Novel criteria for exponential synchronization of inner time-varying complex networks with coupling delay*

Zhang Qun-Jiao(张群娇)[a)†] and Zhao Jun-Chan(赵军产)[a)b)]

[a)] College of Mathematics and Computer Science, Wuhan Textile University, Wuhan 430073, China
[b)] Laboratoire de Mathématiques Appliquées, Université du Havre, Le Havre Cedex, 76058, France

(Received 20 May 2011; revised manuscript received 25 December 2011)

This paper mainly investigates the exponential synchronization of an inner time-varying complex network with coupling delay. Firstly, the synchronization of complex networks is decoupled into the stability of the corresponding dynamical systems. Based on the Lyapunov function theory, some sufficient conditions to guarantee its stability with any given convergence rate are derived, thus the synchronization of the networks is achieved. Finally, the results are illustrated by a simple time-varying network model with a coupling delay. All involved numerical simulations verify the correctness of the theoretical analysis.

Keywords: exponential synchronization, time varying, complex network, coupling delay

PACS: 05.45.–a, 89.75.–k **DOI:** 10.1088/1674-1056/21/4/040502

1. Introduction

Over the past decades, complex dynamical networks have attracting more and more attention due to their ubiquity in the natural world.[1−3] In our daily life, a great number of natural complex networks such as corporate networks, social networks, neural networks, WWW, food webs, electrical power grids, and so on have been widely studied by researchers. Particularly, one of the most interesting and significant phenomena in complex dynamical networks is the synchronization of all dynamical nodes, it has been demonstrated that many real-world problems have close relationships with network synchronization, such as the lights of fireflies, and the spread of an epidemic or computer virus. Recently, the synchronization of networks had been deeply researched by many scientists from various fields, for instance, sociology, biology, mathematics, physics, etc.

Early in 2002, Wang, Chen, and Hong have reported some advanced results about small-world networks and scale-free networks.[4−6] Lately, researchers noticed that time delay is inevitable due to the finite speeds of transmission and spreading, for example, time delays in spreading and response, as well as in traffic congestion. Therefore, in order to simulate more realistic networks, time delays cannot be ignored and should be taken into account. Specific coupled oscillators networks with delayed coupling have been studied widely and a great many theoretical results were achieved in Refs. [7]–[19]. The network models in most of them have a constant coupling matrix and a constant inner matrix.

However, many real complex networks are more likely to be time-varying evolving networks. Lü *et al.* have presented a general time-varying network and studied its synchronization in 2004.[20]

Motivated by the above reasons, in this paper, we introduce an inner time-varying complex dynamical network model with coupling delay and then further investigate the synchronization phenomena in such networks. Based on this network model, we present some exponential synchronization criteria and some remarks, which are described in terms of the solution of Riccati differential equations (RDE) and linear matrix inequalities (LMI). Finally, The results are illustrated by an inner time-varying complex dynamical network model with coupling delay. Numerical simulations verify the correctness of the theoretical analysis.

*Project supported in part by the National Natural Science Foundation of China (Grant No. 11047114), the Key Project of the Chinese Ministry of Education (Grant No. 210141), and the Youth Foundation of the Educational Committee of Hubei Province of China (Grant Nos. Q20111607 and Q20111611).

†Corresponding author. E-mail: qunjiao99@163.com

© 2012 Chinese Physical Society and IOP Publishing Ltd

2. Preliminaries and model description

In order to prescribe the problem more clearly, we introduce some lemmas throughout the paper, which will be used in the next section.

Lemma 1[21] For any vectors x, $y \in R^n$ and positive-definite matrix $S \in R^{n\times n}$, then for every $Q \in R^{n\times n}$, the following matrix inequality holds:

$$2y^{\mathrm{T}}Q^{\mathrm{T}}x \le y^{\mathrm{T}}Sy + x^{\mathrm{T}}Q(S^{-1})^{\mathrm{T}}Q^{\mathrm{T}}x.$$

Lemma 2[22] Assume that the linear control system $[W(t), U(t)]$ is globally nullcontrollable. Then for every symmetric matrices $Q(t) \ge 0$, $P_0 \ge 0$, the Riccati differential equation

$$\begin{aligned}&\dot{P}(t) + W^{\mathrm{T}}(t)P(t) + P(t)W(t)\\ &\quad + P(t)U(t)U^{\mathrm{T}}(t)P(t) + Q(t) = 0,\\ &P(0) = P_0\end{aligned}$$

has the solution $P(t) \ge 0$, $P(t) = P^{\mathrm{T}}(t)$.

Lemma 3[22] Let P, Q, $S \in R^{n\times n}$ be given matrices such that $S > 0$, $S = S^{\mathrm{T}}$. Then

$$\begin{pmatrix} P & Q \\ Q^{\mathrm{T}} & -S \end{pmatrix} < 0 \Longleftrightarrow P + QS^{-1}Q^{\mathrm{T}} < 0.$$

To simulate more realistic complex networks, Lü *et al.*[20] presented a time-varying complex dynamical network model early in 2004, which is described as

$$\begin{aligned}&\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} c_{ij}(t)A(t)x_j(t),\\ &\quad i = 1, 2, \ldots, N,\end{aligned}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^{\mathrm{T}} \in \boldsymbol{R}^n$ are the state variables of node i, the matrix $A(t) = (a_{ij}(t)) \in \boldsymbol{R}^{n\times n}$ is the inner connecting matrix of each node, $C(t) = (c_{ij}(t))_{N\times N}$ is the coupling configuration matrix representing the coupling strength and topological structure of the network at time t, in which $c_{ij}(t)$ is defined as follows: if there is a connection from node i to node j $(j \ne i)$, then the coupling strength $c_{ij}(t) \ne 0$; otherwise, $c_{ij}(t) = 0$ $(j \ne i)$, and the diagonal elements of matrix $C(t)$ are defined by

$$c_{ii}(t) = -\sum_{j=1, j\ne i}^{N} c_{ij}(t), \quad i = 1, 2, \ldots N.$$

Recently, many researchers have studied the synchronization of coupling delay networks with a constant coupling configuration matrix and a constant inner link matrix,[7,9,11,15] the model is often described as:

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} c_{ij}Ax_j(t-\tau),$$

the matrix $C = (c_{ij})_{N\times N}$ is a symmetric nonnegative diffusively coupled matrix, its eigenvalues are real numbers satisfying $0 = \lambda_1 > \lambda_2 \ge \cdots \ge \lambda_N$, $\tau > 0$ is the constant time delay.

In fact, on the one hand, time delay exists everywhere and the stability and convergence properties of dynamical systems are strongly affected by time delays, which are often encountered in many industrial and natural processes due to measurement and computational delays, transmission, and transport lags. At the same time, on the other hand, some real-world complex dynamical networks are more likely to be time-varying evolving networks. Therefore, in this paper, we introduce a time-varying complex dynamical network with coupling delay, which can be modeled as

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^{N} c_{ij}A(t)x_j(t-\tau), \qquad (1)$$

in which $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is a continuously differentiable function; the matrix C, $A(t)$, and τ are defined as given above.

In the following section, we will investigate the exponential synchronization problem with a given convergence rate in such networks. The networks with time-varying coupling matrix $C(t)$ and multiple delays will be discussed in the future.

3. Exponential synchronization of the present network model

3.1. The equivalent synchronization problem of model (1)

Definition 1 The delayed dynamical network (1) is said to achieve (asymptotic) synchronization if

$$x_1(t) = x_2(t) = \cdots = x_N(t) = s(t) \ \text{ as } \ t \to \infty,$$

where $s(t)$ is a solution of an isolate node, satisfying $\dot{s}(t) = f(s(t))$.

Let $e_i(t) = x_i(t) - s(t)$ $(i = 1, 2, \ldots, N)$. Notice that $\sum_{j=1}^{N} c_{ij} = 0$, one can yield the variation equations as follows:

$$\dot{e}_i(t) = \dot{x}_i(t) - \dot{s}(t)$$

$$
\begin{aligned}
&= f(x_i(t)) - f(s(t)) + \sum_{j=1}^{N} c_{ij}A(t)x_j(t-\tau) \\
&\quad - \sum_{j=1}^{N} c_{ij}A(t)s(t-\tau) \\
&= Df(s(t))e_i(t) + \sum_{j=1}^{N} c_{ij}A(t)e_j(t-\tau), \\
&\quad i = 1, 2, \ldots, N. \qquad (2)
\end{aligned}
$$

Because of the symmetric character of matrix C, there exists a generalized eigenvectors matrix $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_N)$ such that it can be diagonalized, that is

$$\Phi^{-1}C^{\mathrm{T}}\Phi = \Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}.$$

Let $E(t) = [e_1(t), \ldots, e_N(t)] \in R^{n\times N}$, introducing a nonsingular linear transformation $E(t) = V(t)\Phi^{-1}$, we rewrite the above equation as

$$\dot{E}(t) = Df(s(t))E(t) + A(t)E(t-\tau)C^{\mathrm{T}},$$

therefore

$$\dot{V}(t) = \dot{E}(t)\Phi = Df(s(t))V(t) + A(t)V(t-\tau)\Lambda. \quad (3)$$

If $\lambda_1 = 0$, it is corresponding to the synchronization states of the variables, as shown in Definition 1. Therefore, from Eq. (3), we only need the following $(N-1)$ n-dimensional linear time-varying delayed differential equations

$$
\begin{aligned}
&\dot{v}_k(t) = Df(s(t))v_k(t) + \lambda_k A(t)v_k(t-\tau), \\
&\quad k = 2, \ldots, N \qquad (4)
\end{aligned}
$$

to be stable at zero. Thus, the complicated synchronization problem of the network (1) is translated into the stabilization problem of $(N-1)$ independent n-dimensional delayed linear systems. Briefly speaking, if the following delayed differential systems

$$
\begin{aligned}
&\dot{v}(t) = Df(s(t))v(t) + \lambda_k A(t)v(t-\tau), \\
&\quad k = 2, \ldots, N, \qquad (5)
\end{aligned}
$$

are local exponential stable about their zero solutions, then the continuous complex dynamical network (1) is local exponential synchronous. Here $Df(s(t)) \in R^{n\times n}$ is the Jacobian of $f(\cdot)$ at $s(t)$.

Definition 2 The time-varying delayed system (5) is said to achieve exponential α-stable if

$$\|v(t,\phi)\| \le M\,\mathrm{e}^{-\alpha t},$$

where M, α are positive constants and $\phi(t) \in C([-\tau, 0], R^n)$ with $\|\phi\| = \sup_{t\in[-\tau,0]}\|\phi(t)\|$, which is the initial condition of $v(t)$ in time interval $[-\tau, 0]$. In other words, $v(t) = \phi(t), t \in [-\tau, 0]$.

3.2. The exponential α-stable of time-varying delayed systems (5)

In what follows, it will be shown that if there is a symmetric matrix $P(t) > 0$ satisfying

$$
\begin{aligned}
&\dot{P}(t) + B^{\mathrm{T}}(t)P(t) \\
&+ P(t)B(t) + P(t)M_k(t)M_k^{\mathrm{T}}(t)P(t) + 2I = 0, \\
&\quad k = 2, \ldots, N, \qquad (6)
\end{aligned}
$$

then the time-invariant delayed systems in Eq. (5) are local exponential α-stable.

For the linear time-varying delayed systems (5), we denote

$$
\begin{aligned}
&B(t) = Df(s(t)) + \alpha I, \\
&M_k(t) = \mathrm{e}^{\alpha\tau}\lambda_k A(t),\ k = 2, \ldots, N,
\end{aligned}
$$

and take $y(t) = \mathrm{e}^{\alpha t}v(t)$ as the change of the variable, then equation (5) are transformed into the following systems

$$\dot{y}(t) = B(t)y(t) + M_k(t)y(t-\tau), \quad k = 2, \ldots, N. \quad (7)$$

Choose the time-varying Lyapunov-like function

$$V(y(t)) = y^{\mathrm{T}}(t)P(t)y(t) + \int_{t-\tau}^{t} \|y(s)\|^2 \mathrm{d}s,$$

its derivative along Eq. (7) is

$$
\begin{aligned}
\dot{V}(y(t)) &= \dot{y}^{\mathrm{T}}(t)P(t)y(t) + y^{\mathrm{T}}(t)\dot{P}(t)y(t) \\
&\quad + y^{\mathrm{T}}(t)P(t)\dot{y}(t) + \|y(t)\|^2 - \|y(t-\tau)\|^2 \\
&= y^{\mathrm{T}}(t)[\dot{P}(t) + B^{\mathrm{T}}(t)P(t) + P(t)B(t)]y(t) \\
&\quad + 2y^{\mathrm{T}}(t-\tau)M_k^{\mathrm{T}}(t)P(t)y(t) \\
&\quad + \|y(t)\|^2 - \|y(t-\tau)\|^2, \qquad (8)
\end{aligned}
$$

according to the condition (6), we have $\dot{P}(t) + B^{\mathrm{T}}(t)P(t) + P(t)B(t) = -P(t)M_k(t)M_k^{\mathrm{T}}(t)P(t) - 2I$, taking it into the above equation, one has

$$
\begin{aligned}
\dot{V}(y(t)) &= y^{\mathrm{T}}(t)[-P(t)M_k(t)M_k^{\mathrm{T}}(t)P(t) - 2I]y(t) \\
&\quad + 2y^{\mathrm{T}}(t-\tau)M_k^{\mathrm{T}}(t)P(t)y(t) \\
&\quad + \|y(t)\|^2 - \|y(t-\tau)\|^2 \\
&= -\|y(t)\|^2 + [2y^{\mathrm{T}}(t-\tau)M_k^{\mathrm{T}}(t)P(t)y(t) \\
&\quad - y^{\mathrm{T}}(t)P(t)M_k(t)M_k^{\mathrm{T}}(t)P(t)y(t) \\
&\quad - y^{\mathrm{T}}(t-\tau)y(t-\tau)].
\end{aligned}
$$

By virtue of Lemma 1, set $S = I$, $Q = P(t)M_k(t)$, one can easily obtain that the second part of the above equation is not lager than 0, so we have

$$\dot{V}(y(t)) \le -\|y(t)\|^2. \quad (9)$$

Integrating both sides of inequality (9) from 0 to t, then

$$V(y(t)) - V(y(0)) \leq -\int_0^t \|y(s)\|^2 \mathrm{d}s,$$

that is to say,

$$\int_0^t \|y(s)\|^2 \mathrm{d}s \leq V(y(0)) - V(y(t)) \leq V(y(0))$$
$$= y^{\mathrm{T}}(0)P^{\mathrm{T}}(0)y(0) + \int_{-\tau}^0 \|y(s)\|^2 \mathrm{d}s,$$

notice that

$$\int_{-\tau}^0 \|y(s)\|^2 \mathrm{d}s = \int_{-\tau}^0 \|\mathrm{e}^{\alpha s}v(s)\|^2 \mathrm{d}s \leq \|\phi\| \int_{-\tau}^0 \mathrm{e}^{\alpha s}\mathrm{d}s$$
$$= \|\phi\|\left(\frac{1-\mathrm{e}^{-\alpha\tau}}{\alpha}\right),$$

so

$$\int_0^t \|y(s)\|^2 \mathrm{d}s \leq y^{\mathrm{T}}(0)P^{\mathrm{T}}(0)y(0)$$
$$+ \|\phi\|\left(\frac{1-\mathrm{e}^{-\alpha\tau}}{\alpha}\right) = G.$$

Let $t \to +\infty$, we also have $\int_0^{+\infty} \|y(s)\|^2 \mathrm{d}s \leq G < +\infty$, which implies that $y(t) \in L_2([0,+\infty), R^n)$ and hence the solution of Eq. (7) is bounded, that is, there exists an $M > 0$ such that

$$\|y(t)\| \leq M, \ \forall \ t \geq 0,$$

along with $y(t) = \mathrm{e}^{\alpha t}v(t)$, we have $\|v(t)\| \leq M\,\mathrm{e}^{-\alpha t}$, which means the time-varying delayed systems in Eq. (5) are exponentially α-stable.

Remark 1 From the linear transformation $E(t)\Phi = V(t)$ in the proof process, the $v(t)$ is exponentially α-stable implies that the $e_i(t)$ $(i = 2, \ldots, N)$ are exponentially α-stable. As a result, the state variables $x_i(t)$ $(i = 2, \ldots, N)$ are exponentially synchronous with the orbit $s(t)$.

Remark 2 In the above proof, we cannot apply the conventional standard stability Lyapunov theorem, since the solution $P(t)$ is not uniformly positive-definite, and hence the function $V(y(t))$ is not uniformly bounded so that it is not the Lyapunov function.

Remark 3 If $B(t) = B$, $M_k(t) = M_k$ $(k = 2, \ldots, N)$ are constant matrices, then the linear time-varying delayed systems (5) are exponentially α-stable if there is a symmetric constant matrix $P > 0$ such that

$$B^{\mathrm{T}}P + PB + PM_kM_k^{\mathrm{T}}P + 2I = 0. \tag{10}$$

In fact, if we rewrite Eq. (8) as follows:

$$\begin{aligned}\dot{V}(y(t)) &= y^{\mathrm{T}}(t)[\dot{P}(t) + B^{\mathrm{T}}(t)P(t) + P(t)B(t) + I]y(t) + 2y^{\mathrm{T}}(t-\tau)M_k^{\mathrm{T}}(t)P(t)y(t) \\ &\quad - y^{\mathrm{T}}(t-\tau)y(t-\tau) \\ &= z^{\mathrm{T}}(t)\begin{pmatrix} \dot{P}(t) + B^{\mathrm{T}}(t)P(t) + P(t)B(t) + I & P(t)M_k(t) \\ M_k^{\mathrm{T}}(t)P(t) & -I \end{pmatrix} z(t),\end{aligned} \tag{11}$$

where

$$z(t) = \begin{pmatrix} y(t) \\ y(t-\tau) \end{pmatrix},$$

from condition (11), there is a sufficient small $\varepsilon > 0$, such that

$$\dot{V}(y(t)) \leq -\varepsilon\|z(t)\|^2 \leq -\varepsilon\|y(t)\|^2, \quad \forall\ t \geq 0,$$

which just has the same form as Eq. (9). Similar to the proof mentioned before, we derive that the time-varying delayed systems in Eq. (5) are local exponentially α-stable if there is a symmetric matrix $P(t) > 0$ satisfying

$$\begin{pmatrix} \dot{P}(t) + B^{\mathrm{T}}(t)P(t) + P(t)B(t) + I & P(t)M_k(t) \\ M_k^{\mathrm{T}}(t)P(t) & -I \end{pmatrix} < 0, \quad k = 2, \ldots, N. \tag{12}$$

Remark 4 If $B(t) = B$, $M_k(t) = M_k$ $(k = 2, \ldots, N)$ are constant matrices, there exists a symmetric constant matrix $P > 0$ such that

$$\begin{pmatrix} B^{\mathrm{T}}P + PB + I & PM_k \\ M_k^{\mathrm{T}}P & -I \end{pmatrix} < 0, \tag{13}$$

then the time-varying delay systems in Eq. (5) are exponentially α-stable.

4. A simulation example

In this section, as an application of the above-derived theoretical criteria, the exponential synchronization problem of the time-varying delayed network (1) is illustrated by a simple example.

Let time delay $\tau = 0.5$, the convergence rate $\alpha = 1$, we consider the continuous network model with 10 nodes, where the isolate node is described by the simple two-dimensional dynamical system, which is similar to those in Refs. [7], [9], and [11], each node is $(\dot{x}_{i1}, \dot{x}_{i2})^{\mathrm{T}} = (-4x_{i1}, -3x_{i2})^{\mathrm{T}}$, then $B(t) = \mathrm{diag}\{-3, -2\}$. Suppose the network is globally connected, the outer coupling matrix $C = (c_{ij})$, and $c_{ii} = -1.8$, $c_{ij} = 0.2$ $(i \neq j)$, then its eigenvalues are 0 and -2. The time-varying inner coupling matrix $A(t)$ is

$$A(t) = \begin{pmatrix} 0.5\,\mathrm{e}^{(-t-0.5)}\sqrt{2.5\,\mathrm{e}^{t}-1} & -0.5\,\mathrm{e}^{(-t-0.5)}\sqrt{2.5\,\mathrm{e}^{t}-1} \\ 0.5\,\mathrm{e}^{-0.5} & 0.5\,\mathrm{e}^{-0.5} \end{pmatrix},$$

then $M_k(t) = -2\,\mathrm{e}^{0.5}A(t)$ $(k = 2, \ldots, N)$, we can check that the positive-definite matrix

$$P(t) = \begin{pmatrix} \mathrm{e}^{t} & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies Eq. (6). Figure 1 shows the exponential synchronization process of network (1) in this case.

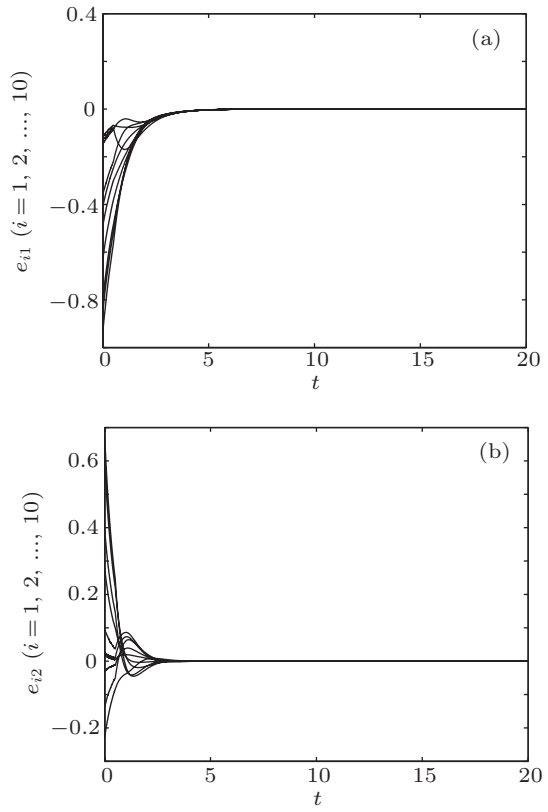


Fig. 1. (colour online) Synchronization of the first and second state variables in network (1).

For the LMI (11), by virtue of Lemma 3, which is equivalent to the following inequality

$$\dot{P}(t) + B^{\mathrm{T}}(t)P(t) + P(t)B(t) + P(t)M_k(t)M_k^{\mathrm{T}}(t)P(t) + I < 0. \quad (14)$$

So when equation (6) holds, it means that

$$\dot{P}(t) + B^{\mathrm{T}}(t)P(t) + P(t)B(t) + P(t)M_k(t)M_k^{\mathrm{T}}(t)P(t) + I = -I,$$

thus condition (14) holds necessarily. Then the state variables in network (1) are synchronous certainly.

5. Conclusions

This paper mainly investigated the synchronization of a general inner time-varying complex network with coupling delay. Firstly, we decoupled the synchronization of a complex network into the stability of the corresponding delayed dynamical systems, based on the time-varying Lyapunov-like function method, we gave two sufficient conditions to guarantee their α-stability with any given convergence rate, thus realizing the exponential synchronization of the presented network. Finally, the results are illustrated by a simple time-varying network model with coupling delay. All involved numerical simulations verify the correctness of the theoretical analysis.

References

[1] Strogatz S H 2001 *Nature* **410** 268
[2] Albert R and Barabási A L 2002 *Rev. Mod. Phys.* **74** 47
[3] Wang X F and Chen G R 2003 *IEEE Cir. Syst. Mag.* **3** 6

[4] Wang X F and Chen G R 2002 *Int. J. Bifur. Chaos* **12** 187
[5] Wang X F and Chen G R 2002 *IEEE Trans. Circ. Syst.* I **49** 54
[6] Hong H, Choi M Y and Kim B J 2002 *Phys. Rev.* E **65** 26
[7] Li C G and Chen G R 2004 *Physica* A **343** 263
[8] Li C G, Xu H, Liao X and Yu J 2004 *Physica* A **335** 359
[9] Chen L, Shi Y D and Wang D S 2010 *Chin. Phys.* B **19** 100503
[10] Gao H J, Lam J and Chen G R 2006 *Phys. Lett.* A **360** 263
[11] Tu L L and Lu J A 2009 *Comput. Math. Appl.* **57** 28
[12] Wang Q Y, Duan Z S, Chen G R and Feng Z S 2008 *Physica* A **387** 5616
[13] Zhang W Y and Li J M 2011 *Chin. Phys.* B **20** 030701
[14] He W L and Cao J D 2009 *Phys. Lett.* A **373** 2682
[15] Wu X Q 2008 *Physica* A **387** 997
[16] Earl M G and Strogatz S H 2003 *Phys. Rev.* E **67** 036204
[17] Tu L L 2011 *Chin. Phys.* B **20** 030504
[18] Du R J, Dong G G, Tian L X, Zheng S and Sun M 2010 *Chin. Phys.* B **19** 070509
[19] Li C, Sun W and Xu D 2005 *Prog. Theor. Phys.* **114** 1
[20] Lü J H, Yu X and Chen G R 2004 *Physica* A **334** 281
[21] Boyd S, Ghaoui El, Feron E and Balakrishnan V 1994 "Linear Matrix Inequalities in System and Control Theory", *SIAM Studies in Applied Mathematics* (Philadelphia: SIAM)
[22] Abou-Kandil H, Freiling G, Ionescu V and Jank G 2003 *Matrix Riccati Equations in Control and Systems Theory* (Basel: Birkhauser)