Nonclassicality of a two-variable Hermite polynomial state*

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The nonclassicality of the two-variable Hermite polynomial state is investigated. It is found that the two-variable Hermite polynomial state can be considered as a two-mode photon subtracted squeezed vacuum state. A compact expression for the Wigner function is also derived analytically by using the Weyl-ordered operator invariance under similar transformations. Especially, the nonclassicality is discussed in terms of the negativity of the Wigner function. Then violations of Bell's inequality for the two-variable Hermite polynomial state are studied.

Keywords: nonclassicality, Hermite polynomial state, Wigner function, Bell's inequality

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1. Introduction

Due to their potential applications, squeezed states and squeezing operators have been an important topic in quantum optics. Recently, more and more researchers have devoted their time to researching the well-behavior nonclassical properties of squeezed states,[1−3] especially for the two-mode squeezed states.[4−11] For example, phase properties for the two-mode squeezed states have been widely investigated by use of the Pegg–Barnett phase formalism.[5−7] Hiroshima also studied the decoherence of two-mode squeezed vacuum states by analyzing the relative entropy of entanglement.[8] The two-mode squeezed-vacuum state is regarded as a quantum channel to realize quantum teleportation and quantum dense coding,[9] respectively. In addition, Hu and Fan[10] found that the two-mode squeezed number state is considered as a two-mode squeezed vacuum state excited by the two-variable Hermite polynomial. Some generalized two-mode squeezed states have also been constructed, which are great progress in quantum optics.[12−14] Because these states exhibit quite significant and unusual physical properties, the study of these states can not only deepen our understanding of the nature of quantum fields but also help us to realize these fields experimentally for possible future applications, such as in weak-signal detection, quantum computing, and quantum coding.

On the other hand, nonclassical states play an important role in quantum computation and quantum information processing. The nonclassicality of quantum states can be characterized by some nonclassical properties, such as negativity of the Wigner function (WF), squeezing in one of the phase quadrature components, and sub-Poissonian photon statistics. Among them, the partial negativity of the WF is a good indication of the highly nonclassical character of the state.[15,16] Therefore it is worthwhile obtaining the WF for any states and to measure their negative values. In addition, the negativity of the WF is also used to describe the decoherence of quantum states, e.g., the excited coherent state in both photon-loss and thermal channels,[17,18] the single-photon subtracted squeezed vacuum state in both amplitude decay and phase damping channels,[19] and so on. In Ref. [20] the WF of a thermo-invariant coherent state is derived by using the thermal Wigner operator of the thermo-field dynamics in the coherent thermal state representation and the integration within ordered products (IWOP) technique.[21,22] For other examples of calculating the WFs, refer to Refs. [23]–[27].

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In this paper, we will investigate the nonclassical properties of the two-variable Hermite polynomial state (THPS) in terms of squeezing properties, photon-number distribution, WF, and Bell's inequality. The rest of this paper is arranged as follows. In Section 2, the normalization factor of the THPS is calculated, and its comparison with the two-mode subtracted squeezed vacuum state (TPSSV) has also been done. In Section 3, the nonclassical properties of the THPS, such as squeezing properties, photon-number distribution, are calculated analytically and then discussed in detail. Especially, In Section 4, the explicit expression of the WF of the THPS is derived by using the Weyl-ordered operator invariance under similar transformations, which is related to two-variable non-Gaussian–Hermite polynomials, and then its nonclassicality is discussed in terms of the negativity of the WF, which implies highly nonclassical properties of the quantum state. In Section 5, violations of the Bell's inequality for the THPS are studied. We can find that the violation of the Bell's inequality becomes more observable with a decrease of J, p, and q (see Eq. (35) below). Conclusions are drawn in the last section.

2. The THPS and its comparison with the TPSSV

2.1. Two-variable Hermite polynomial state

The concept of sum-frequency squeezing for the two variables Z_1 and Z_2 was introduced by Fan *et al.*[28]

$$Z_1 = \frac{1}{2}(a_1^\dagger a_2^\dagger + a_1 a_2), \quad Z_2 = \frac{\mathrm{i}}{2}(a_1^\dagger a_2^\dagger - a_1 a_2), \quad (1)$$

which obey the uncertainty relation $\Delta Z_1 \Delta Z_2 \geq \frac{1}{4}\langle N_1 + N_2 + 1\rangle$ when $N_i = a_i^\dagger a_i$ $(i = 1, 2)$ is the i-th mode photon number operator. If $(\Delta Z_i)^2 < \frac{1}{4}\langle N_1 + N_2 + 1\rangle$, then the state is sum-frequency squeezing in the Z_i direction. The minimum uncertainty states that make the uncertainty relation minimum $\Delta Z_1 \Delta Z_2 = \frac{1}{4}\langle N_1 + N_2 + 1\rangle$ for sum-frequency squeezing are the solutions to the eigenvalue equation:

$$(Z_1 + \mathrm{i}\lambda Z_2)\,|\Psi\rangle = \beta\,|\Psi\rangle\,, \quad (2)$$

where $\beta = \frac{1}{2}\sqrt{\lambda^2 - 1}(M + 1)$, M is nonnegative integer, and $\lambda \geq 1$. Through concise calculations, the THPS $|\Psi\rangle$ can be expressed as

$$|\Psi\rangle = C_{p,q}(\lambda) S(\xi)\, \mathrm{H}_{p,q}(\upsilon a_1^\dagger, \upsilon a_2^\dagger)\,|0, 0\rangle\,, \quad (3)$$

where $S(\xi) = \exp(\xi a_1^\dagger a_2^\dagger - \xi^* a_1 a_2)$ is the two-mode squeezing operator and $\xi = r\,\mathrm{e}^{\mathrm{i}\theta}$ is the squeezing parameter. The parameters r and θ are defined as

$$\upsilon^2 = \begin{cases} -\mathrm{i}\sqrt{1-\lambda^2}, & 0 < \lambda < 1,\ \theta = \dfrac{\pi}{2}, \quad \tanh r = \left(\dfrac{1-\lambda}{1+\lambda}\right)^{1/2}, \\ -\dfrac{\sqrt{\lambda^2-1}}{\lambda}, & \lambda \geq 1,\ \theta = 0, \quad \tanh r = \left(\dfrac{\lambda-1}{1+\lambda}\right)^{1/2}, \end{cases} \quad (4)$$

and $C_{p,q}(\lambda)$ is the normalization constant of the state $|\Psi\rangle$. Using the generating function of $\mathrm{H}_{m,n}(x, y)$,

$$\mathrm{H}_{m,n}(x, y) = \frac{\partial^{m+n}}{\partial t^m \partial t'^n} \exp\left[-tt' + tx + t'y\right]\Big|_{t=t'=0}, \quad (5)$$

we can see that the normalization condition can be given by

$$\begin{aligned} C_{p,q}^{-2}(\lambda) =& \langle 0, 0|\, \mathrm{H}_{p,q}(\upsilon a_1, \upsilon a_2) S(\xi)^\dagger S(\xi)\, \mathrm{H}_{p,q}(\upsilon a_1^\dagger, \upsilon a_2^\dagger)\,|0, 0\rangle \\ =& \frac{\partial^{(p+q)^2}}{\partial t^p \partial t'^q \partial \tau^p \partial \tau'^q} \int \frac{\mathrm{d}^2 z_1\, \mathrm{d}^2 z_2}{\pi^2} \exp\left[-|z_1|^2 + t\upsilon z_1 + \tau \upsilon z_1^*\right] \\ & \times\ \exp[-|z_2|^2 + t'\upsilon z_2 + \tau'\upsilon z_2^* - \tau\tau' - tt']\Big|_{t=t'=\tau=\tau' 0} \\ =& \frac{\partial^{(p+q)^2}}{\partial t^p \partial t'^q \partial \tau^p \partial \tau'^q} \exp\left[t\tau\upsilon^2 + t'\tau'\upsilon^2 - tt' - \tau\tau'\right]\big|_{t=t'=\tau=\tau'=0}, \end{aligned} \quad (6)$$

where we have used the formula

$$\int \frac{\mathrm{d}^2 z}{\pi} \mathrm{e}^{\zeta|z|^2+\xi z+\eta z^*} = -\frac{1}{\zeta} \mathrm{e}^{-\xi\eta/\zeta}, \quad \mathrm{Re}\,(\zeta) < 0. \quad (7)$$

Thus the normalization factor is obtained

$$C_{p,q}^{-2}(\lambda) = \sum_{l=0}^{\min(p,q)} \frac{(p!q!)^2 \,|\upsilon|^{2(p+q-2l)}}{(l!)^2\,(q-l)!\,(p-l)!}. \quad (8)$$

2.2. Relation between the THPS and the TPSSV

The definition of the two-mode subtracted squeezed vacuum state is given by

$$|r_1, m, n\rangle = \frac{1}{N_{r_1,m,n}} a_1^m a_2^n S_2\,(\xi_1)\,|0,0\rangle = \frac{\left(\mathrm{e}^{i\theta_1}\right)^n \sinh^{m+n/2} r_1}{\left(\mathrm{i}\sqrt{2}\right)^{m+n}} S_2\,(r_1) \times \mathrm{H}_{m,n}\left(\mathrm{i}\,\mathrm{e}^{\mathrm{i}\theta_1}\sqrt{\tanh r_1}a_2^\dagger, \mathrm{i}\sqrt{\tanh r_1}a_1^\dagger\right)|0,0\rangle\,, \quad (9)$$

where $\xi_1 = r_1\,\mathrm{e}^{\mathrm{i}\theta_1}$ is the squeezing parameter, and the normalization constant $N_{r_1,m,n}$ for the state $|r_1, m, n\rangle$ is given by

$$N_{r_1,m,n} = m!n!\sinh^{2n}\lambda P_m^{(n-m,0)}\left(\cosh 2r_1\right). \quad (10)$$

Compared Eq. (3) with Eq. (9), we can find that

$$|\psi\rangle = \begin{cases} \dfrac{C_{p,q}(\lambda)\left(\mathrm{i}\sqrt{2}\right)^{p+q}}{\sinh^{(p+q)/2} r_1} S(\mathrm{i}r - r_1) a_1^q a_2^p S_2(r_1)\,|0,0\rangle\,, & 0 < \lambda < 1, r_1 = \arctan\left(\mathrm{i}\sqrt{1-\lambda^2}\right), \theta_1 = 0, \\ \dfrac{C_{p,q}(\lambda)\left(\mathrm{i}\sqrt{2}\right)^{p+q}}{\sinh^{(p+q)/2} r_1} S(r - r_1) a_1^q a_2^p S_2(r_1)\,|0,0\rangle\,, & \lambda \geq 1, r_1 = \arctan\left(\dfrac{\sqrt{\lambda^2-1}}{\lambda}\right), \theta_1 = 0. \end{cases} \quad (11)$$

Next we may use the TPSSV to discuss the THPS and its properties. From Eqs. (10) and (11) we can easily calculate the average photon number in the THPS (denoting $\tau = \cosh 2r_1$, $u = \xi - \xi_1$),

$$\langle a_1^\dagger a_1\rangle = \left(\frac{C_{p,q}(\lambda)\left(\mathrm{i}\sqrt{2}\right)^{p+q}}{\sinh^{(p+q)/2} r_1}\right)^2 [H\cosh^2 u + F\sinh 2u + G\sinh^2 u] + \sinh^2 u, \quad (12)$$

and

$$\langle a_2^\dagger a_2\rangle = \left(\frac{C_{p,q}(\lambda)\left(\mathrm{i}\sqrt{2}\right)^{p+q}}{\sinh^{(p+q)/2} r_1}\right)^2 [G\cosh^2 u + F\sinh 2u + H\sinh^2 u] + \sinh^2 u. \quad (13)$$

In a similar way we have

$$\begin{aligned}\langle a_1^\dagger a_2^\dagger a_1 a_2\rangle = {} & \left(\frac{C_{p,q}(\lambda)\left(\mathrm{i}\sqrt{2}\right)^{p+q}}{\sinh^{(p+q)/2} 2r_1}\right)^2 \times \left[(B+C)\frac{\sinh 4u}{2} + A\frac{\sinh^2 2u}{2}\right. \\ & + \left(\frac{\sinh^2 2u}{2} + 2\sinh^2 u\sinh 2u\right)F \\ & + (E+D)\frac{\sinh^2 2u}{4} \\ & + \left(\frac{\sinh^2 2u}{2} + \sinh^4 u + \cosh^4 u\right)I \\ & \left. + \left(\frac{3\sinh^2 2u}{4} + \sinh^4 u\right)(H+G)\right] \\ & + \sinh^4 u + \frac{1}{4}\sinh^2 2u, \end{aligned} \quad (14)$$

where we have set

$$\begin{aligned} A &= q!\,(p+2)!\sinh^{2p+2} r_1\cosh^2 r_1 P_q^{(p-q,2)}(\tau), \\ B &= q!\,(p+2)!\sinh^{2p+3} r_1\cosh r_1 P_q^{(p-q+1,1)}(\tau), \\ C &= (q+1)!\,(p+1)!\sinh^{2p+1} r_1\cosh r_1 P_{q+1}^{(p-q-1,1)}(\tau), \\ D &= q!\,(p+2)!\sinh^{2p+4} r_1 P_q^{(p-q+2,0)}(\tau), \\ E &= (q+2)!p!\sinh^{2p} r_1 P_{q+2}^{(p-q-2,0)}(\tau), \\ F &= q!\,(p+1)!\sinh^{2p+1} r_1\cosh r_1 P_q^{(p-q,1)}(\tau), \\ G &= q!\,(p+1)!\sinh^{2p+2} r_1 P_q^{(p-q+1,0)}(\tau), \\ H &= (q+1)!p!\sinh^{2p} r_1 P_{q+1}^{(p-q-1,0)}(\tau), \\ I &= (q+1)!\,(p+1)!\sinh^{2p+2} r_1 P_{q+1}^{(p-q,0)}(\tau). \end{aligned} \quad (15)$$

Thus the cross-correlation function $g_{12}^{(2)}$ can be obtained by

$$g_{12}^{(2)}(\lambda) = \frac{\langle a_1^\dagger a_2^\dagger a_1 a_2\rangle}{\langle a_1^\dagger a_1\rangle\langle a_2^\dagger a_2\rangle}. \quad (16)$$

Actually, the cross correlation between the two modes reflects the correlation between photons in two different modes, which plays a key role in rendering many two-mode radiations nonclassically. In Fig. 1, we plot the graph of $g_{12}^{(2)}(\lambda)$ as the function of λ for some different (m, n) values. It is shown that $g_{12}^{(2)}(\lambda)$ is always larger than the unit, thus there exist correlations between the two modes.

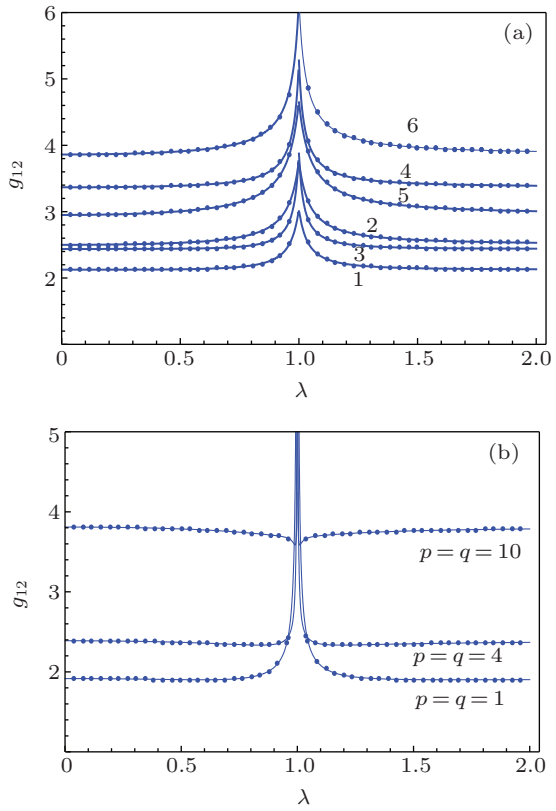


Fig. 1. (colour online) Cross-correlation function between the two modes a_1 and a_2 as a function of λ for different parameters (p, q). The numbers $1, 2, 3, 4, 5, 6$ in panel (a) denote that (q, p) are equal to $(1, 2)$, $(3, 4)$, $(2,4)$, $(6,8)$, $(3,6)$, and $(7,10)$, respectively.

3. Nonclassicality of the two-variable Hermite polynomials state

3.1. Squeezing properties

For a two-mode system, the optical quadrature phase amplitudes can be expressed as follows:

$$Q = \frac{Q_1 + Q_2}{\sqrt{2}}, \quad P = \frac{P_1 + P_2}{\sqrt{2}}, \quad [Q, P] = \mathrm{i}, \quad (17)$$

where $Q_j = (a_j + a_j^\dagger)/\sqrt{2}$, $P_j = (a_j - a_j^\dagger)/(\sqrt{2}\mathrm{i})$ $(j = 1, 2)$ are coordinate and momentum operator, respectively. Their variances are $(\Delta Q)^2 = \langle Q^2\rangle - \langle Q\rangle^2$ and $(\Delta P)^2 = \langle P^2\rangle - \langle P\rangle^2$. The phase amplifications satisfy the uncertainty relation of quantum mechanics $\Delta Q \Delta P \geq 1/2$. By using Eqs. (10) and (11), it is easy to see that $\langle a_1\rangle = \langle a_1^\dagger\rangle = \langle a_2\rangle = \langle a_2^\dagger\rangle = 0$, $\langle a_1^2\rangle = \langle a_1^{\dagger 2}\rangle = \langle a_2^2\rangle = \langle a_2^{\dagger 2}\rangle = 0$, and $\langle a_1 a_2^\dagger\rangle = \langle a_1^\dagger a_2\rangle = 0$, which leads to $\langle Q\rangle = 0$ and $\langle P\rangle = 0$. Moreover, using Eq. (10) one can see

$$\langle a_1 a_2\rangle = \left\langle a_1^\dagger a_2^\dagger\right\rangle = \left(\frac{C_{p,q}(\lambda)\left(\mathrm{i}\sqrt{2}\right)^{p+q}}{\sinh^{(p+q)/2} 2r_1}\right)^2 \times \left[F\cosh 2u + \frac{\sinh 2u}{2}(G+H)\right] + \frac{\sinh 2u}{2}. \quad (18)$$

From Eqs. (12), (13), and (18), it then follows that

$$(\Delta Q)^2 = \frac{\mathrm{e}^{2u}}{2}\left[\left(\frac{C_{p,q}(\lambda)\left(\mathrm{i}\sqrt{2}\right)^{p+q}}{\sinh^{(p+q)/2} r_1}\right)^2 \times (2F + G + H) + 1\right], \quad (19)$$

and

$$(\Delta P)^2 = \frac{\mathrm{e}^{-2u}}{2}\left[\left(\frac{C_{p,q}(\lambda)\left(\mathrm{i}\sqrt{2}\right)^{p+q}}{\sinh^{(p+q)/2} r_1}\right)^2 \times (-2F + G + H) + 1\right]. \quad (20)$$

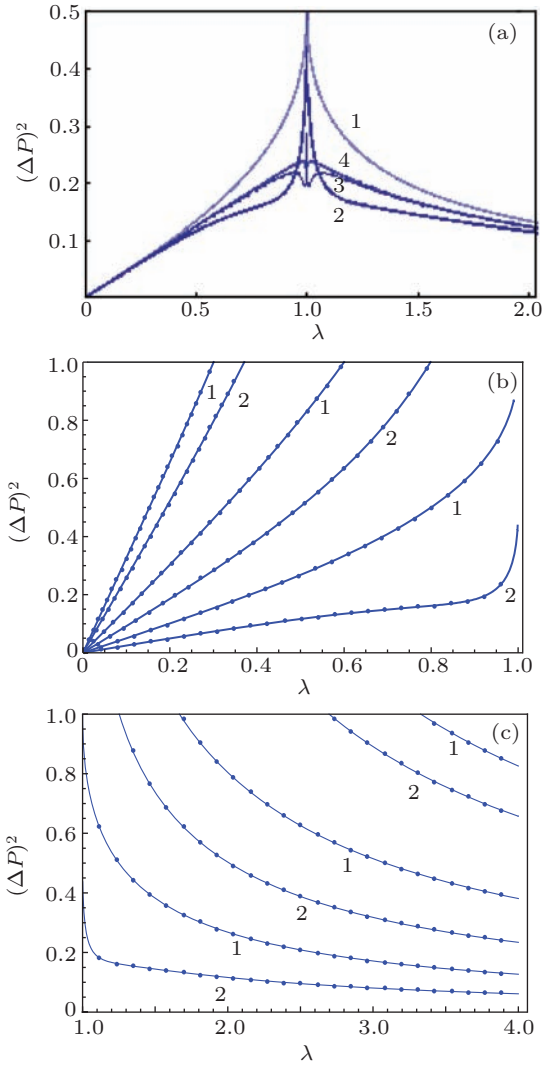


Fig. 2. (colour online) The fluctuations of $(\Delta P)^2$ with other parameters p, q values; in panel (a), the numbers 1, 2, 3, and 4 denote $p = q = 0, 1, 6, 24$; in panels (b) and (c), the numbers 1 and 2 denote $p = 0$ and $p = 1$, respectively, and $q = 1, 5, 12$ from down to up.

Next, let us analyze some special cases. When $p = q = 0$, corresponding to the two-mode squeezed state, equations (19) and (20) become, respectively,

$$(\Delta Q)^2\Big|_{p=0,q=0} = \frac{1}{2}\,\mathrm{e}^{2(r_1+u)},$$
$$(\Delta P)^2\Big|_{p=0,q=0} = \frac{1}{2}\,\mathrm{e}^{-2(r_1+u)}, \quad \Delta Q\Delta P = \frac{1}{2}, \tag{21}$$

which is just the standard squeezing case. One can see that the state $|\Psi\rangle$ is squeezed at the "p-direction". In order to see clearly the fluctuations of $(\Delta P)^2$ with other parameters p, q values, the figures are plotted in Fig. 2.

From Fig. 2(a) one can see that the fluctuations of $(\Delta P)^2$ are always less than 1/2, when $p = q$, say, the state $|\Psi\rangle$ is always squeezed at the "p-direction"; For $0 < \lambda < 1$ and the given p values, there exists the squeezing effect only when the squeezing parameter is less than a certain threshold value which increases with the increase of q (see Fig. 2(b)); However, for $\lambda \geq 1$, there exists the squeezing effect only when the squeezing parameter exceeds a certain threshold value which increases with the increase of q (see Fig. 2(c)). For details about the single-mode photon-subtracted case, one can refer to Ref. [29].

3.2. Photon-number distribution of the THPS

Next we discuss the photon-number distribution (PND) of the THPS. Noticing $a^{\dagger m}|n\rangle = \sqrt{(m+n)!/n!}\,|n+m\rangle$ and using the non-normalized coherent state $|\alpha\rangle = \exp[\alpha a^\dagger]\,|0\rangle$,[28,29] we have

$$|n\rangle = \frac{1}{\sqrt{n!}}\frac{\mathrm{d}^n}{\mathrm{d}\alpha^n}\,|\alpha\rangle\Big|_{\alpha=0}.$$

Considering this result, it is easy to see that

$$\langle n_1, n_2\,|\Psi\rangle = (-1)^{n_2+p+1}\,\frac{\left(\mathrm{i}\sqrt{2}\right)^{p+q}(n_2+p)!\,(\tanh r_1)^{n_1+q}\,\mathrm{sech}(\xi-r_1)\,\mathrm{sech}\,r_1}{\sqrt{n_1!n_2!}\sinh^{(p+q)/2} r_1\,(\tanh(\xi-r_1)\tanh r_1+1)^{(n_2+p+1)}}\delta_{n_1+q,n_2+p}, \tag{22}$$

where we have used the formula

$$\int \frac{\mathrm{d}^2 z_1}{\pi}\exp\left\{-|z_1|^2+\tau z_1\right\} f(z_1^*) = f(\tau), \tag{23}$$

$$\int \frac{\mathrm{d}^2 z_2}{\pi} z^k z^{*n}\exp[-\varsigma|z|^2] = \delta_{n,k}\,(-1)^{k+1}\,(\varsigma)^{-(k+1)}\,k!. \tag{24}$$

Finally it is easy to have the PND of $|\Psi\rangle$

$$P(n_1,n_2) = \frac{C_{p,q}^2(\lambda)2^{p+q}}{n_1!n_2!}\left|\frac{(n_2+p)!\,(\tanh r_1)^{n_1+q}\,\mathrm{sech}(\xi-r_1)\,\mathrm{sech}\,r_1}{\sinh^{(p+q)/2} r_1\,[\tanh(\xi-r_1)\tanh r_1+1]^{(n_2+p+1)}}\delta_{n_1+q,n_2+p}\right|^2. \tag{25}$$

From Eq. (25), for the photon number distribution (see Fig. 3) one can see that there exists a constrained condition, $n_1 + q = n_2 + p$. In particular, when $p = q = 0$, equation (25) becomes

$$P(n_1,n_2) = \begin{cases} \mathrm{sech}^2\,\xi\tanh^{2n_2}\xi, & n_1 = n_2, \\ 0, & n_1 \neq n_2, \end{cases} \tag{26}$$

which is just the PND of two-mode squeezed vacuum states.

In Fig. 3 we plot the distribution $P(n_1,n_2)$ in the Fock space (n_1,n_2) for some given $p = q$ values and squeezing parameter λ. From Fig. 3 it is found that the PND is strained by $n_1 + q = n_2 + p$. It results from the paired presence of photons in a two-mode squeezed state. By subtracting photons, we can move the peak from zero photons to nonzero photons [see Figs. 3(a) and 3(c)]. The position of the peak depends on how many photons are annihilated and how many of the states are squeezed initially. In addition, for example, the PND mainly shifts to the bigger number states and becomes more "flat" and "wide" with the increase of the parameter λ [see Figs. 3(b) and 3(c)].

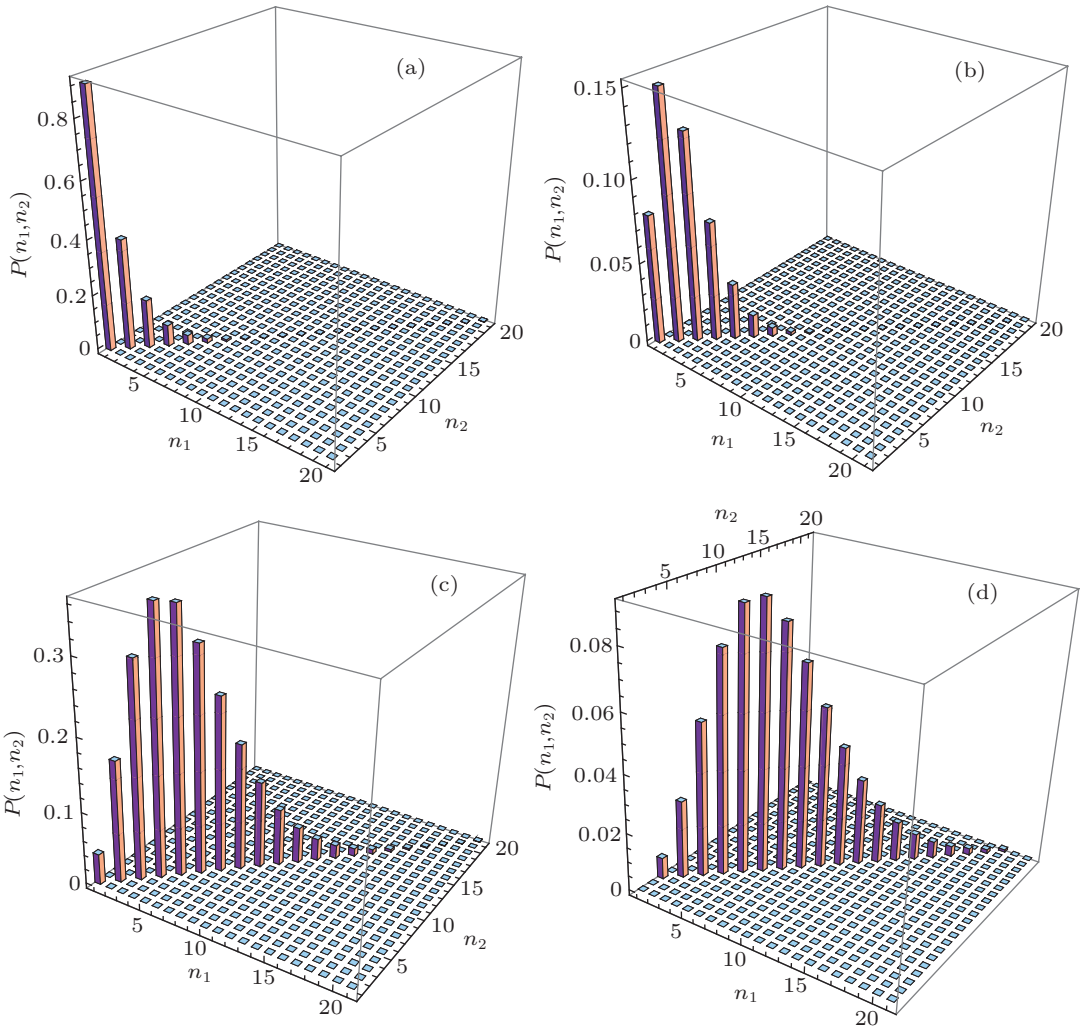


Fig. 3. (colour online) Photon number distribution $P(n_1,n_2)$ in the Fock space (n_1,n_2) for some given $p = q$ values, (a): $p = q = 0$, $\lambda = 1.2$; (b): $p = q = 1$, $\lambda = 1.1$; (c): $p = q = 1$, $\lambda = 1.2$; and (d): $p = 2$, $q = 4$, $\lambda = 1.2$.

4. Wigner function of the THPS

The WF[30−32] is a powerful tool which is used to investigate the nonclassicality of optical fields. Its partial negativity implies the highly nonclassical properties of quantum states. In this section, the analytical expression of the WF for the THPS is derived. For this purpose, we recall the Weyl-ordered form of a single-mode Wigner operator:[33−35]

$$\triangle(\alpha,\alpha^*) = \frac{1}{2}\vdots\,\delta(\alpha - a)\,\delta\left(\alpha^* - a^\dagger\right)\vdots\,, \tag{27}$$

where $\alpha = (x_1 + \mathrm{i}p_1)/\sqrt{2}$ and the symbol $\vdots\ \vdots$ denotes Weyl ordering. The merit of Weyl ordering lies in the Weyl ordered operator invariance under similar transformations proved in Ref. [33], which means

$$S\vdots\,(\ )\,\vdots S^{-1} = \vdots\,S(\ )\,S^{-1}\vdots\,, \tag{28}$$

as if the "fence" $\vdots\ \ \vdots$ did not exist. So S can pass though it. Following this invariance and the above squeezing transformation relations, we have

$$\begin{aligned} &S^\dagger(\xi)\Delta_1(\alpha)\,\Delta_2(\beta)\,S(\xi) \\ &\quad = \frac{1}{4}\vdots\,\delta(\bar{\alpha} - a_1)\,\delta\left(\bar{\alpha}^* - a_1^\dagger\right) \\ &\quad \times\ \delta\left(\bar{\beta} - a_2\right)\delta\left(\bar{\alpha}^* - a_2^\dagger\right)\vdots \\ &\quad = \Delta_1(\bar{\alpha})\,\Delta_2\left(\bar{\beta}\right), \end{aligned} \tag{29}$$

where $\bar{\alpha} = \alpha\cosh\xi - \beta^*\sinh\xi$, $\bar{\beta} = \beta\cosh\xi - \alpha^*\sinh\xi$, and $\beta = (x_2 + \mathrm{i}p_2)/\sqrt{2}$. Thus using Eq. (3) and the coherent state representation of the single-mode

Wigner operator,[38]

$$\Delta_1(\alpha) = \mathrm{e}^{2|\alpha|^2} \int \frac{\mathrm{d}^2 z_1}{\pi^2} |z_1\rangle \langle -z_1| \, \mathrm{e}^{-2(z_1\alpha^* - \alpha z_1^*)}, \tag{30}$$

where

$$|z_1\rangle = \exp\left[-\frac{1}{2}|z_1|^2 + z_1 a_1^\dagger\right]|0\rangle$$

is the Glauber coherent state,[39,40] we can finally obtain the explicit expression of the WF for the THPS:

$$\begin{aligned} W(\alpha,\beta) = {} & \frac{C_{p,q}(\lambda)^2}{\pi^2} \, \mathrm{e}^{-2|\bar{\alpha}|^2 - 2|\bar{\beta}|^2} \\ & \times \sum_{l=0}^{p}\sum_{k=0}^{q} \frac{(-1)^{(l+k)} |\upsilon|^{2(l+k)} [p!q!]^2}{l!k! [(p-l)!(q-k)!]^2} \\ & \times |\mathrm{H}_{p-l,q-k}(B,A)|^2, \end{aligned} \tag{31}$$

where we have set

$$A = 2\bar{\beta}\upsilon^*, \qquad B = 2\bar{\alpha}\upsilon^*.$$

Obviously the WF $W(\alpha,\beta)$ in Eq. (31) is a real function and is non-Gaussian in phase space due to the presence of $\mathrm{H}_{p-l,q-k}(B,A)$. In particular, when $p = q = 0$, equation (31) reduces to

$$W(\alpha,\beta) = \frac{1}{\pi^2} \mathrm{e}^{-2|\bar{\alpha}|^2 - 2|\bar{\beta}|^2}$$

corresponding to the WF of the two-mode squeezed vacuum state. In Figs. 4 and 5, the phase space Wigner distributions are depicted for several different parameter values p, q, and λ. As evidence of the nonclassicality of the state, squeezing in one of the quadrature variables is clear in the plots. In addition, there are some negative regions of the WF in the phase space which is another indicator of the nonclassicality of the state. Figure 4 shows that the negative region becomes greater and visible as the increase of photon number subtracted $p(q)$ and parameter values λ, which may imply that the nonclassicality of the state can be enhanced due to the augment of photon-subtraction number. In addition, the squeezing effect in one of the quadrature variables is clear in the plots (see Figs. 4(a) and 4(b)), which is further evidence of the nonclassicality of this state. In Fig. 5, for the case of $p = 0$ and $q = 1$, it is easily seen that at the center of the phase space ($\alpha = \beta = 0$), the WF is always negative in phase space.

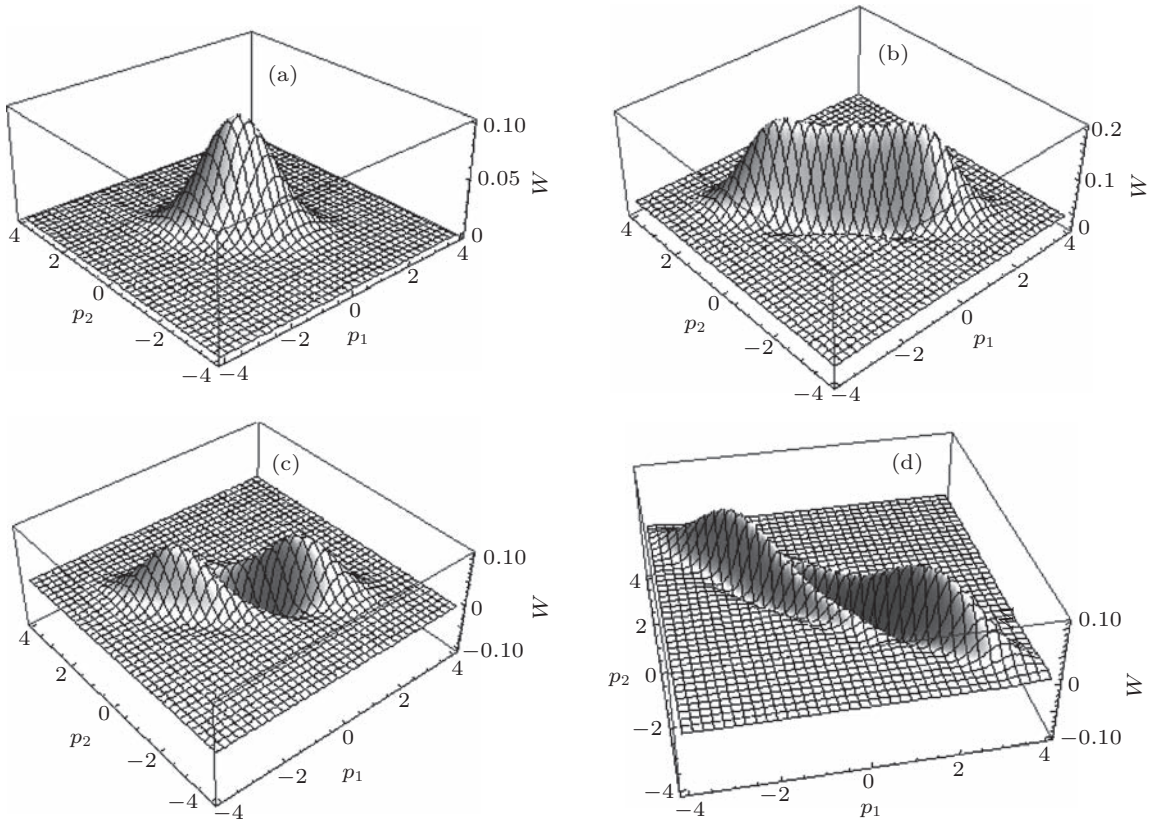


Fig. 4. Wigner function $W(\alpha,\beta)$ in phase space $(0,0,P_1,P_2)$ for several different parameter values p,q and λ, (a): $p = q = 0$, $\lambda = 1.2$; (b): $p = q = 2$, $\lambda = 1.2$; (c): $p = 1$, $q = 2$, $\lambda = 1.2$; and (d): $p = 1$, $q = 2$, $\lambda = 2$.

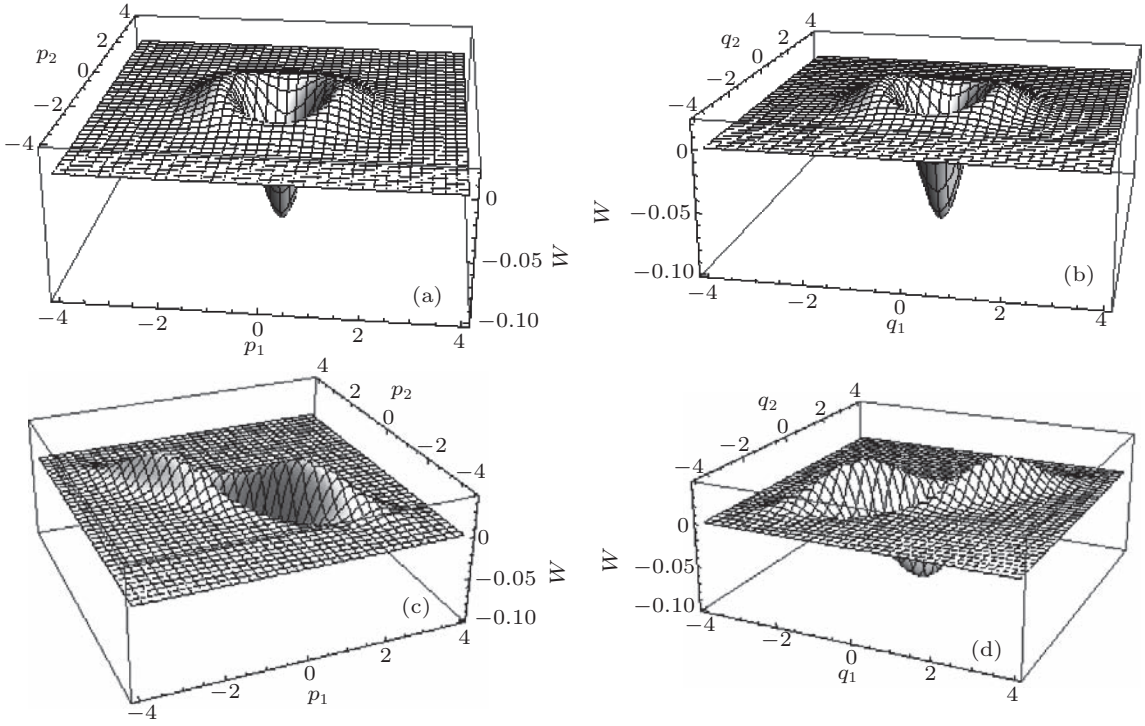


Fig. 5. Wigner function $W(\alpha,\beta)$ in different phase spaces for $p=0$, $q=1$ with $\lambda=0.5$ (first row) and $\lambda=2$ (second row).

5. Violations of Bell's inequality for the THPS

We now turn our attention to the nonlocal properties of the THPS in terms of Bell's inequality. For a two-mode continuous variable system, using correlations between parity measurements, the Bell's inequality is given as follows:

$$|B| = \left|\left\langle \hat{\Pi}^{ab}(\alpha',\beta')\right\rangle + \left\langle \hat{\Pi}^{ab}(\alpha',\beta)\right\rangle + \left\langle \hat{\Pi}^{ab}(\alpha,\beta')\right\rangle + \left\langle \hat{\Pi}^{ab}(\alpha,\beta)\right\rangle\right| \leq 2, \quad (32)$$

where B is the Bell function and the superscripts a and b denote the modes and $\hat{\Pi}^{ab}(\alpha,\beta)$ is the displaced parity operator (wigner operator)[39] defined as[44]

$$\hat{\Pi}^{ab}(\alpha,\beta) = \pi^2 \Delta_a(\alpha)\,\Delta_b(\beta), \quad (33)$$

where $\alpha = (q_1 + \mathrm{i}p_1)/\sqrt{2}$, $\beta = (q_2 + \mathrm{i}p_2)/\sqrt{2}$. The expectation value of this displaced parity operator is just proportional to the two-mode Wigner function, i.e.,

$$\Pi(\alpha,\beta) = \mathrm{tr}[\rho\hat{\Pi}^{ab}(\alpha,\beta)] = \pi^2 W(\alpha,\beta), \quad (34)$$

which shows that the connection between this displaced parity operator and the Wigner function provides an equivalent definition.[42]

The Bell function is measured for any of four combinations of $\alpha = 0$, $\sqrt{J}\,\mathrm{e}^{\mathrm{i}\varphi}$, $\beta = 0$, $\sqrt{J}\,\mathrm{e}^{\mathrm{i}\theta}$, where $J(=|\alpha|^2 = |\beta|^2)$ is a positive constant characterizing the magnitude of the displacement. From these quantities we construct the combination[43]

$$B = \pi^2\Big[W(0,0) + W\left(\sqrt{J}\,\mathrm{e}^{\mathrm{i}\varphi},0\right) + W\left(0,\sqrt{J}\,\mathrm{e}^{\mathrm{i}\theta}\right) - W\left(\sqrt{J}\,\mathrm{e}^{\mathrm{i}\varphi},\sqrt{J}e^{i\theta}\right)\Big]. \quad (35)$$

Then we can test Bell's inequality $-2 \leq B \leq 2$ by means of the two-mode WF measurement. In Figs. 6 and 7, we can say that the THPS is quantum mechanically nonlocal as $|B| > 2$, and the nonlocality is stronger with the increase of $|B|$. We see that the degree of nonlocality depends on the coherent amplitude J, the squeezed parameters λ, p, q, and the phases θ, φ. In Fig. 6, we plot the Bell function in space spanned by parameters J, λ and p, q. One can see clearly that the result of EQ. (35) violates the upper bound imposed by local theories. With the decrease of J, the violation of Bell's inequality becomes more observable for smaller p, q. Especially, when $p = q = 0$ Eq. (35) corresponds to the Bell's inequality of the usual squeezed vacuum state. In Fig. 7, we plot the Bell function B as a function of the phases θ, φ with fixed $J = 0.01$ and several different λ, p, q. One can clearly see from Fig. 7 that B is always greater than zero and for a smaller λ and p, q.

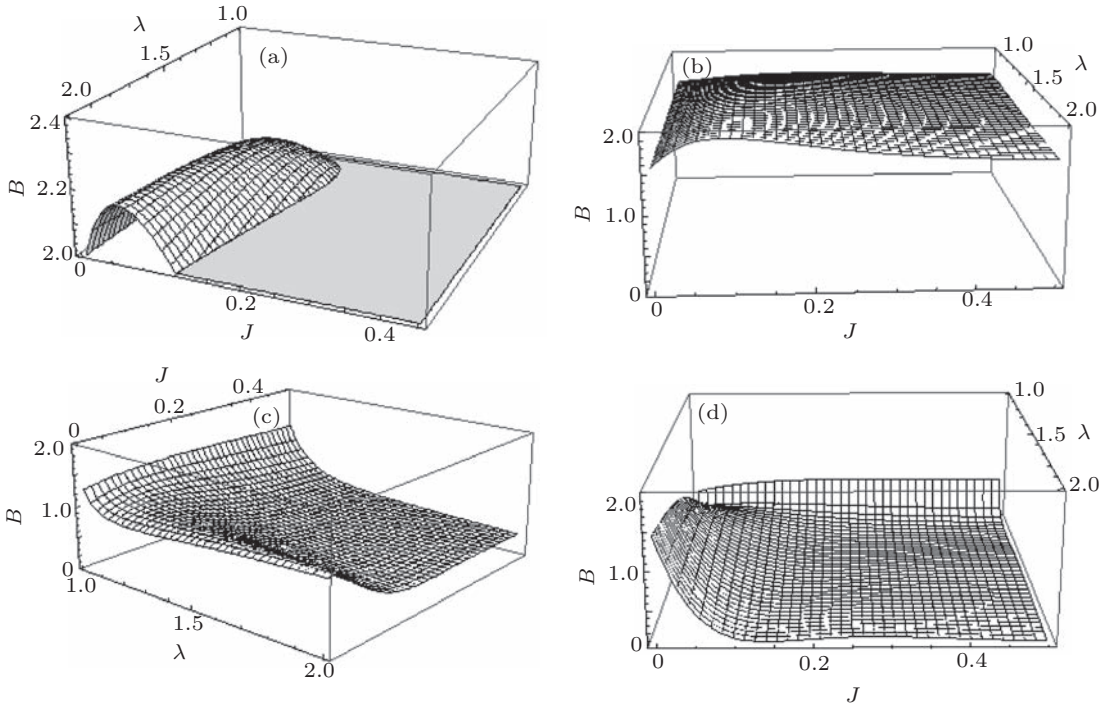


Fig. 6. Plot of the Bell function in the space spanned by parameters J and λ with $\varphi = 0$, $\theta = \pi$, and (a) $p = q = 0$, (b) $p = q = 1$, (c) $p = 0$, $q = 1$, (d) $p = 1$, $q = 2$.

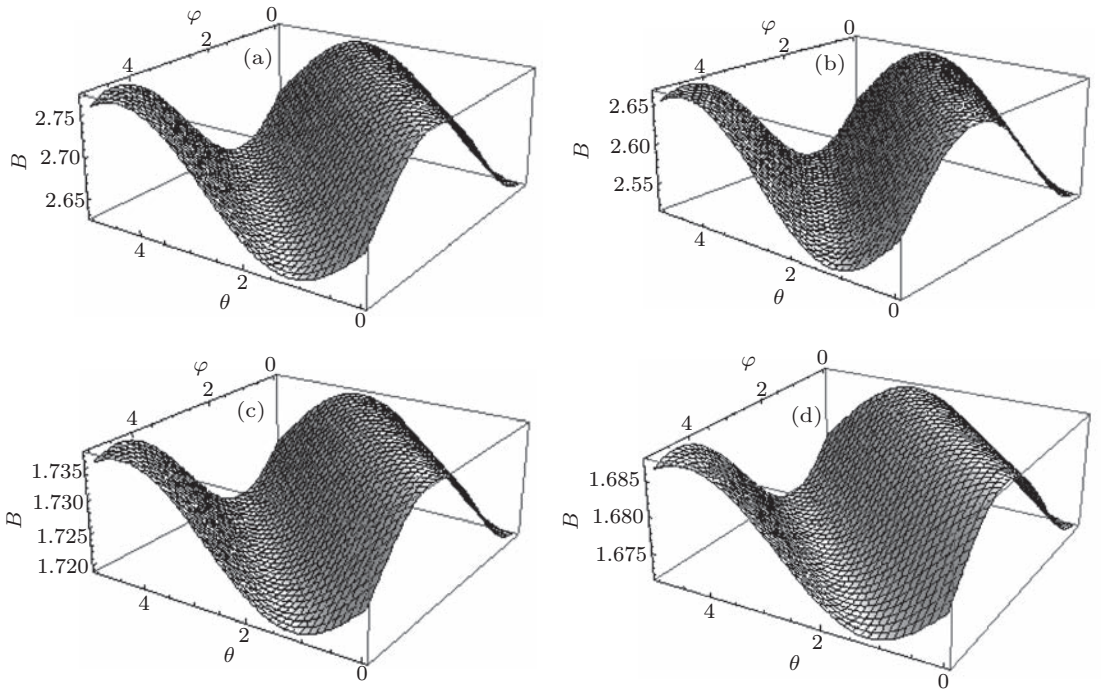


Fig. 7. Plot of the Bell function in the space spanned by parameters θ and φ with (a) $\lambda = 1.5$, $J = 0.01$, $p = q = 1$; (b) $\lambda = 1.5$, $J = 0.01$, $p = q = 2$; (c) $\lambda = 2$, $J = 0.01$, $p = q = 1$; and (d) $\lambda = 2$, $J = 0.01$, $p = q = 2$.

6. Conclusion

In conclusion, we have introduced the two-variable Hermite polynomial state which can be considered as a generalized two-mode squeezed state and investigated the quantum nonlocality of this state in terms of squeezing properties, photon-number distribution, and the Bell's inequality. Especially, the nonclassicality of the THPS is discussed in terms of the

negativity of the WF after deriving the explicit expression of the WF, and the result shows that the nonclassicality of the state is stronger as the increase of the photon number subtracted $p(q)$ and parameter values λ. Then violations of the Bell's inequality for the THPS are also studied. It is shown that in a certain smaller regime of λ, it is found that the violation of the Bell's inequality becomes more observable with the decrease of J, which indicates that the nonlocality of the THPS can be stronger than that of the TPSSV due to the presence of J.

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