Further improvement of the Lyapunov functional and the delay-dependent stability criterion for a neural network with a constant delay*

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This paper investigates the asymptotical stability problem of a neural system with a constant delay. A new delay-dependent stability condition is derived by using the novel augmented Lyapunov–Krasovskii function with triple integral terms, and the additional triple integral terms play a key role in the further reduction of conservativeness. Finally, a numerical example is given to demonstrate the effectiveness and lower conservativeness of the proposed method.

Keywords: neural system, globally asymptotical stability, time delay

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1. Introduction

Neural networks have been intensively investigated in the past decade due to the fact that neural networks can be applied to signal processing, pattern recognition, static image processing, associative memory, combinatorial optimization, and other engineering and scientific areas. It is well known that time delay systems are frequently encountered in various areas, and the existence of time delay may cause oscillations and instability in a system. Therefore, dynamics in a neural network often have time delay due, for example, to the finite switching speed of amplifiers in electronic neural networks or to the finite signal propagation time in biological networks. Consequently, a vast amount of efforts have been devoted to deriving stability criteria for delayed neural networks. The criteria for delayed neural networks can be roughly classified into two categories, namely delay-independent criteria[1−5] and delay-dependent criteria,[6−22] and it is generally recognized that the former cases tend to be conservative especially for small delay. Therefore, much attention has been paid to the delay-dependent type.

Generally speaking, study of the delay-dependent stability problem has two main purposes. One is to provide a maximal allowable delay that is as large as possible, and the other is to use as few as possible decision variables while keeping the maximal allowable delay. In recent years, various methods have been presented by many researchers to obtain delay-dependent stability criteria for time-delay systems, such as matrix decomposition approaches, the improved bounding techniques,[23,24] and model transformation techniques.[25] However, the model transformations often introduce additional dynamics, which leads to relatively conservative results. In Ref. [26], the free-weighting matrices method is proposed to deal with the time-delay system, in which the method reduces the conservativeness of methods involving a fixed model transformation. In addition, a novel extensively augmented Lyapunov functional is introduced by Parlakci[27] without employing any free weighting matrices. The method in Ref. [27] includes a consideration of an additional delayed state and a delayed form of the state integral terms, in which the stability criterion derived is less conservative than some existing ones. Recently, in Ref. [28], a novel delay-dependent stability criterion was derived for delayed neural networks by dividing the delay interval into multiple segments and choosing proper functions with different weighted matrices corresponding to different segments. Up to now, the delay-

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dividing method and the free-weighting matrices approach have been used very frequently in obtaining delay-dependent stability criteria for time-delay systems. However, these proposed methods need to decide more possible variables which may increase the complexity of the computation. In Ref. [29], the suitable augmented Lyapunov functional is introduced to obtain the delay-dependent stability criteria for neutral system with constant delays. The same idea is also introduced to study the stability of uncertain neutral systems with time-varying delays.[30] However, it can be found that the existing Lyapunov functional introduced in the literature only contains some integral terms, such as $\int_{t-\tau}^{t} x^{\mathrm{T}}(s)\boldsymbol{Z}x(s)\mathrm{d}s$, and double-integral terms, such as $\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)\boldsymbol{Z}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta$. Then, inspired by Ref. [31], we add the triple integral terms $2\tau^2\int_{-\tau}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{x}^{\mathrm{T}}(s)\boldsymbol{U}\dot{x}(s)\mathrm{d}s\mathrm{d}\lambda\mathrm{d}\theta$ into the augmented Lyapunov functional. This idea motivates this study.

In this paper, the asymptotic stability of the neural network with a constant delay is studied. It turns out that the stability results for such a system in Ref. [32] can be further improved. By employing a novel Lyapunov functional with triple integral terms, a new delay-dependent stability condition is derived in terms of the integral inequality technique. Compared with some existing methods, the resulting stability criterion proves to be less conservative, which is shown with a numerical example.

Notation: Throughout this paper, T stands for matrix transposition. $\boldsymbol{R}^n$ is the n-dimensional Euclidean space. $\boldsymbol{R}^{n\times m}$ is the set of all $(n\times m)$-dimensional matrice. $\boldsymbol{P}>0$ means that $\boldsymbol{P}$ is positive-definite. Symbol $*$ represents the elements below the main diagonal of a symmetric matrix.

2. Problem statement

The neural network with a constant delay, in this paper, can be described by the following normalized equation

$$\dot{u}(t) = -\boldsymbol{A}u(t) + \boldsymbol{W}g(u(t)) + \boldsymbol{W}^{\tau}g(u(t-\tau)) + \boldsymbol{J}, \quad (1)$$

where $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^{\mathrm{T}} \in \boldsymbol{R}^n$ is the state vector for the n neurons; $g(u(t)) = (g_1(u(t)), g_2(u(t)), \ldots, g_n(u(t)))^{\mathrm{T}} \in R^n$ represents an activation function indicating how the j-th neuron responses to its input; $\boldsymbol{A} = \mathrm{diag}(a_1, \ldots, a_n)$ is a diagonal matrix with each $a_i > 0$ controlling the rate with which the i-th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs; $\boldsymbol{W} = (a_{ij})_{n\times n} \in \boldsymbol{R}^{n\times n}$ and $\boldsymbol{W}^{\tau} = (a_{ij}^{\tau})_{n\times n} \in \boldsymbol{R}^{n\times n}$ are the feedback and the delayed feedback matrices, respectively; $\boldsymbol{J} = (J_1, J_2, \ldots, J_n)^{\mathrm{T}} \in R^n$ is a constant input vector, and τ is a constant delay. Throughout this paper, each of the activation functions g_j $(j = 1, 2, \ldots, n)$ is bounded and satisfies the following conditions

$$0 \le \frac{g_i(u) - g_i(v)}{u - v} \le k_i, \quad u \ne v \in R, \quad i = 1, 2, \ldots, n, \quad (2)$$

where k_i $(i = 1, 2, \cdots, n)$ are positive constants.

Next, the equilibrium point $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ of system (1) is shifted to the origin through the transformation $x(t) = u(t) - u^*$, then system (1) can be equivalently written as the following system

$$\dot{x}(t) = -\boldsymbol{A}x(t) + \boldsymbol{W}f(x(t)) + \boldsymbol{W}^{\tau}f(x(t-\tau)), \quad (3)$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\mathrm{T}}$ is the state vector of the transformed system (3), $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^{\mathrm{T}}$ with

$$f_i(x_i(t)) = g_i\big(x_i(t) + u_i^*\big) - g_i(u_i^*), \quad i = 1, 2, \ldots, n.$$

It is obvious that the function $f_j(t)$ $(j = 1, 2, \ldots, n)$ satisfies the following conditions

$$0 \le \frac{f_i(x_i)}{x_i} \le k_i, \quad f_i(0) = 0, \quad \forall x_i \ne 0, \quad i = 1, 2, \ldots, n, \quad (4)$$

which is equivalent to

$$f_i(x_i)(f_i(x_i) - k_i x_i) \le 0, \quad f_i(0) = 0, \quad \forall x_i \ne 0, \quad i = 1, 2, \ldots, n. \quad (5)$$

Lemma 1[31] Given constant matrices Ω_1, Ω_2, and Ω_3 with appropriate dimensions, where $\Omega_1^{\mathrm{T}} = \Omega_1$ and $\Omega_2^{\mathrm{T}} = \Omega_2 > 0$, then

$$\Omega_1 + \Omega_3^{\mathrm{T}}\Omega_2^{-1}\Omega_3 < 0$$

if and only if

$$\begin{pmatrix} \Omega_1 & \Omega_3^{\mathrm{T}} \\ * & -\Omega_2 \end{pmatrix} < 0, \quad \text{or} \quad \begin{pmatrix} -\Omega_2 & \Omega_3 \\ * & \Omega_1 \end{pmatrix} < 0.$$

Lemma 2[33] For any constant appropriately dimensioned matrices $\boldsymbol{Z}$ and scalar $\tau > 0$ such that the following integrations are well defined,

$$-\int_{t-\tau}^{t} x^{\mathrm{T}}(s)\boldsymbol{Z}x(s)\mathrm{d}s$$

$$\leq -\frac{1}{\tau}\left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right)^{\mathrm{T}} \boldsymbol{Z}\left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right), \quad (6)$$

$$-\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)\boldsymbol{Z}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta$$

$$\leq -\frac{2}{\tau}\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}(s)\mathrm{d}s\mathrm{d}\theta\right)^{\mathrm{T}}$$

$$\times \boldsymbol{Z}\left(\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}(s)\mathrm{d}s\mathrm{d}\theta\right), \quad (7)$$

3. Stability analysis

In this section, we will present a delay-dependent asymptotical stability criterion for system (3) with a constant delay.

To obtain the stability condition, we choose a Lyapunov functional candidate for system (3), which is

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \quad (8)$$

with

$$V_1(t) = \zeta^{\mathrm{T}}(t)\boldsymbol{P}\zeta(t) + 2\sum_{i=1}^{n} s_i \int_0^{x_i(t)} f_i(s)\mathrm{d}s + \int_{t-\tau}^{t} f^{\mathrm{T}}(x(s))\boldsymbol{Q}f(x(s))\mathrm{d}s, \quad (9)$$

$$V_2(t) = \int_{t-\tau}^{t} \rho^{\mathrm{T}}(s)\boldsymbol{R}\rho(s)\mathrm{d}s, \quad (10)$$

$$V_3(t) = \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} x^{\mathrm{T}}(s)\boldsymbol{V}x(s)\mathrm{d}s\mathrm{d}\theta, \quad (11)$$

$$V_4(t) = 2\tau^2\int_{-\tau}^{0}\int_{\theta}^{0}\int_{t+\lambda}^{t} \dot{x}^{\mathrm{T}}(s)\boldsymbol{U}\dot{x}(s)\mathrm{d}s\mathrm{d}\lambda\theta, \quad (12)$$

where

$$\zeta^{\mathrm{T}}(t) = \left[x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-\tau), \left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right)^{\mathrm{T}}\right]^{\mathrm{T}},$$

$$\rho^{\mathrm{T}}(s) = \left[x^{\mathrm{T}}(s), \dot{x}^{\mathrm{T}}(s)\right]^{\mathrm{T}},$$

$$\boldsymbol{P} = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ * & P_{22} & P_{23} \\ * & * & P_{33} \end{pmatrix} = \boldsymbol{P}^{\mathrm{T}} > 0,$$

$$\boldsymbol{R} = \begin{pmatrix} R_{11} & R_{12} \\ * & R_{22} \end{pmatrix} = \boldsymbol{R}^{\mathrm{T}} > 0,$$

$$\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}} > 0, \quad \boldsymbol{V} = \boldsymbol{V}^{\mathrm{T}} > 0, \quad \boldsymbol{U} = \boldsymbol{U}^{\mathrm{T}} > 0,$$

and diagonal matrices $\boldsymbol{S} = \mathrm{diag}\{s_1, s_2, \ldots, s_n\} \geq 0$ are to be determined.

Now, calculating the derivative of $V(t)$ with respect to t along with the solution of system (3) yields

$$\begin{aligned}
\dot{V}_1(t) &= 2\zeta^{\mathrm{T}}(t)\boldsymbol{P}\dot{\zeta}(t) + 2f^{\mathrm{T}}(x(t))\boldsymbol{S}\dot{x}(t) + f^{\mathrm{T}}(x(t))\boldsymbol{Q}f(x(t)) - f^{\mathrm{T}}(x(t-\tau))\boldsymbol{Q}f(x(t-\tau)) \\
&= x^{\mathrm{T}}(t)(-P_{11}\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}}P_{11} + P_{13} + P_{13}^{\mathrm{T}})x(t) + x^{\mathrm{T}}(t)(P_{11}\boldsymbol{W} + \boldsymbol{W}^{\mathrm{T}}P_{11} - \boldsymbol{S}\boldsymbol{A} \\
&\quad - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S})f(x(t)) + x^{\mathrm{T}}(t)(P_{11}\boldsymbol{W}^{\tau} + \boldsymbol{W}^{\tau\mathrm{T}}P_{11})f(x(t-\tau)) \\
&\quad + x^{\mathrm{T}}(t)(-P_{12}^{\mathrm{T}}\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}}P_{12} - P_{13} - P_{13}^{\mathrm{T}} + P_{23} + P_{23}^{\mathrm{T}})x(t-\tau) + x^{\mathrm{T}}(t)(-A^{\mathrm{T}}P_{13} - P_{13}^{\mathrm{T}}\boldsymbol{A} \\
&\quad + P_{33} + P_{33}^{\mathrm{T}})\int_{t-\tau}^{t} x(s)\mathrm{d}s + x^{\mathrm{T}}(t)(P_{12} + P_{12}^{\mathrm{T}})\dot{x}(t-\tau) \\
&\quad + x^{\mathrm{T}}(t-\tau)(P_{12}^{\mathrm{T}}\boldsymbol{W} + \boldsymbol{W}^{\mathrm{T}}P_{12})f(x(t)) + x^{\mathrm{T}}(t-\tau)(P_{12}^{\mathrm{T}}\boldsymbol{W}^{\tau} + \boldsymbol{W}^{\tau\mathrm{T}}P_{12})f(x(t-\tau)) \\
&\quad + x^{\mathrm{T}}(t-\tau)(P_{22} + P_{22}^{\mathrm{T}})\dot{x}(t-\tau) + x^{\mathrm{T}}(t-\tau)(-P_{33} - P_{33}^{\mathrm{T}})\int_{t-\tau}^{t} x(s)\mathrm{d}s \\
&\quad + x^{\mathrm{T}}(t-\tau)(-P_{23} - P_{23}^{\mathrm{T}})x(t-\tau) + \left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right)^{\mathrm{T}}(P_{13}^{\mathrm{T}}\boldsymbol{W} + \boldsymbol{W}^{\mathrm{T}}P_{13})f(x(t)) \\
&\quad + \left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right)^{\mathrm{T}}(P_{23} + P_{23}^{\mathrm{T}})\dot{x}(t-\tau) + \left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right)^{\mathrm{T}}(P_{13}^{\mathrm{T}}\boldsymbol{W}^{\tau} + \boldsymbol{W}^{\tau\mathrm{T}}P_{13})f(x(t-\tau)) \\
&\quad + f^{\mathrm{T}}(x(t))(\boldsymbol{W}^{\mathrm{T}}\boldsymbol{S} + \boldsymbol{S}\boldsymbol{W} + \boldsymbol{Q})f(x(t)) + f^{\mathrm{T}}(x(t))(\boldsymbol{W}^{\tau\mathrm{T}}\boldsymbol{S} + \boldsymbol{S}\boldsymbol{W}^{\tau})f(x(t-\tau)) \\
&\quad + f^{\mathrm{T}}(x(t-\tau))(-\boldsymbol{Q})f(x(t-\tau)), \quad (13)
\end{aligned}$$

$$\begin{aligned}
\dot{V}_2(t) &\leq \rho^{\mathrm{T}}(t)\boldsymbol{R}\rho(t) - \rho^{\mathrm{T}}(t-\tau)\boldsymbol{R}\rho(t-\tau) \\
&= x^{\mathrm{T}}(t)R_{11}x(t) + x^{\mathrm{T}}(t)(R_{12} + R_{12}^{\mathrm{T}})\dot{x}(t) + \dot{x}^{\mathrm{T}}(t)R_{22}\dot{x}(t) + x^{\mathrm{T}}(t-\tau)(-R_{11})x(t-\tau) \\
&\quad + x^{\mathrm{T}}(t-\tau)(-R_{12} - R_{12}^{\mathrm{T}})\dot{x}(t-\tau) + \dot{x}^{\mathrm{T}}(t-\tau)(-R_{22})\dot{x}(t-\tau), \quad (14)
\end{aligned}$$

$$\dot{V}_3(t) = \tau^2 x^{\mathrm{T}}(t)\boldsymbol{V}x(t) - \tau\int_{t-\tau}^{t} x^{\mathrm{T}}(s)\boldsymbol{V}x(s)\mathrm{d}s \leq \tau^2 x^{\mathrm{T}}(t)\boldsymbol{V}x(t) - \left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right)^{\mathrm{T}}\boldsymbol{V}\left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right), \quad (15)$$

$$\dot{V}_4(t) \leq \dot{x}^{\mathrm{T}}(t)(\tau^4\boldsymbol{U})\dot{x}(t) - 2\tau^2\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)\boldsymbol{U}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta$$

$$= \dot{x}^{\mathrm{T}}(t)(\tau^4\boldsymbol{U})\dot{x}(t) + x^{\mathrm{T}}(t)(-4\tau^2\boldsymbol{U})x(t) + x^{\mathrm{T}}(t)(4\tau\boldsymbol{U} + 4\tau\boldsymbol{U}^{\mathrm{T}})\int_{t-\tau}^{t} x(s)\mathrm{d}s$$
$$+ \left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right)^{\mathrm{T}}(-4\boldsymbol{U})\left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right). \tag{16}$$

From formula (5), for appropriately dimensioned diagonal matrices $\Lambda_j \geq 0$ $(j = 1, 2)$ and $\Sigma = \mathrm{diag}\{k_1, k_2, \ldots, k_n\}$, we have

$$-2f^{\mathrm{T}}(x(t))\Lambda_1\Big[f(x(t)) - \Sigma x(t)\Big] \geq 0, \tag{17}$$
$$-2f^{\mathrm{T}}(x(t-\tau))\Lambda_2\Big[f(x(t-\tau)) - \Sigma x(t-\tau)\Big] \geq 0. \tag{18}$$

Then combining Eqs. (13)–(18), we derive

$$\begin{aligned}
\dot{V}(t) \leq\ & \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t) - 2f^{\mathrm{T}}(x(t))\Lambda_1\Big[f(x(t)) - \Sigma x(t)\Big] \\
& - 2f^{\mathrm{T}}(x(t-\tau))\Lambda_2\Big[f(x(t-\tau)) - \Sigma x(t-\tau)\Big] \\
\leq\ & x^{\mathrm{T}}(t)(-P_{11}\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}}P_{11} + P_{13} + P_{13}^{\mathrm{T}} + \tau^2\boldsymbol{V} - 4\tau^2\boldsymbol{U} + R_{11} - R_{12}\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}}R_{12}^{\mathrm{T}})x(t) \\
& + x^{\mathrm{T}}(t)(P_{11}\boldsymbol{W} + \boldsymbol{W}^{\mathrm{T}}P_{11}^{\mathrm{T}} - \boldsymbol{S}\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{S} + \Sigma\Lambda_1 + \Lambda_1^{\mathrm{T}}\Sigma^{\mathrm{T}} + R_{12}\boldsymbol{W} + \boldsymbol{W}^{\mathrm{T}}R_{12}^{\mathrm{T}})f(x(t)) \\
& + x^{\mathrm{T}}(t)(P_{11}\boldsymbol{W}^{\tau} + \boldsymbol{W}^{\tau\mathrm{T}}P_{11} + R_{12}\boldsymbol{W}^{\tau} + \boldsymbol{W}^{\tau\mathrm{T}}R_{12}^{\mathrm{T}})f(x(t-\tau)) \\
& + x^{\mathrm{T}}(t)(-P_{12}^{\mathrm{T}}\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}}P_{12} - P_{13} - P_{13}^{\mathrm{T}} + P_{23} + P_{23}^{\mathrm{T}})x(t-\tau) \\
& + x^{\mathrm{T}}(t)(-\boldsymbol{A}^{\mathrm{T}}P_{13} - P_{13}^{\mathrm{T}}\boldsymbol{A} + P_{33} + P_{33}^{\mathrm{T}} + 4\tau\boldsymbol{U} + 4\tau\boldsymbol{U}^{\mathrm{T}})\int_{t-\tau}^{t} x(s)\mathrm{d}s \\
& + x^{\mathrm{T}}(t)(P_{12} + P_{12}^{\mathrm{T}})\dot{x}(t-\tau) + x^{\mathrm{T}}(t-\tau)(P_{12}^{\mathrm{T}}\boldsymbol{W} + \boldsymbol{W}^{\mathrm{T}}P_{12})f(x(t) \\
& + x^{\mathrm{T}}(t-\tau)(P_{12}^{\mathrm{T}}\boldsymbol{W}^{\tau} + \boldsymbol{W}^{\tau\mathrm{T}}P_{12} + \Lambda_2\Sigma + \Sigma^{\mathrm{T}}\Lambda_2)f(x(t-\tau)) \\
& + x^{\mathrm{T}}(t-\tau)(P_{22} + P_{22}^{\mathrm{T}} - R_{12} - R_{12}^{\mathrm{T}})\dot{x}(t-\tau) + x^{\mathrm{T}}(t-\tau)(-P_{33} - P_{33}^{\mathrm{T}})\int_{t-\tau}^{t} x(s)\mathrm{d}s \\
& + x^{\mathrm{T}}(t-\tau)(-P_{23} - P_{23}^{\mathrm{T}} - R_{11})x(t-\tau) + \left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right)^{\mathrm{T}}(P_{13}^{\mathrm{T}}\boldsymbol{W} + \boldsymbol{W}^{\mathrm{T}}P_{13})f(x(t)) \\
& + \left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right)^{\mathrm{T}}(P_{13}^{\mathrm{T}}\boldsymbol{W}^{\tau} + \boldsymbol{W}^{\tau\mathrm{T}}P_{13})f(x(t-\tau)) + \dot{x}^{\mathrm{T}}(t)(\tau^4\boldsymbol{U} + R_{22})\dot{x}(t) \\
& + \left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right)^{\mathrm{T}}(P_{23} + P_{23}^{\mathrm{T}})\dot{x}(t-\tau) + \left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right)^{\mathrm{T}}(-4\boldsymbol{U} - \boldsymbol{V})\left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right) \\
& + \dot{x}^{\mathrm{T}}(t-\tau)(-R_{22})\dot{x}(t-\tau) + f^{\mathrm{T}}(x(t))(\boldsymbol{W}^{\mathrm{T}}\boldsymbol{S} + \boldsymbol{S}\boldsymbol{W} + \boldsymbol{Q} - 2\Lambda_1)f(x(t)) \\
& + f^{\mathrm{T}}(x(t))(\boldsymbol{W}^{\tau\mathrm{T}}\boldsymbol{S} + \boldsymbol{S}\boldsymbol{W}^{\tau})f(x(t-\tau)) + f^{\mathrm{T}}(x(t-\tau))(-\boldsymbol{Q} - 2\Lambda_2)f(x(t-\tau)) \\
=\ & \xi^{\mathrm{T}}(t)(\Theta_1 + \Gamma^{\mathrm{T}}\Lambda\Gamma)\xi(t),
\end{aligned} \tag{19}$$

where

$$\xi^{\mathrm{T}}(t) = \left[x^{\mathrm{T}}(t), x^{\mathrm{T}}(t-\tau), \dot{x}^{\mathrm{T}}(t-\tau), f^{\mathrm{T}}(x(t)), f^{\mathrm{T}}(x(t-\tau)), \left(\int_{t-\tau}^{t} x(s)\mathrm{d}s\right)^{\mathrm{T}}\right],$$
$$\Gamma = [-\mathbf{A}, 0, 0, \boldsymbol{W}, \boldsymbol{W}^{\tau}, 0], \qquad \Lambda = \tau^4\boldsymbol{U} + R_{22},$$
$$\Theta_1 = \begin{pmatrix}
\Theta_{11} & \Theta_{12} & P_{12} & \Theta_{14} & \Theta_{15} & \Theta_{16} \\
* & \Theta_{22} & P_{22} - R_{12} & P_{12}^{\mathrm{T}}\boldsymbol{W} & P_{12}^{\mathrm{T}}\boldsymbol{W}^{\tau} + \Lambda_2\Sigma & -P_{33} \\
* & * & -R_{22} & 0 & 0 & P_{23} \\
* & * & * & \Theta_{44} & \boldsymbol{S}\boldsymbol{W}^{\tau} & P_{13}^{\mathrm{T}}\boldsymbol{W} \\
* & * & * & * & -\boldsymbol{Q} - 2\Lambda_2 & P_{13}^{\mathrm{T}}W^{\tau} \\
* & * & * & * & * & -4\boldsymbol{U} - \boldsymbol{V}
\end{pmatrix},$$

with

$$\begin{aligned}
\Theta_{11} &= -P_{11}\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}}P_{11} + P_{13} + P_{13}^{\mathrm{T}} + \tau^2\boldsymbol{V} \\
&\quad - 4\tau^2\boldsymbol{U} + R_{11} - R_{12}\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}}R_{12}^{\mathrm{T}}, \\
\Theta_{12} &= -P_{12}^{\mathrm{T}}\boldsymbol{A} - P_{13} + P_{23}, \\
\Theta_{14} &= P_{11}\boldsymbol{W} - \boldsymbol{S}\boldsymbol{A} + \Sigma\Lambda_1 + R_{12}\boldsymbol{W}, \\
\Theta_{15} &= P_{11}\boldsymbol{W}^{\tau} + R_{12}\boldsymbol{W}^{\tau}, \\
\Theta_{16} &= -\boldsymbol{A}^{\mathrm{T}}P_{13} + P_{33} + 4\tau\boldsymbol{U}, \\
\Theta_{22} &= -P_{23} - P_{23}^{\mathrm{T}} - R_{11}, \\
\Theta_{44} &= \boldsymbol{W}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}} + \boldsymbol{S}\boldsymbol{W} + \boldsymbol{Q} - 2\Lambda_1.
\end{aligned}$$

By Lemma 1, $\Theta_1 + \Gamma^{\mathrm{T}}\Lambda\Gamma < 0$ is equivalent to

$$\Theta = \begin{pmatrix} \Theta_1 & \Gamma^{\mathrm{T}}\Lambda \\ * & -\Lambda \end{pmatrix} < 0. \tag{20}$$

So, if there exist positive-definite matrices $\boldsymbol{P} > 0$, $\boldsymbol{R} > 0$, $\boldsymbol{Q} > 0$, $\boldsymbol{V} > 0$, $\boldsymbol{U} > 0$ and diagonal matrices $S \geq 0$, $\Lambda_j \geq 0$ $(j = 1, 2)$ such that the symmetric linear matrix inequality (LMI) (20) holds, then $\dot{V}(t) \leq 0$. Therefore, according to Hale,[34] for given scalar τ, the origin of system (3) is globally asymptotically stable.

Remark 1 The above result gives a delay-dependent asymptotical stability criterion for system (1) based on the novel augmented Lyapunov–Krasovskii function without introducing any free-weighting matrices. Compared with some existing results, Lyapunov functional in this paper contains some additional triple integral terms, which plays a leading role in the further reduction of conservativeness.

Remark 2 A novel augmented Lyapunov–Krasovskii function with triple integral terms technique in this paper can be extended to solve many problems, such as robust stability, absolute stability, exponential stability, and so on, for time delay system with constant delays or time-varying delays.

4. Illustrative example

In this section, we use an example and compare our result with the previous ones to show the effectiveness of ours.

Example Consider the following delayed neural network as in Ref. [28] with:

$$\boldsymbol{A} = \mathrm{diag}\{1.2769,\ 0.6231,\ 0.9230,\ 0.4480\},$$

$$\boldsymbol{W} = \begin{pmatrix} -0.0373 & 0.4852 & -0.3351 & 0.2336 \\ -1.6033 & 0.5988 & -0.3224 & 1.2352 \\ 0.3394 & -0.0860 & -0.3824 & -0.5785 \\ -0.1311 & 0.3253 & -0.9534 & -0.5015 \end{pmatrix},$$

$$\boldsymbol{W}^{\tau} = \begin{pmatrix} 0.8674 & -1.2405 & -0.5325 & 0.0220 \\ 0.0474 & -0.9164 & 0.0360 & 0.9816 \\ 1.8495 & 2.6117 & -0.3788 & 0.8428 \\ -2.0413 & 0.5179 & 1.1734 & -0.2775 \end{pmatrix}.$$

In this example, the activation functions satisfy condition (4) with

$$\Sigma = \mathrm{diag}\{0.1137,\ 0.1279,\ 0.7994,\ 0.2386\}.$$

According to our method, the numerical result on the the maximum bound by calculating the LMI (20) are listed in Table 1 along with the ones obtained in Refs. [23], [24], [26], [28], and [32]. Obviously, it can be seen from Table 1 that the proposed method in this work provides a considerably less conservative delay bound in comparison with some reported results.

Table 1. Allowable time delay τ (Example).

Method	Xu[23]	Hua[24]	He[26]	Shao[28] $(N = 2)$	Shao[28] $(N = 3)$	Qiu[32] $(N = 2)$	Our result
τ	1.4224	3.5841	3.5841	4.0204	4.1032	6.2278	6.5378

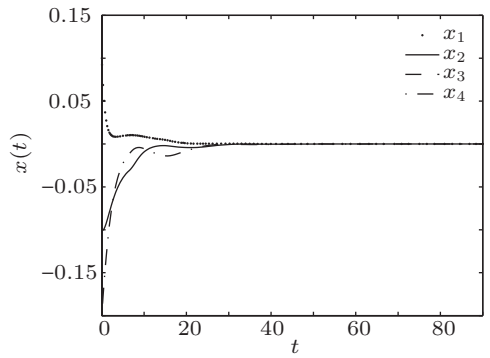


Fig. 1. Transient response of state variables $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$.

From condition (4), we can denote $f(x(t)) = 0.1137x(t)$. Then, when $\tau = 6.5378$, the state response curves of the example are shown in Fig. 1. This illustrates that the solutions of the obove system globally converge to the equilibrium point u^* even for $\tau = 6.5378$.

5. Conclusion

This paper discussed the stability criterion for a neural system with a constant delay. A novel aug-

mented Lyapunov–Krasovskii function approach with triple integral terms is introduced to make the stability condition less conservative as compared with the existing results. A numerical example is given to show the effectiveness of the proposed method.

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