Verification of the spin-weighted spheroidal equation in the case of $s = 1$*

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The spin-weighted spheroidal equation in the case of $s = 1$ is studied. By transforming the independent variables, we make it take the Schrödinger-like form. This Schrödinger-like equation is very interesting in itself. We investigate it by using super-symmetric quantum mechanics and obtain the ground eigenvalue and eigenfunction, which are consistent with the results previously obtained.

Keywords: spheroidal wave equation, supersymmetric quantum mechanics, super-potential, eigenvalue and eigenfunction

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1. Introduction

Spin-weighted spheroidal equations first appeared in the study of the stable problem of Kerr black holes, which were obtained by the separate variable method. There are two Teukolsky equations, a radial one and an angular one. The latter is called the spin-weighted spheroidal equation, that is[1,2]

$$\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right) + s + \beta^2\cos^2\theta - 2s\beta\cos\theta - \frac{(m + s\cos\theta)^2}{\sin^2\theta} + E\right]\Theta(\theta) = 0 \tag{1}$$

with $\theta \in (0, \pi)$. It is a singular Sturm–Liuvelle problem with the boundary condition that Θ is bounded at the endpoints of the interval $\theta = 0, \pi$.

In Eq. (1), $\beta = a\omega$, and a is the angular moment of the unit mass of the Kerr black holes. s, ω, and m correspond to the quantum numbers of the perturbation fields, spin, frequency, and magnetic quantum number. $s = 0, \pm1, \pm1/2, \pm2$ represent the spins of scalar and electromagnetic, neutrino, and gravitational field, respectively.[1−8]

Equation (1) will reduce to the spin-weighted spherical equation when $\beta = a\omega = 0$, whose solution is known. Equation (1) is a spheroidal equation when $s = 0$,[3] which has been studied recently by using the perturbation method in super-symmetric quantum mechanics (SUSYQM).[9−13]

References [14]–[18] extended it to study Eq. (1) in the case of $s = 1, 2, 1/2$. They transformed Eq. (1) into the Schrödinger-like form by $\Theta = \Psi/\sin^{1/2}\theta$, that is

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\theta^2} + \left[\frac{1}{4} + s + \beta^2\cos^2\theta - 2s\beta\cos\theta - \frac{(m + s\cos\theta)^2 - 1/4}{\sin^2\theta} + E\right]\Psi = 0, \tag{2}$$

and the perturbation method in SUSYQM was applied and the result was obtained there.

In this paper, we will transform Eq. (1) by $z = \ln\tan(\theta/2)$ and restudy it again. As stated above, the transformation $z = \ln\tan(\theta/2)$ make Eq. (1) in the case of $s = 1$ become

$$-\frac{\mathrm{d}^2\Theta}{\mathrm{d}z^2} + V(z, \beta, 1)\Theta = -m^2\Theta, \tag{3}$$

where $V(z, \beta, 1) = V(z, \beta, s)|_{s=1}$ is

$$\begin{aligned} V(z, \beta, 1) = &-[(E + 1)\operatorname{sech}^2 z + \beta^2\operatorname{sech}^2 z\tanh^2 z \\ &+ 2\beta\tanh z\operatorname{sech}^2 z \\ &- 2m\tanh z - \tanh^2 z]. \end{aligned} \tag{4}$$

Because of the fact that $\operatorname{sech} z \to 0$, $\tanh z \to \pm1$ as $z \to \pm\infty$. $V(z, \beta, 1)$ approaches $(\pm2m + 1)$ as $z \to \pm\infty$ and $-(E + 1)$ at $z = 0$.

Just as Fig. 1 shows when Eq. (4) is studied from the viewpoint of quantum mechanics, it is the bound state that is really interesting to us and it corresponds to the solution of eigenvalue problems for Eq. (1). Therefore the boundary condition becomes $\Theta \to 0$

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as $z \to \pm\infty$. Some discussion on the boundary conditions is relevant. In Eq. (1), $\Theta(\theta)$ must be zero at the boundaries $\theta = 0$, $\theta = \pi$ because of the fact that $\theta = 0$ and $\theta = \pi$ are singular points for Eq. (1). Though we just state that Θ is finite at $\theta = 0, \pi$, the real meaning is that $\Theta|_{\theta=0,\pi} = 0$. As $\theta = 0, \pi$ correspond to $z = \pm\infty$ respectively, we should have $\Theta|_{z\to\pm\infty} = 0$.

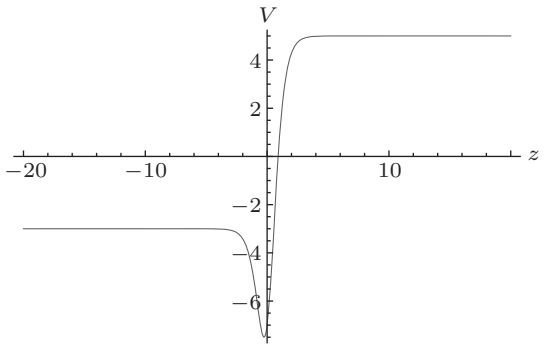


Fig. 1. The figure of $V(z, \beta, 1)$ with $m = 2$, $\beta = 0.01$, $E = 6$.

Equation (3) is very interesting as a quantum system and requires us to take some efforts to study it. It will have applications in the area related to nonlinear equations, especially in soliton theory. In this paper, we will apply the perturbation method of supersymmetry quantum mechanics to investigate Eq. (3) and give the results concerning its ground eigenvalue and eigenfunction, we will also confirm the results in Ref. [15]. As the calculations involved in Refs. [15] and [16] are very complex, their results need to be confirmed. Though their results are identical with those already existing in Refs. [4]–[6], they still lack complete confirmation due to the fact that Ref. [6] only gave the first several terms concerning the revelent quantities. On the other hand, references [15] and [16] gave all the terms concerning the ground eigenfunction and ground eigenvalue. These results need to be checked. Here we supply the comparison.

As has been done before in Refs. [9]–[18], we also apply the perturbation method in SUSYQM to study Eq. (3). For details about SUSYQM, readers can refer to Ref. [19]. The super-potential W is a key concept in SUSYQM in Refs. [20] and [21], which is connected with the ground eigenfunction Θ_0 and the potential $V(z, \beta, 1)$ respectively by the following equations:

$$\Theta_0(z) = N \exp\left[-\int W \,\mathrm{d}z \right], \tag{5}$$

$$W^2 - W' = V(z, \beta, s) + m^2. \tag{6}$$

In SUSYQM, it is Eq. (6) that needs to be solved to obtain the super-potential W. The ground function Θ_0 is then given by Eq. (5).

Similarly, equation (6) is treated by the perturbation methods. The super-potential W and the original eigenvalue E_0 are expanded into the series form of the parameter β

$$W = \sum_{n=0}^{\infty} \beta^n W_n, \qquad E_0 = \sum_{n=0}^{\infty} E_{0,n;m} \beta^n, \tag{7}$$

where the first suffix 0 means that it is the ground state energy, the suffix m refers to the parameter m in Eq. (2), and the suffix n certainly refers to the expansion order of the parameter β.

We can expand both sides of Eq. (6) as series of the parameter β as

$$\begin{aligned} W^2 - W' = {} & (W_0^2 - W_0') \\ & + \beta(2W_0W_1 - W_1') \\ & + \beta^2(2W_0W_2 + W_1^2 - W_2') \\ & + \beta^3(2W_0W_3 + 2W_1W_2 - W_3') \\ & + \cdots \\ & + \beta^n \sum_{k=1}^{n-1} W_k W_{n-k} + \cdots, \end{aligned} \tag{8}$$

$$\begin{aligned} & V(z, \beta, s) + m^2 \\ = {} & -(E_{0,0;m} + 1)\operatorname{sech}^2 z - 2m \tanh z + \tanh^2 z + m^2 \\ & - \beta(E_{0,1;m} \operatorname{sech}^2 z + 2 \tanh z \operatorname{sech}^2 z) \\ & - \beta^2(E_{0,2;m} \operatorname{sech}^2 z + \operatorname{sech}^2 z \tanh^2 z) \\ & \cdots \\ & - \beta^n E_{0,n,m} \operatorname{sech}^2 z + \cdots. \end{aligned} \tag{9}$$

Comparing the coefficient of β of Eq. (8) with Eq. (9), we can easily obtain

$$W_0' - W_0^2 = (E_{0,0;m} + 1)\operatorname{sech}^2 z - 2m \tanh z - \tanh^2 z - m^2 = f_0(z), \tag{10}$$

$$W_1' - 2W_0W_1 = E_{0,1;m} \operatorname{sech}^2 z + 2 \tanh z \operatorname{sech}^2 z = f_1(z), \tag{11}$$

$$W_2' - 2W_0W_2 = E_{0,2;m} \operatorname{sech}^2 z + \operatorname{sech}^2 z \tanh^2 z + W_1^2 = f_2(z), \tag{12}$$

$$\cdots$$

$$W_n' - 2W_0W_n = E_{0,n;m}\operatorname{sech}^2 z + \sum_{k=1}^{n-1} W_kW_{n-k} = f_n(z), \quad (n \geqslant 3). \tag{13}$$

So we can conclude W_0 and $E_{0,0;m}$ for Eq. (10) as

$$W_0 = m\tanh z - 1, \qquad E_{0,0;m} = m^2 + m - 2. \tag{14}$$

From Eq. (13), it is easy to deduce

$$\begin{aligned} W_n(z) &= \cosh^{2m} z\, \mathrm{e}^{-2z} A_n(z) \\ &= A_n(z)(2\cosh^2 z \\ &\quad - 2\cosh z \sinh z - 1)\cosh^{2m} z, \end{aligned} \tag{15}$$

where

$$\begin{aligned} A_n(z) &= \int f_n(z)\cosh^{-2m} z\, \mathrm{e}^{2z}\, \mathrm{d}z \\ &= \int f_n(z)\cosh^{-2m} z \\ &\quad \times [2\cosh^2 z + 2\cosh z \sinh z - 1]\, \mathrm{d}z. \end{aligned} \tag{16}$$

Defining the following quantities[22]

$$\begin{aligned} &Q(m,z) \\ &= \int \operatorname{sech}^{2m} z\, \mathrm{d}z = \int \cosh^{-2m} z\, \mathrm{d}z \\ &= \frac{\sinh z}{m} \sum_{k=0}^{m-1} \bar{I}(m,k)\cosh^{-2m+2k+1} z, \end{aligned} \tag{17}$$

$$\begin{aligned} &\bar{I}(m,k) \\ &= \frac{2^k m(m-1)(m-2)\cdots(m-k)}{(2m-1)(2m-3)(2m-5)\cdots(2m-2k-1)} \\ &= \frac{m(2m-2)(2m-4)\cdots(2m-2k)}{(2m-1)(2m-3)(2m-5)\cdots(2m-2k-1)} \end{aligned} \tag{18}$$

will make the following calculations simple.

By making use of the relation between $Q(m,z)$ and $Q(m+1,z)$ in Appendix A, we can obtain $A_1(z)$ with unknown $E_{0,1;m}$ as

$$\begin{aligned} &A_1(z) \\ &= (2E_{0,1;m} + 4)Q(m,z) - (4 + E_{0,1;m})Q(m+1,z) \\ &\quad - \frac{E_{0,1;m}+2}{m}\cosh^{-2m} z + \frac{1}{m+1}\cosh^{-2m-2} z \\ &= \left[\frac{2E_{0,1;m}(m+1)+4}{2m+1}\right] Q(m,z) \\ &\quad - \frac{4 + E_{0,1;m}}{2m+1}\sinh z \cosh^{-2m-1} z \\ &\quad - \left(\frac{E_{0,1;m}+2}{m}\right)\cosh^{-2m} z + \frac{1}{m+1}\cosh^{-2m-2} z \\ &= \left[\frac{2E_{0,1;m}(m+1)+4}{2m+1}\right] Q(m,z) \\ &\quad + (a_1 \sinh z \cosh^{-1} z + a_2 \\ &\quad + a_3 \cosh^{-2} z)\cosh^{-2m} z, \end{aligned} \tag{19}$$

where

$$a_1 = -\frac{4 + E_{0,1;m}}{2m+1}, \tag{20}$$

$$a_2 = -\frac{2 + E_{0,1;m}}{m}, \tag{21}$$

$$a_3 = \frac{1}{m+1}, \tag{22}$$

$$a_1 - a_2 = \frac{(m+1)E_{0,1;m} + 2}{m(2m+1)}, \tag{23}$$

$$a_1 + 2a_3 = -\frac{(m+1)E_{0,1;m} + 2}{(m+1)(2m+1)}. \tag{24}$$

According to Eq. (15), W_1 can be written as

$$\begin{aligned} W_1 &= \left[\frac{2E_{0,1;m}(m+1)+4}{2m+1}\right](2\cosh^2 z - 2\cosh z \sinh z - 1)\left[\frac{\sinh z}{m}\sum_{k=0}^{m-1}\bar{I}(m,k)\cosh^{2k+1} z\right] \\ &\quad + (a_1 \sinh z \cosh^{-1} z + a_2 + a_3 \cosh^{-2} z)(2\cosh^2 z - 2\cosh z \sinh z - 1) \\ &= \frac{2E_{0,1;m}(m+1)+4}{2m+1}\left[\frac{\sinh z}{m}\sum_{k=0}^{m-1}\bar{I}(m,k)\cosh^{2k+1} z\right](2\cosh^2 z - 2\cosh z \sinh z - 1) \\ &\quad + 2(a_1 - a_2)\sinh z \cosh z - 2(a_1 - a_2)\cosh^2 z - (a_1 + 2a_3)\sinh z \cosh^{-1} z \\ &\quad - a_3 \cosh^{-2} z + 2a_1 - a_2 + 2a_3 \\ &= \frac{2E_{0,1;m}(m+1)+4}{2m+1}\left[\frac{\sinh z}{m}\sum_{k=0}^{m-1}\bar{I}(m,k)\cosh^{2k+1} z\right](2\cosh^2 z - 2\cosh z \sinh z - 1) + \frac{2E_{0,1;m}(m+1)+4}{2m+1} \\ &\quad \times \left[\frac{\sinh z \cosh z}{m} - \frac{\cosh^2 z}{m} - \frac{\sinh z \cosh^{-1} z}{2(m+1)}\right] - \frac{1}{m+1}\cosh^{-2} z + \frac{(m+1)E_{0,1;m}+2}{m(m+1)(2m+1)}. \end{aligned} \tag{25}$$

The quantity $E_{0,1,m}$ is obtained by the condition of finite Θ_0 at $\pm\infty$

$$\begin{aligned}\Theta_0 &= N\exp\left[-\int W\,\mathrm{d}z\right]\\ &= N\exp\left[-\int W_0\,\mathrm{d}z-\int\sum_{n=1}^{\infty}\beta^n W_n\,\mathrm{d}z\right]\\ &= N\cosh^{-m}z\,\mathrm{e}^z\exp\left[-\beta\int W_1\,\mathrm{d}z\right]\\ &\quad\times\exp o(\beta^2).\end{aligned}\tag{26}$$

It is easy to see that the first two terms in Eq. (25) will result in the divergence in Θ_0 at $z\to\pm\infty$. Therefore, the common coefficients $E_{0,1;m}(m+1)+2$ should be zero to make Θ_0 be finite at the infinities. So we have

$$E_{0,1;m}=-\frac{2}{m+1},\tag{27}$$

$$W_1(z)=-\frac{1}{m+1}\cosh^{-2}z.\tag{28}$$

Some comments are needed for the following proof. We see that the coefficients of the first two terms have common factor $(m+1)E_{0,1;m}+2$. This fact makes is easy for us to first set the coefficient of main divergence, the first term, to be zero, then the secondary divergence, the second term will be zero through calculations. This process gives the same result as that is obtained above. However, the process can be further simplified as to require the coefficient of $Q(m,z)$ in $A_1(z)$ to be zero, which will make computation be more easy. In the following, we will adopt this method to obtain W_2, W_3, W_4, and even the W_n, $n=5,6\ldots$. The results for $n\leqslant 4$ are given below

$$E_{0,2;m}=-\frac{m^3+7m^2+11m+3}{(m+1)^3(2m+3)},\tag{29}$$

$$W_2(z)=a_{2,1}\sinh z\cosh^{-3}z+b_{2,1}\cosh^{-2}z,\tag{30}$$

$$a_{2,1}=-\frac{m^2+2m}{(m+1)^2(2m+3)},\tag{31}$$

$$b_{2,1}=-\frac{m^2+2m}{(m+1)^3(2m+3)},\tag{32}$$

$$E_{0,3;m}=-\frac{2m^2(m+2)}{(m+1)^5(2m+3)},\tag{33}$$

$$\begin{aligned}W_3(z)&=a_{3,1}\sinh z\cosh^{-3}z\\ &\quad+b_{3,1}\cosh^{-2}z+b_{3,2}\cosh^{-4}z,\end{aligned}\tag{34}$$

$$a_{3,1}=\frac{m}{(m+1)^4(2m+3)},\tag{35}$$

$$b_{3,1}=\frac{m}{(m+1)^5(2m+3)},\tag{36}$$

$$b_{3,2}=-\frac{m}{2(m+1)^3(2m+3)},\tag{37}$$

$$E_{0,4;m}=-\frac{2m^2(m+2)(m^5+18m^4+78m^3+112m^2+24m-36)}{(m+1)^7(2m+3)^3(2m+5)},\tag{38}$$

$$W_4(z)=a_{4,1}\sinh z\cosh^{-3}+a_{4,2}\sinh z\cosh^{-5}+b_{4,1}\cosh^{-2}+b_{4,2}\cosh^{-4},\tag{39}$$

$$a_{4,1}=\frac{m(m^5+18m^4+78m^3+112m^2+24m-36)}{(m+1)^6(2m+3)^3(2m+5)},\tag{40}$$

$$a_{4,2}=-\frac{m(m+3)(m^2+m-1)}{(m+1)^4(2m+3)^2(2m+5)},\tag{41}$$

$$b_{4,1}=\frac{m(m^5+18m^4+78m^3+112m^2+24m-36)}{(m+1)^7(2m+3)^3(2m+5)},\tag{42}$$

$$b_{4,2}=-\frac{3m(m+3)(m^2+m-1)}{(m+1)^5(2m+3)^2(2m+5)}.\tag{43}$$

2. General expression of $W_n(z)$

According to W_0, W_1, W_2, W_3, and W_4, we can calculate all the other terms W_n, $n\geqslant 5$ as before. However, we will not adopt this way. Instead, we summarize the general formula of super-potential $W_n(z)$ as

$$\begin{aligned}W_n(z)&=\sinh z\sum_{k=1}^{[\frac{n}{2}]}a_{n,k}\cosh^{-2k-1}z\\ &\quad+\sum_{k=1}^{[\frac{n+1}{2}]}b_{n,k}\cosh^{-2k}z,\end{aligned}\tag{44}$$

where the meaning of $[\frac{n}{2}]$ is as follows:

$$\begin{aligned}[\tfrac{n}{2}]&=n/2, &&\text{for even } n,\\ [\tfrac{n}{2}]&=(n-1)/2, &&\text{for odd } n.\end{aligned}\tag{45}$$

If we suppose the coefficients to be taken as

$$\begin{aligned}a_{n,k}&=0 \text{ if } k<1 \text{ or } k>[\tfrac{n}{2}],\\ b_{n,k}&=0 \text{ if } k<1 \text{ or } k>[\tfrac{n+1}{2}],\end{aligned}\tag{46}$$

then equation (44) can be concisely written as

$$W_n(z) = \sum_{k=1}^{[\frac{n+1}{2}]} (a_{n,k} \tanh z + b_{n,k}) \cosh^{-2k} z. \quad (47)$$

This general formula needs proof to guarantee its correctness. We will use induction method to gain our goal. First, it is easy to see that equation (44) is true in the case of $n = 1$. We will suppose its correctness up to $n-1$, and will prove its validness at n. For W_n, of course, we will directly calculate W_n by making use of Eqs. (15) and (16). Obviously, the terms W_{n-k} and W_k in $\sum_{k=1}^{n-1} W_k W_{n-k}$ satisfy the requirement that $k < n$ and $n - k < n$, so, by the use of Eq. (44), they can be written as

$$W_k(z) = \sinh z \sum_{i=1}^{[\frac{k}{2}]} a_{k,i} \cosh^{-2i-1} z + \sum_{i=1}^{[\frac{k+1}{2}]} b_{k,i} \cosh^{-2i} z, \quad (48)$$

$$W_{n-k}(z) = \sinh z \sum_{j=1}^{[\frac{n-k}{2}]} a_{n-k,j} \cosh^{-2j-1} z + \sum_{j=1}^{[\frac{n-k+1}{2}]} b_{n-k,j} \cosh^{-2j} z. \quad (49)$$

From Eqs. (13)–(16), we need to calculate $\sum_{k=1}^{n-1} W_k W_{n-k}$ to obtain W_n

$$\begin{aligned}
&\sum_{k=1}^{n-1} W_k W_{n-k} \\
&= \sum_{k=1}^{n-1} \Bigg\{ \sinh^2 z \sum_{i=1}^{[\frac{k}{2}]} \sum_{j=1}^{[\frac{n-k}{2}]} a_{k,i} a_{n-k,j} \cosh^{-2(i+j)-2} z \\
&\quad + \sum_{i=1}^{[\frac{k+1}{2}]} \sum_{j=1}^{[\frac{n-k+1}{2}]} b_{k,i} b_{n-k,j} \cosh^{-2(i+j)} z \Bigg\} \\
&\quad + \sum_{k=1}^{n-1} \sinh z \Bigg\{ \sum_{i=1}^{[\frac{k}{2}]} \sum_{j=1}^{[\frac{n-k+1}{2}]} a_{k,i} b_{n-k,j} \cosh^{-2(i+j)-1} z \\
&\quad + \sum_{i=1}^{[\frac{k+1}{2}]} \sum_{j=1}^{[\frac{n-k}{2}]} b_{k,i} a_{n-k,j} \cosh^{-2(i+j)-1} z \Bigg\}. \quad (50)
\end{aligned}$$

Assuming $p = i + j$, and with some simplification, equation (50) becomes

$$\begin{aligned}
&\sum_{k=1}^{n-1} W_k W_{n-k} \\
&= \sum_{k=1}^{n-1} \sum_{p=2}^{[\frac{n}{2}]+1} \sum_{j=1}^{p-2} [(a_{k,p-j} a_{n-k,j} - a_{k,p-1-j} a_{n-k,j} \\
&\quad + b_{k,p-j} b_{n-k,j}) \cosh^{-2p} z \\
&\quad + \sinh z (a_{k,p-j} b_{n-k,j} \\
&\quad + b_{k,p-j} a_{n-k,j}) \cosh^{-2p-1} z], \quad (51)
\end{aligned}$$

where equation (46) has been used.

Then the series formula of the product $W_k W_{n-k}$ is simplified as

$$\sum_{k=1}^{n-1} W_k W_{n-k} = \sum_{p=2}^{[\frac{n}{2}]+1} (c_{n,p} \cosh^{-2p} z + d_{n,p} \sinh z \cosh^{-2p-1} z), \quad (52)$$

where c_p and d_p are constant coefficients,

$$c_{n,p} = \sum_{k=1}^{n-1} \sum_{j=1}^{p-2} (a_{k,p-j} a_{n-k,j} - a_{k,p-1-j} a_{n-k,j} + b_{k,p-j} b_{n-k,j}), \quad (53)$$

$$d_{n,p} = \sum_{k=1}^{n-1} \sum_{j=1}^{p-2} (a_{k,p-j} b_{n-k,j} + b_{k,p-j} a_{n-k,j}). \quad (54)$$

Due to Eq. (46), the bounds of c_p and d_p can be obtained as follows:

$$\begin{aligned}
c_{n,p} &= 0, \text{ if } p < 2 \text{ or } p > [\tfrac{n}{2}] + 1, \\
d_{n,p} &= 0, \text{ if } p < 2 \text{ or } p > [\tfrac{n+1}{2}],
\end{aligned} \quad (55)$$

Referring to $A_n(z)$ with Eq. (52), equation 16) can be written as

$$\begin{aligned}
A_n(z) &= \int \left(E_{0,n;m} \cosh^{-2} z + \sum_{p=2}^{[\frac{n}{2}]+1} c_{n,p} \cosh^{-2p} z + \sum_{p=2}^{[\frac{n}{2}]+1} d_{n,p} \sinh z \cosh^{-2p-1} z \right) \\
&\quad \times \cosh^{-2m} z [(2\cosh^2 z - 1) + 2 \sinh z \cosh z] \, \mathrm{d}z \\
&= -\frac{E_{0,n;m}}{m} \cosh^{-2m} z - \sum_{p=2}^{[\frac{n}{2}]+1} \frac{c_{n,p} + d_{n,p}}{m+p-1} \cosh^{-2m-2p+2} z + \sum_{p=2}^{[\frac{n}{2}]+1} \frac{d_{n,p}}{2m+2p} \cosh^{-2m-2p} z \\
&\quad + 2E_{0,n;m} Q(m, z) - E_{0,n;m} Q(m+1, z)
\end{aligned}$$

$$+\sum_{p=2}^{[\frac{n}{2}]+1}2(c_{n,p}+d_{n,p})Q(m+p-1,z)-\sum_{p=2}^{[\frac{n}{2}]+1}(c_{n,p}+2d_{n,p})Q(m+p,z). \tag{56}$$

With the help of the formulae in Appendix A, equation (56) can be rewritten as

$$\begin{aligned}A_n(z) =\ & h_1\times Q(m,z)-E_{0,n;m}\left[\frac{\cosh^{-2m}z}{m}+\frac{\sinh z}{2m+1}\cosh^{-2m-1}z\right]\\ &+\sum_{p=2}^{[\frac{n}{2}]+1}\left[\frac{d_{n,p}}{2m+2p}\cosh^{-2m-2p}z-\frac{c_{n,p}+d_{n,p}}{m+p-1}\cosh^{-2m-2p+2}z\right]\\ &-\sum_{p=2}^{[\frac{n}{2}]+1}(c_{n,p}+2d_{n,p})\left[\frac{\sinh z}{m+p}\sum_{j=1}^{p}\bar{I}(m+p,p-j)\cosh^{-2m-2j+1}z\right]\\ &+\sum_{p=2}^{[\frac{n}{2}]+1}2(c_{n,p}+d_{n,p})\left[\frac{\sinh z}{m+p-1}\sum_{j=1}^{p-1}\bar{I}(m+p-1,p-1-j)\cosh^{-2m-2j+1}z\right],\end{aligned} \tag{57}$$

where

$$\begin{aligned}h_1 =\ & \frac{2(m+1)}{2m+1}E_{0,n;m}+\sum_{p=2}^{[\frac{n}{2}]+1}2(c_{n,p}+d_{n,p})\frac{2m\bar{I}(m+p-1,p-2)}{m+p-1}\\ &-\sum_{p=2}^{[\frac{n}{2}]+1}(c_{n,p}+2d_{n,p})\frac{2m\bar{I}(m+p,p-1)}{m+p}.\end{aligned} \tag{58}$$

We analyse $A_n(z)$ in the same way as that we did on $A_1(z)$. We first assume that the coefficient h_1 becomes zero, so $E_{0,n;m}$ can be obtained

$$\begin{aligned}E_{0,n;m} =\ & \frac{m(2m+1)}{m+1}\left[\sum_{p=2}^{[\frac{n}{2}]+1}\frac{c_{n,p}+2d_{n,p}}{m+p}\bar{I}(m+p,p-1)\right]\\ &-\frac{2m(2m+1)}{m+1}\left[\sum_{p=2}^{[\frac{n}{2}]+1}\frac{c_{n,p}+d_{n,p}}{m+p-1}\bar{I}(m+p-1,p-2)\right].\end{aligned} \tag{59}$$

In the following, the main purpose is to calculate W_n by making use of Eq. (59). Because equation (57) is much more complicated in form than that in Eq. (25), the formula for the similar form of W_n as that of W_1 in Eq. (25) are much longer. This is what we will do in the following.

From Eqs. (57) and (59), $A_n(z)$ can be written as

$$A_n(z) = R_n(z)+\sinh z T_n(z), \tag{60}$$

where $R_n(z)$ and $T_n(z)$ are

$$\begin{aligned}R_n(z) =\ & -\frac{E_{0,n,m}}{m}\cosh^{-2m}z\\ &+\sum_{p=2}^{[\frac{n}{2}]+1}\left[\frac{d_{n,p}}{2m+2p}\cosh^{-2m-2p}z\right.\\ &\left.-\frac{c_{n,p}+d_{n,p}}{m+p-1}\cosh^{-2m-2p+2}z\right]\end{aligned}$$

$$=\sum_{j=0}^{[\frac{n}{2}]+1}R_{n,j}\cosh^{-2m-2j}z \tag{61}$$

with $R_{n,j}$ as

$$\begin{aligned}R_{n,0} &= -\frac{E_{0,n;m}}{m},\\ R_{n,1} &= -\frac{c_{n,2}+d_{n,2}}{m+1},\\ R_{n,j} &= \frac{d_{n,j}-2c_{n,j+1}-2d_{n,j+1}}{2m+2j}\quad (j\geq 2),\end{aligned} \tag{62}$$

and

$$\begin{aligned}&T_n(z)\\ &= -\frac{E_{0,n;m}}{2m+1}\cosh^{-2m-1}z\\ &\quad+\sum_{j=1}^{[\frac{n}{2}]}\sum_{p=j+1}^{[\frac{n}{2}]+1}\left[\frac{2(c_{n,p}+d_{n,p})}{m+p-1}\bar{I}(m+p-1,p-1-j)\right.\end{aligned}$$

$$\times \cosh^{-2m-2j+1} z\Big]$$
$$- \sum_{j=1}^{[\frac{n}{2}]+1} \sum_{p=j}^{[\frac{n}{2}]+1} \Big[\frac{c_{n,p}+2d_{n,p}}{m+p} \bar{I}(m+p, p-j)$$
$$\times \cosh^{-2m-2j+1} z\Big]$$
$$= \sum_{j=1}^{[\frac{n}{2}]+1} T_{n,j} \cosh^{-2m-2j+1} z. \quad (63)$$

In order to obtain Eqs. (63), the orders of sum for the last two terms have been exchanged in Eq. (55). From Eq. (63), it is easy to see that

$$T_{n,1} = -\frac{E_{0,n;m}}{2m+1} + \sum_{p=2}^{[\frac{n}{2}]+1} \Big[\frac{2(c_{n,p}+d_{n,p})}{m+p-1} \times \bar{I}(m+p-1, p-2)\Big] - \sum_{p=1}^{[\frac{n}{2}]+1} \Big[\frac{c_{n,p}+2d_{n,p}}{m+p} \bar{I}(m+p, p-1)\Big], \quad (64)$$

$$T_{n,j} = \sum_{p=j+1}^{[\frac{n}{2}]+1} \Big[\frac{2(c_{n,p}+d_{n,p})}{m+p-1} \bar{I}(m+p-1, p-1-j)\Big] - \sum_{p=j}^{[\frac{n}{2}]+1} \Big[\frac{c_{n,p}+2d_{n,p}}{m+p} \bar{I}(m+p, p-j)\Big] \quad (j \geqslant 2). \quad (65)$$

Due to Eq. (55), we can conclude the upper and lower limits,

$$R_{n,j} = 0, \text{ if } j < 0 \text{ or } j > [\tfrac{n}{2}] + 1,$$
$$T_{n,j} = 0, \text{ if } j < 1 \text{ or } j > [\tfrac{n}{2}] + 1. \quad (66)$$

According to Eq. (15), we can obtain

$$\begin{aligned} W_n(z) &= [R_n(z) + \sinh z T_n(z)] \cosh^{2m} z (\cosh z - \sinh z)^2 \\ &= (2R_n \cosh^2 z - R_n - 2T_n \cosh^3 z \\ &\quad + 2T_n \cosh z) \cosh^{2m} z \\ &\quad + \sinh z (2T_n \cosh^2 z - 2R_n \cosh z - T_n) \cosh^{2m} z \\ &= X_n + \sinh z Y_n, \end{aligned} \quad (67)$$

where

$$\begin{aligned} X_n &= (-2T_n \cosh^3 z + 2T_n \cosh z \\ &\quad + 2R_n \cosh^2 z - R_n) \cosh^{2m} z, \\ Y_n &= (2T_n \cosh^2 z - T_n - 2R_n \cosh z) \cosh^{2m} z. \end{aligned} \quad (68)$$

From Eq. (61) and (63), it is easy to see that X_n is the sum of the kind of $\cosh^{-2j} z$. Whereas Y_n is the sum of the kind of $\cosh^{-2j-1} z$. They really show that $W_n = X_n + \sinh z Y_n$ will have the similar form as that needs to be proof. Hence, we only need to proof that the upper and lower limits in X_n, Y_n are the same as that in Eq. (47). This will be done in the following. X_n, Y_n could be rewritten as

$$\begin{aligned} X_n &= \sum_{j=-1}^{[\frac{n}{2}]+1} (2R_{n,j+1} - 2T_{n,j+2} \\ &\quad + 2T_{n,j+1} - R_{n,j}) \cosh^{-2j} z \\ &= \sum_{j=-1}^{[\frac{n}{2}]+1} X_{n,j} \cosh^{-2j} z, \end{aligned} \quad (69)$$

$$\begin{aligned} Y_n &= \sum_{j=-1}^{[\frac{n}{2}]} (2T_{n,j+2} - T_{n,j+1} - 2R_{n,j+1}) \cosh^{-2j-1} z \\ &= \sum_{j=-1}^{[\frac{n}{2}]} Y_{n,j} \cosh^{-2j-1} z, \end{aligned} \quad (70)$$

where

$$\begin{aligned} X_{n,j} &= 2R_{n,j+1} - 2T_{n,j+2} + 2T_{n,j+1} - R_{n,j} \\ &= \sum_{p=j+1}^{[\frac{n}{2}]+1} \frac{(2m+2j-1)[(m+p)c_{n,p} + d_{n,p}] \bar{I}(m+p-1, p-j-1)}{(m+j)(m+p-1)(2m+2p-1)(m+j+1)} + \frac{d_{n,j}}{2m+2}, \end{aligned} \quad (71)$$

$$\begin{aligned} Y_{n,j} &= 2T_{n,j+2} - T_{n,j+1} - 2R_{n,j+1} \\ &= \sum_{p=j+1}^{[\frac{n}{2}]+1} \frac{(2m+2j-1)[(m+p)c_{n,p} + d_{n,p}] \bar{I}(m+p-1, p-j-1)}{(m+p-1)(2m+2p-1)(m+j+1)}. \end{aligned} \quad (72)$$

It is easy to calculate and obtain

$$X_{n,-1} = X_{n,0} = 0, \quad Y_{n,-1} = Y_{n,0} = 0. \quad (73)$$

Then equation (67) can be rewritten as

$$\begin{aligned} W_n(z) &= \sinh z \sum_{j=1}^{[\frac{n}{2}]} Y_{n,j} \cosh^{-2j-1} z \\ &\quad + \sum_{j=1}^{[\frac{n}{2}]+1} X_{n,j} \cosh^{-2j} z \\ &= \sinh z \sum_{k=1}^{[\frac{n}{2}]} a_{n,k} \cosh^{-2k-1} z \\ &\quad + \sum_{k=1}^{[\frac{n}{2}]+1} b_{n,k} \cosh^{-2k} z, \end{aligned} \tag{74}$$

where $a_{n,k} = Y_{n,k}$, $b_{n,k} = X_{n,k}$. We can find that equation (74) is the same as that of Eq. (44), the general expression $W_n(z)$ in Eq. (74) satisfies our assumption in Eq. (44). This ends our induction proof.

3. Conclusion

According to Eq. (5), $\Theta_0(z)$ can be obtained as follows:

$$\begin{aligned} \Theta_0(z) &= N \exp\left[-\int W \mathrm{d}z \right] \\ &= N \exp\left[-\int W_0 \mathrm{d}z - \int \sum_{n=1}^{\infty} \beta^n W_n \mathrm{d}z \right] \\ &= N \exp\left[-\int (m \tanh z - 1) \mathrm{d}z \right. \\ &\quad \left. - \int \sum_{n=1}^{\infty} \beta^n W_n \mathrm{d}z \right] \\ &= N e^z \cosh^{-m} z \exp\left\{ \sum_{n=1}^{\infty} \beta^n \right. \\ &\quad \left. \times \left[\sum_{k=1}^{[\frac{n}{2}]} \frac{a_{n,k}}{2k} \cosh^{-2k} z - \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k} Q(k,z) \right] \right\}, \end{aligned} \tag{75}$$

where we have used Eq. (17) in the above equation. The ground eigenvalue is

$$E_{0;m} = m^2 + m - 2 + \sum_{n=1}^{\infty} \beta^n E_{0,n;m}, \tag{76}$$

where $E_{0,n;m}$ is determined by Eq. (59). The ground eigenfunction is obtained as in Eq. (75). These results will be very important for their application in future.

In the following, we will compare our result with that in Ref. [15]. In order to compare Eq. (75) with the result previously obtained in Ref. [15], we transform all the functions of independent variable z back to those of independent variable θ by $z = \ln\tan(\theta/2)$, $\sin\theta = \cosh^{-1} z$, $\tanh z = -\cos\theta$, $\sinh z = -\cot\theta$, and $\exp(z) = (1-\cos\theta)/\sin\theta$.

At first we rewrite $Q(m,z)$ as a function of θ as

$$\begin{aligned} Q(k,z) &= \frac{\sinh z}{k} \sum_{i=0}^{k-1} \bar{I}(k,i) \cosh^{-2k+2i+1} z \\ &= -\cos\theta \sum_{i=0}^{k-1} \bar{I}(k,i) \sin^{2k-2i}\theta, \end{aligned} \tag{77}$$

$\Theta_0(z)$ can be rewritten as

$$\begin{aligned} \Theta_0(z) &= N(1-\cos\theta)\sin^{m-1}\theta \\ &\quad \times \exp\left\{ \sum_{n=1}^{\infty} \beta^n \left[\sum_{k=1}^{[\frac{n}{2}]} \frac{a_{n,k}}{2k} \sin^{2k}\theta \right.\right. \\ &\quad \left.\left. + \cos\theta \sum_{k=1}^{[\frac{n+1}{2}]} \sum_{i=0}^{k-1} b_{n,k} \bar{I}(k,i) \sin^{2k-2i}\theta \right] \right\}. \end{aligned} \tag{78}$$

Notice that our quantities $a_{n,k}$ are opposite to those in Ref. [15], as they could be shown in comparison in Eqs. (28)–(37), (72) with that in Eqs. (23)–(25), (47) in Ref. [15]. From Ref. [15], $P(2m-1,\theta)$ is

$$\begin{aligned} & P(2m-1,\theta) \\ &= \int \sin^{2m-1}\theta \mathrm{d}\theta \\ &= -\frac{\cos\theta}{2m} \left[\sum_{k=0}^{m} I(2m-1,k) \sin^{2m-2k-2}\theta \right], \end{aligned} \tag{79}$$

where

$$\begin{aligned} I(2m-1,k) &= \frac{2m(2m+2)\cdots(2m-2k)}{(2m-1)(2m-3)\cdots(2m-2k-1)} \\ &= 2\bar{I}(m,k). \end{aligned} \tag{80}$$

Except for a small error due to a slip of the pen (in Eq. (58) the correct super index of $\sin\theta$ is $m-1/2$, and $\Theta_0 = \Psi_0/\sin^{1/2}\theta$) in Ref. [15], our result equation (78) is the same as that in Ref. [15]. We have verified the correctness result in Ref. [15] of the spin-weighted spheroidal equation in the case of $s = 1$.

Appendix A

From Ref. [22], we have

$$\begin{aligned} Q(m,z) &= \int \mathrm{sech}^{2m} z \mathrm{d}z = \int \cosh^{-2m} z \mathrm{d}z \\ &= \frac{\sinh z}{m} \sum_{k=0}^{m-1} \bar{I}(m,k) \cosh^{-2m+2k+1} z, \end{aligned} \tag{A1}$$

where

$$\begin{aligned} & \bar{I}(m,k) \\ &= \frac{2^k m(m-1)(m-2)\cdots(m-k)}{(2m-1)(2m-3)(2m-5)\cdots(2m-2k-1)} \\ &= \frac{m(2m-2)(2m-4)\cdots(2m-2k)}{(2m-1)(2m-3)(2m-5)\cdots(2m-2k-1)}. \end{aligned} \tag{A2}$$

In the following, we will give the relation between $Q(m+n,z)$ and $Q(m,z)$.

First,

$$\begin{aligned} &Q(m+n,z)\\ &=\frac{\sinh z}{2m+2n}\sum_{k=0}^{m+n-1}\bar{I}(m+n,k)\,\mathrm{sech}^{2m+2n-2k-1}\,z\\ &=Q_1(m+n,z)+Q_2(m+n,z), \end{aligned} \tag{A3}$$

where

$$\begin{aligned} &Q_1(m+n,z)\\ &=\frac{\sinh z}{m+n}\sum_{k=0}^{n-1}\bar{I}(m+n,k)\cosh^{-2(m+n)+2k+1}z, \end{aligned} \tag{A4}$$

$$\begin{aligned} &Q_2(m+n,z)\\ &=\frac{\sinh z}{m+n-1}\sum_{k=n}^{m+n-1}\bar{I}(m+n,k)\\ &\quad\times\cosh^{-2(m+n)+2k+1}z. \end{aligned} \tag{A5}$$

By making use of $j=n-k$ in $Q_1(m+n,z)$, one obtains

$$\begin{aligned} &Q_1(m+n,z)\\ &=\frac{\sinh z}{m+n}\sum_{k=0}^{n-1}\bar{I}(m+n,k)\,\mathrm{sech}^{2m+2n-2k-1}\,z\\ &=\frac{\sinh z}{m+n}\sum_{j=1}^{n}\bar{I}(m+n,n-1-j)\,\mathrm{sech}^{2m+2j+1}\,z. \end{aligned} \tag{A6}$$

Similarly, by making use of $j=n-k$ in $Q_2(m+n,z)$ and the relation

$$\begin{aligned} \bar{I}(m+n,j+n)=&\frac{(m+n)(2m+2n-2)(2m+2n-4)\cdots(2m-2j)}{(2m+2n-1)(2m+2n-3)(2m+2n-5)\cdots(2m-2j-1)}\\ =&2\frac{(m+n)(2m+2n-2)\cdots(2m+2)m\ldots(2m-2j)}{(2m+2n-1)(2m+2n-3)\cdots(2m+1)(2m-1)\ldots(2m-2j-1)}\\ =&2\bar{I}(m+n,n-1)\bar{I}(m,j), \end{aligned} \tag{A7}$$

one has

$$\begin{aligned} &Q_2(m+n,z)\\ &=\frac{\sinh z}{m+n}\sum_{k=n}^{m+n-1}\bar{I}(m+n,k)\,\mathrm{sech}^{2m+2n-2k-1}\,z\\ &=\frac{\sinh z}{m+n}\sum_{j=0}^{m-1}\bar{I}(m+n,n+j)\,\mathrm{sech}^{2m+2j-1}\,z\\ &=2\frac{m}{m+n}\bar{I}(m+n,n-1)Q(m,z). \end{aligned} \tag{A8}$$

Applying the above formula, we have

$$\begin{aligned} Q(m,z)=&\frac{\sinh z}{2m-1}\cosh^{-2m+1}z\\ &+\frac{2m-2}{2m-1}Q(m-1,z), \end{aligned} \tag{A9}$$

$$\begin{aligned} Q(m+1,z)=&\frac{\sinh z}{2m+1}\cosh^{-2m-1}z\\ &+\frac{2m}{2m+1}Q(m,z). \end{aligned} \tag{A10}$$

In the calculation of W_n, $Q_1(m+n,z)$ makes Θ_0 be convergent, whereas $Q_2(m+n,z)$ does not.

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