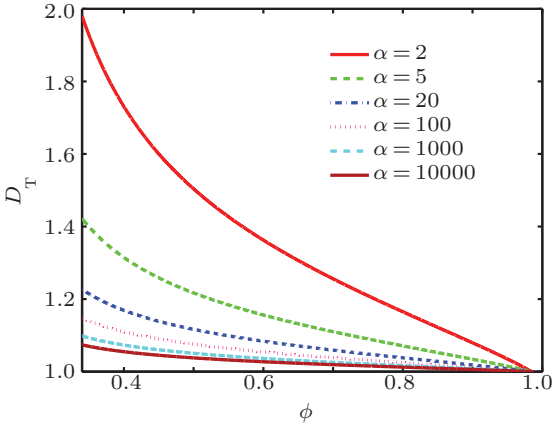


Fig. 6. (colour online) The tortuosity fractal dimension D_T versus porosity ϕ at different ratios α.

ϕ range is $\phi_c \leq \phi \leq 1$. At the point $\phi_c = 0.320$, the tortuosity fractal dimension D_T is not equal to zero, it is limit $D_{T_c} = 1.07$. Therefore, we can obtain that the tortuosity fractal dimension is in the range of $D_{T_c} \leq D_T \leq 2$.

4. Conclusion

We have investigated the tortuosity of fluid flow through porous media. We think that the porous media should be filled with circular and square particles. The tortuosity can be expressed as a function of porosity. There is no empirical constant in the proposed expression. In addition, the correlation between the tortuosity fractal dimension and the porosity is also expressed as a function in the present study, and the thresholds of tortuosity fractal dimension and the porosity are obtained, i.e., $D_{T_c} = 1.07$, $\phi_c = 0.32$. The predicted results of the present model coincide with experimental and simulated results. Therefore, we expect that our work will be helpful for the further studies on flow in porous media.

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