Bound states of spatial optical dark solitons in nonlocal media*

Ou-Yang Shi-Gen(欧阳世根)†, Hu Wei(胡 巍), and Guo Qi(郭 旗)

Laboratory of Nanophotonic Functional Materials and Devices, South China Normal University, Guangzhou 510631, *China*

(Received 9 September 2011; revised manuscript received 16 November 2011)

It is shown that multiple dark solitons can form bound states in a series of balance distances in nonlocal bulk media. Dark solitons can either attract or repel each other depending on their separated distance. The stability of such bound states are studied numerically. There exist unstable degenerate bound states decaying in different ways and having different lifetimes.

Keywords: nonlocal nonlinear Schrödinger equation, dark soliton, bound state

PACS: 05.45.Yv, 42.65.Tg **DOI:** 10.1088/1674-1056/21/4/040505

1. Introduction

Dark solitons are intensity dips embedded in a cw background.[1] For dark solitons in nonlinear media with a local response, the background intensity and the soliton's velocity relative to the background (or equivalently, the darkness) are two characteristic parameters. Dark soltions always repel each other and cannot form bound states in nonlinear media with a local response. Nonlocality can greatly change the interaction patterns, and dark solitons can form bound states due to their long-range attraction in a nonlocal nonlinear medium.[2−6] However the stabilities of such bound states are not clear and the bound states presented in Refs. [2] and [4] were assumed to be stable. On the contrary, in this paper we will show that the bound states presented in Refs. [2] and [4] are both unstable and will decay into a group of single solitons at a rather long propagation distance. As indicated previously,[7] nonlocal dark solitons have exponentially decaying oscillatory tails, which in turn give rise to a widely distributed light-induced perturbed refractive index. As a result, in this paper, we will find that the force between two dark solitons may either be attractive or repulsive depending on their separated distance, and nonlocal dark solitons can form bound states in a series of balance distances. The bound states presented in Refs. [2] and [4] are just one type of this family of bound states formed in the shortest balance distance. For such dark soliton bound states, other characteristic parameters are needed to describe their configurations. Moreover, there exist unstable degenerate bound states moving with the same nonvanishing velocity and having the same Hamiltonian, particle numbers, and momentum but decaying in different ways and having different lifetimes.

2. Bound states of dark soltions in the shortest balance distance

Nonlocality plays an important role in several physical systems, including the interacting particle systems with two-body interaction described by Hartree–Fock equation,[8] photorefractive media,[9,10] thermal nonlinear media,[11,12] and nematic liquid crystals.[13,14] In this paper, we focus on the propagation of a one-dimensional optical beam in a self-defocusing nonlinear medium with a nonlocal response. The evolution of the beam can be described by the dimensionless nonlocal nonlinear Schrödinger equation (NNLSE)[2,4,7]

$$\mathrm{i}\frac{\partial\psi}{\partial z}+\frac{1}{2}\frac{\partial^2\psi}{\partial x^2}-\psi\int R(|x-x'|)|\psi(x',z)|^2\,\mathrm{d}x'=0, \quad (1)$$

where $\psi(x,z)$ is the complex amplitude envelop of the light beam, $I(x,z)=|\psi(x,z)|^2$ is the light intensity, x and z are transverse and longitude coordinates respectively, $R(x)$ is the real nonlocal response function and satisfies the normalization condition $\int R(x)\,\mathrm{d}x=1$,

*Project supported by the National Natural Science Foundation of China (Grant No. 61008007) and the Specialized Research Fund for Growing Seedlings of the Higher Education in Guangdong Province of China (Grant No. LYM10066).

†Corresponding author. E-mail: ouyang_shigen@163.com

© 2012 Chinese Physical Society and IOP Publishing Ltd

and $n(x,z) = -\int R(|x-x'|)I(x',z)\mathrm{d}x'$ is the perturbed refractive index nonlocally depending on the light beam $I(x,z)$. Note that not stated otherwise all integrals in this paper will extend over the whole x axis. Equation (1) is invariant under the scaled transformation

$$\widetilde{\psi}(x,z) = \frac{1}{\alpha}\psi\left(\frac{x}{\alpha}, \frac{z}{\alpha^2}\right)$$

and $\widetilde{R}(x) = \alpha^{-1}R(x/\alpha)$, where α is an arbitrary positive real number. As a result, it is adequate to consider dark solitons having a unity background intensity $I(\pm\infty, z) = 1$. There are three integrals of motion[15,16] for this model, namely, the (renormalized) particle numbers N, momentum P, and Hamiltonian H

$$N = \int (1-|\psi|^2)\mathrm{d}x, \tag{2}$$

$$P = \frac{\mathrm{i}}{2}\int\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right)\left(1-\frac{1}{|\psi|^2}\right)\mathrm{d}x, \tag{3}$$

$$\begin{aligned} H = {} & \frac{1}{2}\int\left|\frac{\partial\psi}{\partial x}\right|^2\mathrm{d}x + \int(1-|\psi|^2)\mathrm{d}x \\ & + \frac{1}{2}\iint R(|x-x'|) \\ & \times\left[|\psi(x)|^2|\psi(x')|^2 - 1\right]\mathrm{d}x\mathrm{d}x'. \end{aligned} \tag{4}$$

Equation (1) has dark soliton solutions

$$\psi(x,z) = \phi(x-vz)\,\mathrm{e}^{\mathrm{i}\,\theta(x-vz)}, \tag{5}$$

where v is the velocity of the dark soliton relative to the cw background, and when $x \to \pm\infty$ the real functions ϕ and θ asymptotically behave as $\phi(x) \to 1$, $\theta(x) \to \mp\theta_0$, where θ_0 is a real constant. Insertion of Eq. (5) into Eq. (1) results in

$$\theta' = v\left(1-\frac{1}{\phi^2}\right), \tag{6}$$

$$\begin{aligned} & \frac{\phi''}{2} + \frac{v^2+2}{2}\phi - \frac{v^2}{2\phi^3} \\ & - \phi\int R(x-x')\phi^2(x')\mathrm{d}x' = 0, \end{aligned} \tag{7}$$

which can be numerically solved provided that $R(x)$ is given. As an important example, we consider the following nonlocal case in which the light-induced perturbed refractive index is governed by[2,4,7]

$$n - w^2\frac{\partial^2 n}{\partial x^2} = -|\psi|^2, \tag{8}$$

or, equivalently,

$$n(x,z) = -\int R(|x-x'|)|\psi(x',z)|^2\mathrm{d}x',$$

where

$$R(|x|) = \frac{1}{2w}\exp\left(-\frac{|x|}{w}\right)$$

is the nonlocal response function and w is the characteristic nonlocal response length of the media.

In Fig. 1 we show the intensity profiles of fundamental dark solitons ψ_0 and bound states $\psi_1, \psi_{1,1}, \psi_{1,1,1}$ and their counterpart numerical simulations when $w = 5$, $v = 0$. The lower indexes $i, j, \ldots, k$ of a bound state $\psi_{i,j,\ldots,k}$ denote the distances between two adjacent fundamental dark soltions coupled in it and are additional characteristic parameters besides its background intensity and velocity. Number 1 denotes the shortest distance and number 2 the second shortest distance and so on. No observable changes in the intensity profile of ψ_0 can be found during its propagation. Numerical simulation (not shown here) indicates that an initially broadened beam $\psi(x,0) = \psi_0(x/1.5, 0)$ will converge into a fundamental soliton quickly during its propagation. So the fundamental soliton ψ_0 is stable. Bound states $\psi_1, \psi_{1,1}, \psi_{1,1,1}$ were assumed to be stable previously in Refs. [2] and [4]. On the contrary, as shown by Fig. 1 in this paper, they are all unstable and will decay into a group of fundamental solitons after a rather long propagation distance. Numerical simulations (not shown here) also indicate that bound states $\psi_1, \psi_{1,1}, \psi_{1,1,1}$ moving with another non-vanishing velocity v are also unstable. The noise embedded in such bound states is not suppressed but slowly accumulates nonlinearly during the propagation and ultimately results in the decay of these bound states. It is worthwhile noting that the numerical simulation results in Refs. [2] and [4] do not contradict with our results since the numerical simulation in Refs. [2] and [4] are performed only over tens[2] or hundreds[4] of diffraction length, and in such relative short propagation distances the noise embedded in bound states does not accumulate large enough to be observed at all. Actually, to observe instability one must perform simulations over thousands of diffraction length in this case.

Now let us focus on the criterion of dark soliton instability suggested in Refs. [17]–[19]. It says that if $\mathrm{d}P/\mathrm{d}v > 0$, viz., the derivative of the momentum P defined in Eq. (3) with respect to the velocity v is positive, the dark solitons are instable. As figure 2 indicates, one obtains $\mathrm{d}P/\mathrm{d}v < 0$ both for ψ_0 and ψ_1. The fundamental dark soliton ψ_0 is stable while the bound state ψ_1 is instable. So this criterion of dark soliton instability may be a sufficient but not necessary a condition for the dark soliton instability, namely, we cannot tell if a dark soliton state whether stable or not when $\mathrm{d}P/\mathrm{d}v < 0$.

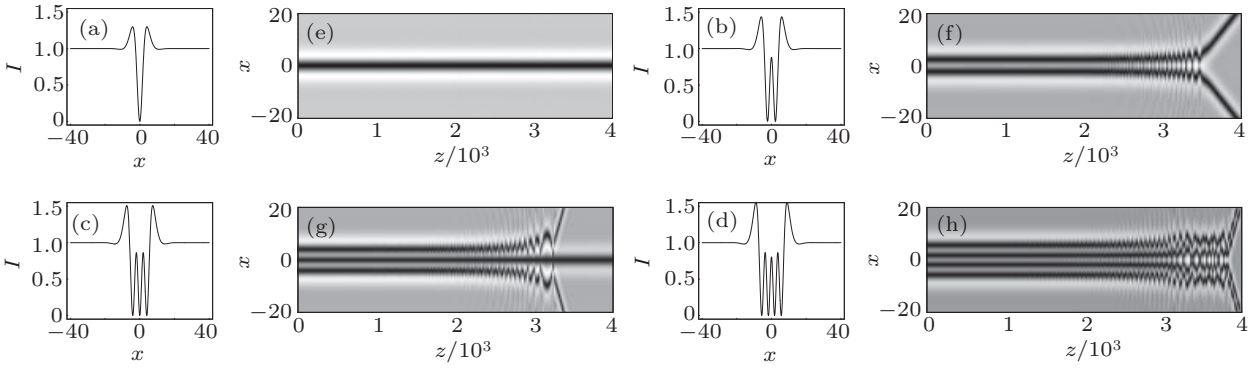


Fig. 1. Panels (a), (b), (c), and (d) are intensity profiles of ψ_0, ψ_1, $\psi_{1,1}$, $\psi_{1,1,1}$ and (e), (f), (g), (h) are their corresponding propagations when $w = 5$, $v = 0$.

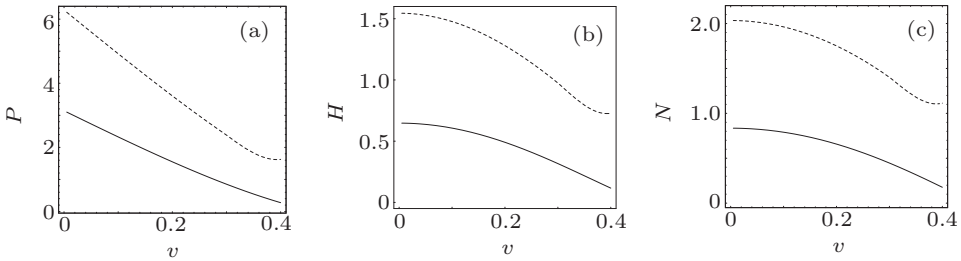


Fig. 2. (a) Momentum $P(v)$, (b) Hamiltonian $H(v)$, and (c) particle numbers $N(v)$ for ψ_0 (solid line) and ψ_1 (dashing line) when $w = 5$.

3. Bound states of dark soltions in another balance distance

According to Ref. [7], the nonlocal dark solitons have exponentially decaying oscillatory tails when $|v| \leq \sqrt{(4w-1)/4w^2}$ for $w > 1/4$. Namely, when $|x| \to \infty$, one has

$$\phi(x) \approx 1 - c_1 \exp(-\lambda_1|x|)\cos(\lambda_2|x| + c_2), \qquad (9)$$

where c_1 and c_2 are two constants depending on the specific soliton state, and

$$\lambda_1 = \sqrt{\frac{1 - 4w^2v^2 + 4w\sqrt{1-v^2}}{4w^2}}, \qquad (10)$$

$$\lambda_2 = \sqrt{\frac{-1 + 4w^2v^2 + 4w\sqrt{1-v^2}}{4w^2}} \qquad (11)$$

are two parameters depending only on the velocity v and the characteristic nonlocal response length w. We note that $1/\lambda_1$ characterizes the size of the tail and $2\pi/\lambda_2$ is its oscillatory spatial period. As a result of this oscillatory feature, the intensity of the soliton solution will have a series of maxima and minima. For example, as shown in Fig. 3, when $w = 5$, $v = 0.4$, the intensity of fundamental soliton ψ_0 has minima and maxima interdigitally located at $x_0 = 0$, $x_1 = 4.57$, $x_2 = 10.29$, $x_3 = 15.61$, $x_4 = 21.10$, $x_5 = 26.52$, $x_6 = 31.97$, $x_7 = 37.41$, $x_8 = 42.85$. Such a decaying oscillatory tail will in turn give rise to an oscillatory light-induced perturbed refractive index. As a consequence the force between two fundamental dark solitons may either be attractive or repulsive depending on their separated distance. For example, as figure 4 shows, the force is initially attractive when $d = 7$ and $d = 24$, or initially repulsive when $d = 12$ and $d = 26$. If the distance increases, the force continually changes from repulsive to attractive, and one will expect that there is a balance distance where the force vanishes, and two fundamental dark solitons, initially

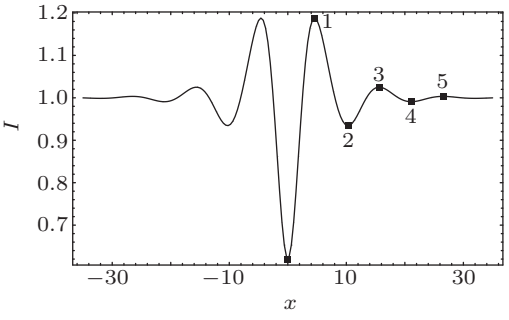


Fig. 3. The intensity profile of $\psi_0(x)$ and its series of minimums and maximums (black dots) for $w = 5$, $v = 0.4$.

separated in a distance slightly different from it, will be pulled back to this balance distance. On the other hand, if the distance increases, the force continually changes from attractive to repulsive, and there is also a balance distance. However, in the latter case two dark solitons initially separated at a distance slightly different from such a balance distance will be pulled away from this balance distance.

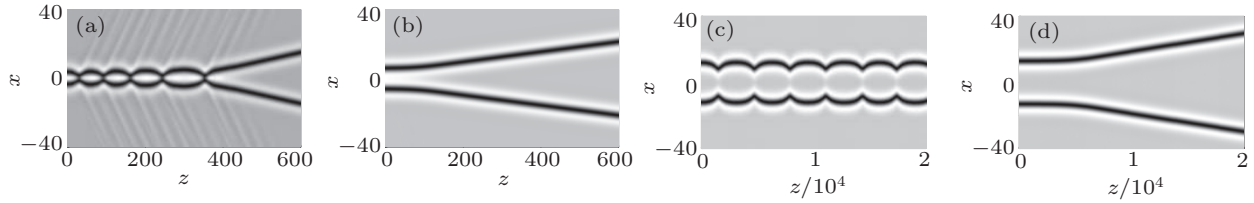


Fig. 4. Interactions of two fundamental dark solitons initially separated in distances (a) $d = 7$, (b) $d = 12$, (c) $d = 24$, (d) $d = 26$, when $w = 5, v = 0$.

These interaction patterns imply that there exist other bound states formed in different balance distances. In fact, by numerically solving Eqs. (6) and (7) we find other soliton bound states ψ_i ($i = 1, 2, \ldots, 8$) which are shown in Fig. 5. The bound states previously presented in Refs. [2] and [4] belong to ψ_1 and are just one type of this family of soliton states. Soliton states ψ_i can be approximately viewed as bound states of two fundamental dark solitons ψ_0 non-local-nonlinearly coupled in a series of balance distances $d_i = 4.88, 10.48, 15.97, 21.33, 26.82, 32.24, 37.69, 43.13$, where d_i are the distances between the two deepest intensity dips in ψ_i. Then the difference of such balance distances are $\Delta d_i \equiv d_{i+1} - d_i = 5.6$, 5.49, 5.36, 5.49, 5.42, 5.45, 5.44. And from Eq. (11) we have $\pi/\lambda_2 = 5.44$ for $w = 5$, $v = 0.4$. Obviously one obtains $d_i \approx x_i$ and $\Delta d_i \approx \pi/\lambda_2$. Numerical simulations shown in Fig. 5 indicate that $\psi_2, \psi_4, \psi_6, \psi_8$ are all instable, and ψ_1 is weakly instable. No observable changes in the intensity profiles of ψ_3, ψ_5, ψ_7 can be found during a propagation distance of $z = 5000$. However the stabilities of ψ_3, ψ_5, ψ_7 are still unclear.

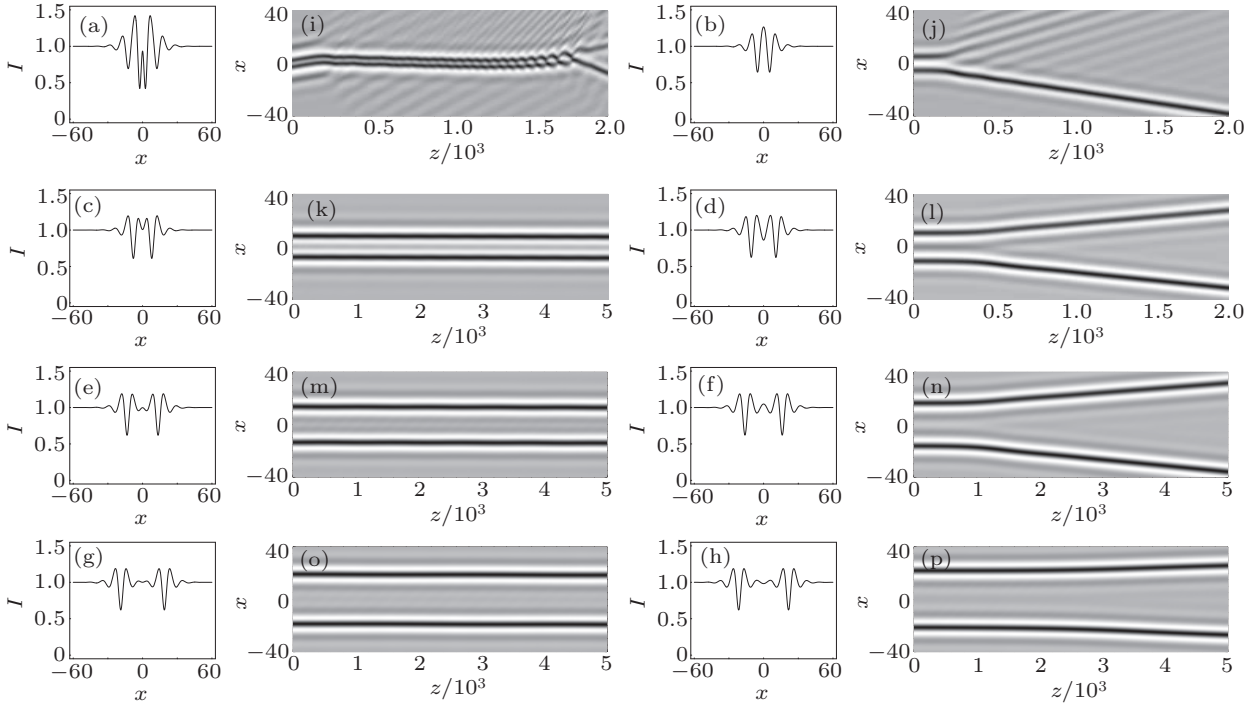


Fig. 5. Panels (a), (b), (c), (d), (e), (f), (g), and (h) are intensity profiles of bound states ψ_1, ψ_2, ψ_3, ψ_4, ψ_5, ψ_6, ψ_7, ψ_8 respectively, and (i), (j), (k), (l), (m), (n), (o), (p) are their corresponding propagations when $w = 5$, $v = 0.4$.

4. Degenerate bound states of dark solitons

Since two fundamental dark solitons can form bound states ψ_i in a series of balance distances d_i, we can use three or more fundamental dark solitons separated in different balance distances to construct bound states like $\psi_{1,2,1}$ or $\psi_{3,3}$, and so on. In particular, when $i \neq j$ and $v \neq 0$, $\psi_{i,j}$ and $\psi_{j,i}$ are two degenerate bound states having the same momentum P, Hamiltonian H, and particle numbers N. Interestingly, as shown in Fig. 6, some of such degenerate states, like $\psi_{1,2}$ and $\psi_{2,1}$, are instable and will decay in different ways and will have different lifetimes. Nonetheless, some degenerate states, like $\psi_{3,5}$ and $\psi_{5,3}$, may be stable, and their stability is still an open problem.

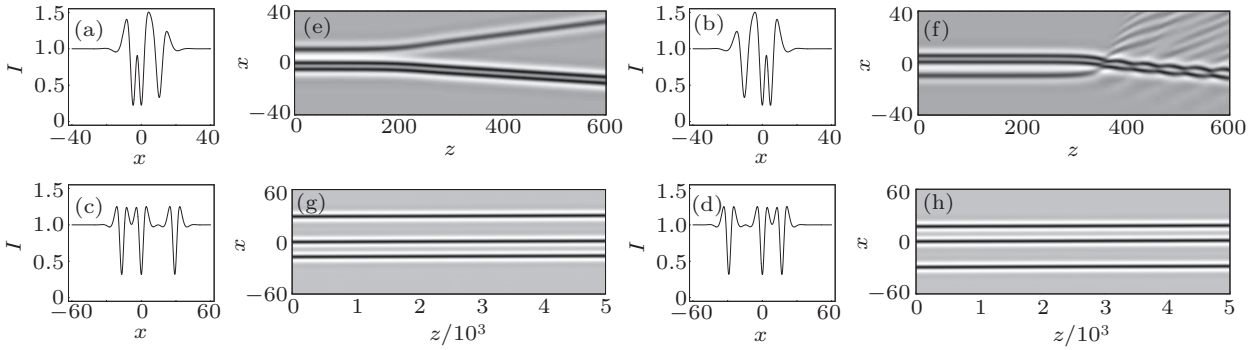


Fig. 6. Panels (a), (b), (c), and (d) are intensity profiles of $\psi_{1,2}$, $\psi_{2,1}$, $\psi_{3,5}$, $\psi_{5,3}$ and (e), (f), (g), (h) are their corresponding propagations when $w = 5$, $v = 0.3$.

5. Conclusion

In nonlocal bulk media, multiple dark solitons can form bound states in a series of balance distances. The force between two dark solitons may either be attractive or repulsive depending on the separated distance between them. Some such bound states are unstable and will decay into a group of fundamental solitons, while others may be stable. There exist degenerate bound states moving with the same velocity and have the same Hamiltonian, particle numbers, and momentum but decaying in different ways and having different lifetimes.

References

[1] Agrawal G P 2002 *Nonlinear Fiber Optics and Applications of Nonlinear Fiber Optics* 3rd edn. (Beijing: Publishing House of Electronics Industry) pp. 101–104 (in Chinese)

[2] Nikolov N I, Neshev D, Krolikowski W, Bang O, Rasmussen J J and Christiansen P L 2004 *Opt. Lett.* **29** 286

[3] Dreischuh A, Neshev D N, Petersen D E, Bang O and Krolikowski W 2006 *Phys. Rev. Lett.* **96** 043901

[4] Kartashov Y V and Torner L 2007 *Opt. Lett.* **32** 946

[5] Zhou L, Gao X, Yang Z, Lu D, Guo Q, Cao W and Hu W 2011 *Acta Phys. Sin.* **60** 044208 (in Chinese)

[6] Gao X, Yang Z, Zhou L, Zheng Y, Lu D and Hu W 2011 *Acta Phys. Sin.* **60** 084213 (in Chinese)

[7] Ouyang S and Guo Q 2009 *Opt. Express* **17** 5170

[8] Greiner W and Reinhardt J 2003 *Field Quantization* (Berlin: Spriner-Verlag) pp. 69–74

[9] Mamaev A V, Zozulya A A, Mezentsev V K, Anderson D Z and Saffman M 1997 *Phys. Rev.* A **56** R1110

[10] Krolikowski W, Saffman M, Luther-Davies B and Denz C 1998 *Phys. Rev. Lett.* **80** 3240

[11] Rotschild C, Cohen O, Manela O, Segev M and Carmon T 2005 *Phys. Rev. Lett.* **95** 213904

[12] Rotschild C, Segev M, Xu Z, Kartashov Y V, Torner L and Cohen O 2006 *Opt. Lett.* **31** 3312

[13] Peccianti M, Brzdkiewicz K A and Assanto G 2002 *Opt. Lett.* **27** 1460

[14] Hu W, Zhang T, Guo Q, Xuan L and Lan S 2006 *Appl. Phys. Lett.* **89** 071111

[15] Barashenkov I V and Harin A O 1995 *Phys. Rev.* D **52** 2471

[16] Barashenkov I V and Harin A O 1993 *Phys. Rev. Lett.* **72** 1575

[17] Kivshar Y S and Krolikowski W 1995 *Opt. Lett.* **20** 1527

[18] Barashenkov I V 1996 *Phys. Rev. Lett.* **77** 1193

[19] Pelinovsky D E, Kivshar Y S and Afanasjev V V 1996 *Phys. Rev.* E **54** 2015