Analytic solutions of the ground and excited states of the spin-weighted spheroidal equation in the case of $s = 2$*

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By using the super-symmetric quantum mechanics (SUSYQM) method, this paper obtains the analytical solutions for the spin-weighted spheroidal wave equation in the case of $s = 2$. Based on the derived W_0 to W_4 the general form for the n-th-order super-potential is summarized and is proved correct by mathematical induction. Hence the ground eigenvalue problem is completely solved. Particularly, the novel solutions of the excited state are investigated according to the shape-invariance property.

Keywords: spheroidal wave equation, supersymmetric quantum mechanics, super-potential, eigenvalue and eigenfunction

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1. Introduction

Spin-weighted spheroidal harmonics (SWSHs) have entered into various branches of the physics field, including light scattering, nuclear modeling, signal processing, electromagnetic wave propagation, black hole perturbation theory in four and higher dimensions, and so on.[1−6] Recently, SWSHs have played a significant role in research on cosmological anisotropies and computing black hole quasinormal modes.[7] SWSHs also find applications in describing the electromagnetic radiation and polarization in the Newman–Penrose (NP) formalism and quantum field theory in curved space–time especially the renormalized expectation value of the stress-energy tensor (RSET).[8]

SWSHs are first separated from the Teukolsky equation by introducing the Kinnersley tetrad and Boyer–Lindquist coordinates, the solutions of the angular function ${}_sS_{lm\omega}(\theta)$ are called the spin-weighted spheroidal harmonics, which is normalized as[9]

$$\int_0^\pi |{}_sS_{lm\omega}(\theta)|^2 \sin\theta \mathrm{d}\theta = 1. \tag{1}$$

We take as a starting point of the differential equation of ${}_sS_{lm\omega}$,

$$\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right) + s + a^2\omega^2\cos^2\theta - 2as\omega\cos\theta - \frac{(m + s\cos\theta)^2}{\sin^2\theta} + {}_sE_{lm\omega}\right] \times {}_sS_{lm\omega}(\theta) = 0, \tag{2}$$

where a is the angular momentum per unit mass of the rotating black hole and ω is the frequency of the mode, ${}_sE_{lm\omega}$ is the eigenvalue of ${}_sS_{lm\omega}(\theta)$ and in the Schwarzschild limit, it reduces to the SWSHs with ${}_sE_{lm\omega} \to l(l+1)$. We are only interested in the independent variable θ that lies in the interval $(0, \pi)$. The parameter s is the helicity of the fields, $s = 0, \pm\frac{1}{2}, \pm1, \pm2$ correspond to the scalar, neutrino, electromagnetic or gravitational perturbation fields, respectively.

The SWSHs has been solved by various approaches in previous works,[10−17] Sasaki and Nakamura's work is restricted to the choice of $s = -2$ and this is an appropriate choice for studies of gravitational perturbations, which we will not present in this paper. Instead, we present a novel formalism, namely SUSYQM, to solve the SWSHs in a more systematic manner, albeit very mathematical. In this paper, we study the low frequency ($a\omega \to 0$) of the solution and eigenvalues of the spin-weighted spheroidal differential equation in the case of $s = 2$. And such results can be extensively used in the studies of gravitational wave generation and gravitational radiation reaction. The main purpose of this paper is to provide a complete

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analysis of the eigenvalues and eigenfunctions of SWSHs. This research has not been developed any further hence this paper is significant to further study.

The SWSHs satisfy the following symmetries[6,7]

$$ {}_sS_{lm\omega}(\theta) = (-1)^{l+m} \times_{-s} S_{lm\omega}(\pi-\theta), \quad (3)$$

$$ {}_sS_{lm\omega}(\theta) = (-1)^{l+s} \times_{s} S_{l-m-\omega}(\pi-\theta), \quad (4)$$

$$ {}_sS_{lm\omega}(\theta) = (-1)^{h+m} \times_{-s} S_{l-m-\omega}(\theta), \quad (5)$$

and the eigenvalues also satisfy the symmetries

$$ {}_sE_{lm\omega} = {}_{-s}E_{lm\omega}, \quad (6)$$

$$ {}_sE_{lm\omega} = {}_sE_{l-m-\omega}. \quad (7)$$

Hence the helicity s is related to the SWSH with helicity $-s$. That is to say, in the case if $s=-2$ the relevant results could be derived similarly to the results of $s=2$ and we will not discuss it here. This latter aspect is a very important one since it allows to study some fundamental and conceptual introduction of SUSYQM method.

2. Solving the super-potential by using the SUSYQM method

In this section, we present a method of SUSYQM based on the above formalism in the case of a Kerr black hole background. In the past ten years, the idea of supersymmetry has been applied to nonrelativitic quantum mechanical problems and is first studied in the simplest case of SUSYQM. Using $\Theta(\theta)$ to denote ${}_sS_{lm\omega}(\theta)$ and E denote ${}_sE_{lm\omega}$ for simplicity, one can write a simple transformation as before[10−17]

$$\Theta(\theta) = \frac{\Psi(\theta)}{\sqrt{\sin\theta}}. \quad (8)$$

Inserting such an expression into Eq. (2), one can obtain the Schrödinger-like equation

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\theta^2} + \left[\frac{1}{4} + s + \beta^2\cos^2\theta - 2s\beta\cos\theta - \frac{(m+s\cos\theta)^2 - 1/4}{\sin^2\theta} + E\right]\Psi = 0, \quad (9)$$

where $\beta = a\omega$ and the corresponding boundary conditions are $\Psi|_{\theta=0} = \Psi|_{\theta=\pi} = 0$. The explicit expression for the potential in Eq. (9) is[10−17]

$$V(\theta,\beta,s) = -\left[\frac{1}{4} + s + \beta^2\cos^2\theta - 2s\beta\cos\theta - \frac{(m+s\cos\theta)^2 - 1/4}{\sin^2\theta}\right]. \quad (10)$$

The super-potential W is an important concept in SUSYQM and is further linked via potential V and E_0[18]

$$W^2 - W' = V(\theta,\beta,s) - E_0. \quad (11)$$

We can see that equation (11) is as difficult to solve as Eq. (9). As discussed in detail in Refs. [10]–[17], we look for the approximation methods: the parameter β is assumed to be small, then it is straightforward to obtain a spheroidal harmonic $\Theta(\theta)$ and its eigenvalue E perturbatively by the standard method (and it also can be expanded in terms of the Jacobi function, and we can obtain the results of arbitrary value of $a\omega$[9]). Considering the above equations to be different enough and expanding $\Psi(\theta)$ being not very advantageous, in contrast, in the W case, it is much easier to seek the ground state wave function Ψ_0. Thus, we can expand the super-potential W as a Taylor series with an artificial small parameter β by

$$W = \sum_{n=0}^{\infty} \beta^n W_n. \quad (12)$$

And the W-expansion method can greatly simplify the ground eigenfunction by

$$\begin{aligned}\Psi_0 &= N\exp\left[-\int W\,\mathrm{d}\theta\right] \\ &= N\exp\left[-\int (W_0 + \beta W_1 \right. \\ &\quad \left. + \beta^2 W_2 + \cdots)\,\mathrm{d}\theta\right]. \end{aligned} \quad (13)$$

Hence we will make use of the easier method of W-expansion to resolve the spin-weighted spheroidal equations. As done before, the ground eigenvalue E_0 is expanded in the parameter β as

$$E_0 = \sum_{n=0}^{\infty} E_{0,n;m}\beta^n, \quad (14)$$

where the lower suffix m in $E_{0,n;m}$ is referred to the parameter m in Eq. (9) and the index 0 means belonging to the ground state energy and n refers to the n-th-order item of the series expansion. Substituting Eqs. (12) and (14) into Eq. (11), it is easy to see that

$$\begin{aligned} &W_0^2 - W_0' + \beta(2W_0W_1 - W_1') + \cdots \\ &+ \beta^n\left[\sum_{k=1}^{n-1} W_kW_{n-k} - W_n'\right] + \cdots \\ &= -\left(E_{0,0;m} + s + \frac{1}{4} - \frac{(m+s\cos\theta)^2 - 1/4}{\sin^2\theta}\right) \\ &\quad - \beta(E_{0,1;m} - 2s\cos\theta) - \beta^2(E_{0,2;m} \end{aligned}$$

$$+\cos^2\theta) - \beta^3(E_{0,3;m}) - \cdots . \quad (15)$$

From Eq. (15), one can easily obtain

$$\begin{aligned}
W_0' - W_0^2 &= E_{0,0;m} + s + \frac{1}{4} - \frac{(m + s\cos\theta)^2 - 1/4}{\sin^2\theta} \\
&\equiv f_0(\theta), \\
W_1' - 2W_0W_1 &= E_{0,1;m} - 2s\cos\theta \equiv f_1(\theta), \\
W_2' - 2W_0W_2 &= E_{0,2;m} + \cos^2\theta + W_1^2 \equiv f_2(\theta), \\
W_n' - 2W_0W_n &= E_{0,n;m} + \sum_{k=1}^{n-1} W_kW_{n-k} \\
&\equiv f_n(\theta)(n > 2). \quad (16)
\end{aligned}$$

When $s = 2$ it is straightforward to obtain

$$\begin{aligned}
E_{0,0;m} &= m(m+1) - s(s+1) = m^2 + m - 6, \\
W_0 &= -\frac{s + (m + \frac{1}{2})\cos\theta}{\sin\theta} \\
&= -\frac{2 + (m + \frac{1}{2})\cos\theta}{\sin\theta}. \quad (17)
\end{aligned}$$

So, it is easy to see that

$$\begin{aligned}
W_n(\theta) &= \left(\tan\frac{\theta}{2}\right)^{-4} \sin^{-2m-1}\theta A_n(\theta) \\
&= (1+\cos\theta)^4 \sin^{-2m-5}\theta A_n(\theta), \quad (18)
\end{aligned}$$

where

$$A_n(\theta) = \int f_n(\theta)(1-\cos\theta)^4 \sin^{2m-3}\theta \mathrm{d}\theta. \quad (19)$$

In order to calculate Eqs. (18) and (19), one needs the following formulae

$$\begin{aligned}
&P(2m+1,\theta) \\
&= \int \sin^{2m+1}\theta \mathrm{d}\theta \\
&= -\frac{\cos\theta}{2m+2}\left[\sum_{k=0}^{m} I(2m+1,k)\sin^{2m-2k}\theta\right] \quad (20)
\end{aligned}$$

with

$$I(2m+1,k) = \begin{cases} \dfrac{(2m+2)(2m)\cdots(2m+2-2k)}{(2m+1)(2m-1)\cdots(2m-2k+1)}, & \text{if } k >= 0; \\ 0, & \text{if } k < 0. \end{cases} \quad (21)$$

Here we give some recurrence relations between different $P(2m+2n+1,\theta)$ and $P(2m-3,\theta)$ for later use

$$\begin{aligned}
P(2m+2n+1,\theta) &= \frac{-\cos\theta}{2m+2n+2}\sum_{k=0}^{n+1} I(2m+2n+1,k)\sin^{2m+2n-2k}\theta \\
&\quad + \frac{(m-1)I(2m+2n+1,n+1)}{m+n+1} P(2m-3,\theta). \quad (22)
\end{aligned}$$

By the help of Eq. (14), the calculation of A_1 comes easily as

$$\begin{aligned}
A_1(\theta) &= \int (E_{0,1;m} - 4\cos\theta)(1-\cos\theta)^4 \sin^{2m-3}\theta \mathrm{d}\theta \\
&= 8(E_{0,1;m}+4)P(2m-3,\theta) \\
&\quad - 8(E_{0,1;m}+6)P(2m-1,\theta) \\
&\quad + (E_{0,1;m}+16)P(2m+1,\theta) \\
&\quad - \frac{4(E_{0,1;m}+4)\sin^{2m-2}\theta}{m-1} \\
&\quad + \frac{2(E_{0,1;m}+8)\sin^{2m}\theta}{m} - \frac{2\sin^{2m+2}\theta}{m+1}, \quad (23)
\end{aligned}$$

and it can be further simplified as

$$\begin{aligned}
A_1(\theta) &= a_1 \times P(2m-3,\theta) \\
&\quad - \cos\theta\frac{(E_{0,1;m}+16)\sin^{2m}\theta}{2m+1} \\
&\quad - \frac{4(E_{0,1;m}+4)\sin^{2m-2}\theta}{m-1} \\
&\quad + \frac{2(E_{0,1;m}+8)\sin^{2m}\theta}{m} - \frac{2\sin^{2m+2}\theta}{m+1} \\
&\quad + \cos\theta\frac{8(E_{0,1;m}+6)\sin^{2m-2}\theta}{2m-1} \\
&\quad - \cos\theta\frac{2m(E_{0,1;m}+16)\sin^{2m-2}\theta}{(2m+1)(2m-1)}, \quad (24)
\end{aligned}$$

where the coefficient of the term $P(2m-3,\theta)$ is

$$\begin{aligned}
a_1 &= 8(E_{0,1;m}+4) - 8(E_{0,1;m}+6)\frac{2m-2}{2m-1} \\
&\quad + \frac{2m(2m-2)(E_{0,1;m}+16)}{(2m+1)(2m-1)}. \quad (25)
\end{aligned}$$

From Eqs. (18) and (20) the term $P(2m-3,\theta)$ in $A_1(\theta)$ would make $W_1(\theta)$ blow up at the boundaries $\theta = 0,\ \pi$, which induce the unboundedness of the ground state Ψ_0 at $\theta = 0,\ \pi$. That is, unless the coefficient a_1 of the divergent term $P(2m-3,\theta)$ in A_1 is zero, the eigenfunction $\Psi(\theta)$ could not remain finite at the boundaries. Thus, $a_1 = 0$ gives the following

result

$$E_{0,1;m} = -\frac{8}{m+1}, \tag{26}$$

and it further provides the concise form for A_1

$$\begin{aligned} A_1(\theta) = & -\frac{2\sin^{2m-2}\theta}{m+1}[8 - 8\cos\theta - 8\sin^2\theta \\ & + 4\cos\theta\sin^2\theta + \sin^4\theta]. \end{aligned} \tag{27}$$

There is only one parameter, that is, the eigenvalue $E_{0,1;m}$, to select in calculation of the ground state Ψ_0, this selection of the eigenvalue $E_{0,1;m}$ makes $A_1(\theta)$ as that of Eq. (27). Could equation (27) give the required Ψ_0 finite at $\theta = 0$, π? Calculation shows that this can be the case. With the help of Eq. (18), it is easy to obtain the first-order $W_1(\theta)$

$$\begin{aligned} W_1(\theta) &= A_1(\theta)(1+\cos\theta)^4(\sin\theta)^{-2m-5} \\ &= -\frac{2}{m+1}\sin\theta, \end{aligned} \tag{28}$$

and it obviously satisfies our requirements for Ψ_0. By the same tedious calculation as that of $W_1(\theta)$, $E_{0;2;m}$, W_2, $E_{0;3;m}$, W_3, and $E_{0;4;m}$, W_4 can also be obtained

$$E_{0,2;m} = -\frac{m^3+19m^2+35m-15}{(m+1)^3(2m+3)},$$

$$W_2(\theta) = a_{2,1}\sin\theta\cos\theta + b_{2,1}\sin\theta, \tag{29}$$

$$a_{2,1} = \frac{(m-1)(m+3)}{(2m+3)(m+1)^2}, \tag{30}$$

$$b_{2,1} = -\frac{2(m-1)(m+3)}{(2m+3)(m+1)^3}, \tag{31}$$

$$E_{0,3;m} = -\frac{16(m-1)^2(m+3)^2}{(m+1)^5(2m+3)(m+2)},$$

$$W_3(\theta) = a_{3,1}\cos\theta\sin\theta + b_{3,1}\sin\theta + b_{3,2}\sin^3\theta, \tag{32}$$

$$a_{3,1} = -\frac{8(m-1)(m+3)}{(m+1)^4(2m+3)(m+2)}, \tag{33}$$

$$b_{3,1} = \frac{16(m-1)(m+3)}{(m+1)^5(2m+3)(m+2)}, \tag{34}$$

$$b_{3,2} = -\frac{2(m-1)(m+3)}{(m+1)^3(2m+3)(m+2)}, \tag{35}$$

$$E_{0,4;m} = -\frac{2(m-1)^2(m+3)^2(m^5+70m^4+358m^3-80m^2-1839m-1710)}{(m+1)^7(2m+3)^3(m+2)(2m+5)},$$

$$W_4(\theta) = a_{4,1}\cos\theta\sin\theta + a_{4,2}\cos\theta\sin^3\theta + b_{4,1}\theta\sin\theta + b_{4,2}\sin^3\theta, \tag{36}$$

$$a_{4,1} = -\frac{(m-1)(m+3)(m^5+70m^4+358m^3-80m^2-1839m-1710)}{(m+1)^6(2m+3)^3(m+2)(2m+5)}, \tag{37}$$

$$a_{4,2} = \frac{(m-1)(m+3)(m^3+4m^2-15m-30)}{(m+1)^4(2m+3)^2(m+2)(2m+5)}, \tag{38}$$

$$b_{4,1} = \frac{2(m-1)(m+3)(m^5+70m^4+358m^3-80m^2-1839m-1710)}{(m+1)^7(2m+3)^3(m+2)(2m+5)}, \tag{39}$$

$$b_{4,2} = -\frac{6(m-1)(m+3)(m^3+4m^2-15m-30)}{(m+1)^5(2m+3)^2(m+2)(2m+5)}. \tag{40}$$

3. Summarizing and proving the general formula of super-potential

Although using the SWSHs method to compute the first five super-potentials is mathematically rather complicated, we summarize the beautiful formulae for the n-th-order super-potential by the first five terms

$$\begin{aligned} W_n(\theta) = & \cos\theta\sum_{k=1}^{[\frac{n}{2}]} a_{n,k}\sin^{2k-1}\theta \\ & + \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k}\sin^{2k-1}\theta, \end{aligned} \tag{41}$$

where $a_{n,k}$ and $b_{n,k}$ meet the requirement

$$\begin{aligned} &a_{n,k} = 0 \text{ if } k < 1 \text{ or } k > [\tfrac{n}{2}], \\ &b_{n,k} = 0 \text{ if } k < 1 \text{ or } k > [\tfrac{n+1}{2}]. \end{aligned} \tag{42}$$

Here we use mathematical induction to prove that when n is greater than 1, $W_n(\theta)$ can also be expressed as the general formula. Assuming $W_k(\theta)$ and $W_\varepsilon(\theta)$ $(k+\varepsilon = n)$ are both meeting the general form, we have

$$\begin{aligned} W_k(\theta) = & \cos\theta\sum_{i=1}^{[\frac{k}{2}]} a_{k,i}\sin^{2i-1}\theta \\ & + \sum_{i=1}^{[\frac{k+1}{2}]} b_{k,i}\sin^{2i-1}\theta, \end{aligned}$$

$$W_\varepsilon(\theta) = \cos\theta \sum_{j=1}^{[\frac{\varepsilon}{2}]} a_{\varepsilon,j} \sin^{2j-1}\theta + \sum_{j=1}^{[\frac{\varepsilon+1}{2}]} b_{\varepsilon,j} \sin^{2j-1}\theta. \quad (43)$$

Hence our goal is to verify that $W_n(\theta)$ meets the general formula. Since equation to equation (16) takes the above $W_k(\theta)$ and $W_\varepsilon(\theta)$ into $\sum_{k=1}^{n-1} W_k W_{n-k}$, that is[14,15]

$$\sum_{k=1}^{n-1} W_k W_\varepsilon = \sum_{p=2}^{[\frac{n}{2}]+1} h_{n,p} \sin^{2p-2}\theta + \sum_{p=2}^{[\frac{n}{2}]+1} g_{n,p} \sin^{2p-2}\theta\cos\theta, \quad (44)$$

where $p = i+j$, $h_{n,p}$ and $g_{n,p}$ are constant coefficients, we have

$$h_{n,p} = \sum_{k=1}^{n-1}\sum_{j=1}^{p-1}(a_{k,p-j}a_{n-k,j} - a_{k,p-1-j}a_{n-k,j} + b_{k,p-j}b_{n-k,j}), \quad (45)$$

$$g_{n,p} = \sum_{k=1}^{n-1}\sum_{j=1}^{p-1}(a_{k,p-j}b_{n-k,j} + b_{k,p-j}a_{n-k,j}). \quad (46)$$

By using Eq. (42), we can obtain

$$\begin{aligned} &h_{n,p} = 0, \text{ if } p < 2 \text{ or } p > [\tfrac{n}{2}] + 1; \\ &g_{n,p} = 0, \text{ if } p < 2 \text{ or } p > [\tfrac{n+1}{2}]; \end{aligned} \quad (47)$$

when we do not distinguish the parity of n, the maximum limit of p can be written as $[\frac{n}{2}]+1$. Substituting $f_n(\theta) = E_{0,n;m} + \sum_{k=1}^{n-1} W_k W_{n-k}$ into Eq. (19), we obtain

$$\begin{aligned} A_n(\theta) = {} & E_{0,n;m}\Big[8P(2m-3,\theta) - 8P(2m-1,\theta) \\ & + P(2m+1,\theta) - \frac{4\sin^{2m-2}\theta}{m-1} + \frac{2\sin^{2m}\theta}{m}\Big] \\ & + \sum_{p=2}^{[\frac{n}{2}]+1}\Big[\frac{g_{n,p}\sin^{2m+2p}\theta}{2m+2p} \\ & + \frac{8(g_{n,p}-h_{n,p})\sin^{2m+2p-4}\theta}{2m+2p-4} \\ & + \frac{(4h_{n,p}-8g_{n,p})\sin^{2m+2p-2}\theta}{2m+2p-2}\Big] \\ & + \sum_{p=2}^{[\frac{n}{2}]+1}\Big[8(h_{n,p}-g_{n,p})P(2m+2p-5,\theta) \\ & + (12g_{n,p}-8h_{n,p})P(2m+2p-3,\theta)\Big] \\ & + \sum_{p=2}^{[\frac{n}{2}]+1}(h_{n,p}-4g_{n,p})P(2m+2p-1,\theta). \end{aligned} \quad (48)$$

We simplify the above result according to Eq. (22), then we have

$$\begin{aligned} A_n(\theta) = {} & P(2m-3,\theta)\times b_1 \\ & + E_{0,n;m}\Big[-\frac{4\sin^{2m-2}\theta}{m-1} + \frac{2\sin^{2m}\theta}{m} + \cos\theta\left(\frac{2(7m+4)\sin^{2m-2}\theta}{(2m-1)(2m+1)} - \frac{\sin^{2m}\theta}{2m+1}\right)\Big] \\ & + \sum_{p=2}^{[\frac{n}{2}]+1}\Big[\frac{8(g_{n,p}-h_{n,p})\sin^{2m+2p-4}\theta}{2m+2p-4} + \frac{g_{n,p}\sin^{2m+2p}\theta}{2m+2p} + \frac{(4h_{n,p}-8g_{n,p})\sin^{2m+2p-2}\theta}{2m+2p-2}\Big] \\ & + \sum_{p=2}^{[\frac{n}{2}]+1} 8(h_{n,p}-g_{n,p})\Big[\frac{-\cos\theta}{2m+2p-4}\sum_{j=0}^{p-2} I(2m+2p-5,p-2-j)\sin^{2m+2j-2}\theta\Big] \\ & + \sum_{p=2}^{[\frac{n}{2}]+1}(-8h_{n,p}+12g_{n,p})\Big[\frac{-\cos\theta}{2m+2p-2}\sum_{j=0}^{p-1} I(2m+2p-3,p-1-j)\sin^{2m+2j-2}\theta\Big] \\ & + \sum_{p=2}^{[\frac{n}{2}]+1}(h_{n,p}-4g_{n,p})\Big[\frac{-\cos\theta}{2m+2p}\sum_{j=0}^{p} I(2m+2p-1,p-j)\sin^{2m+2j-2}\theta\Big], \end{aligned} \quad (49)$$

where b_1 is the coefficient of the term $P(2m-3,\theta)$

$$\begin{aligned} b_1 = {} & \Big[\frac{4(m+1)(m+2)E_{0,n;m}}{(2m+1)(2m-1)} + \sum_{p=2}^{[\frac{n}{2}]+1}\frac{(m-1)(12g_{n,p}-8h_{n,p})}{m+p-1} I(2m+2p-3,p-1)\Big] \\ & + \sum_{p=2}^{[\frac{n}{2}]+1}\Big[\frac{8(m-1)(h_{n,p}-g_{n,p})}{m+p-2} I(2m+2p-5,p-2)\Big] \end{aligned}$$

$$+\sum_{p=2}^{[\frac{n}{2}]+1}\left[\frac{(m-1)(h_{n,p}-4g_{n,p})}{m+p}I(2m+2p-1,p)\right]. \tag{50}$$

From a similar analysis to $A_1(\theta)$, $W_n(\theta)$ blows up at the boundaries $\theta=0,\ \pi$, and the coefficient b_1 of the term $P(2m-3,\theta)$ in A_n must be zero, then the eigenfunction $\Psi_0(\theta)$ could be finite at the boundaries. Thus, at $b_1=0$, $E_{0,n;m}$ can be obtained[14,15]

$$\begin{aligned}E_{0,n;m} =& -\frac{(2m+1)(2m-1)}{4(m+1)(m+2)}\left[\sum_{p=2}^{[\frac{n}{2}]+1}\frac{(m-1)(12g_{n,p}-8h_{n,p})}{m+p-1}I(2m+2p-3,p-1)\right]\\ &-\frac{(2m+1)(2m-1)}{4(m+1)(m+2)}\left[\sum_{p=2}^{[\frac{n}{2}]+1}\frac{8(m-1)(h_{n,p}-g_{n,p})}{m+p-2}I(2m+2p-5,p-2)\right]\\ &-\frac{(2m+1)(2m-1)}{4(m+1)(m+2)}\left[\sum_{p=2}^{[\frac{n}{2}]+1}\frac{(m-1)(h_{n,p}-4g_{n,p})}{m+p}I(2m+2p-1,p)\right],\end{aligned} \tag{51}$$

$$E_{0,n;m} = -\sum_{p=2}^{[\frac{n}{2}]+1}\frac{(m-1)(4m^2-1)(m+p+1)[(m+p)h_{n,p}-2g_{n,p}]I(2m+2p-5,p-2)}{(m+1)(m+2)(m+p-2)(2m+2p-1)(2m+2p-3)}. \tag{52}$$

By use of Eq. (52), the following focus on the simplification of $A_n(\theta)$ and calculation of $W_n(\theta)$. From Eqs. (49) and (52), $A_n(\theta)$ can be simplified as $A_n(\theta)=R_n(\theta)+\cos\theta T_n(\theta)$, where[14,15]

$$\begin{aligned}R_n(\theta) =& E_{0,n;m}\left[-\frac{4\sin^{2m-2}\theta}{m-1}+\frac{2\sin^{2m}\theta}{m}\right]+\sum_{p=2}^{[\frac{n}{2}]+1}\left[\frac{8(-h_{n,p}+g_{n,p})\sin^{2m+2p-4}\theta}{2m+2p-4}\right]\\ &+\sum_{p=2}^{[\frac{n}{2}]+1}\left[\frac{g_{n,p}\sin^{2m+2p}\theta}{2m+2p}+\frac{(4h_{n,p}-8g_{n,p})\sin^{2m+2p-2}\theta}{2m+2p-2}\right]=\sum_{p=-1}^{[\frac{n}{2}]+1}R_{n,p}\sin^{2m+2p}\theta,\end{aligned} \tag{53}$$

$$\begin{aligned}T_n(\theta) =& E_{0,n;m}\left[\frac{(14m+8)\sin^{2m-2}\theta}{(2m-1)(2m+1)}-\frac{\sin^{2m}\theta}{2m+1}\right]\\ &+\sum_{j=0}^{[\frac{n}{2}]-1}\sum_{p=j+2}^{[\frac{n}{2}]+1}\left[\frac{8(-h_{n,p}+g_{n,p})}{2m+2p-4}I(2m+2p-5,p-2-j)\sin^{2m+2j-2}\theta\right]\\ &+\sum_{j=0}^{[\frac{n}{2}]}\sum_{p=j+1}^{[\frac{n}{2}]+1}\left[\frac{(8h_{n,p}-12g_{n,p})}{2m+2p-2}I(2m+2p-3,p-1-j)\sin^{2m+2j-2}\theta\right]\\ &+\sum_{j=0}^{[\frac{n}{2}]+1}\sum_{p=j}^{[\frac{n}{2}]+1}\left[\frac{(-h_{n,p}+4g_{n,p})}{2m+2p}I(2m+2p-1,p-j)\sin^{2m+2j-2}\theta\right]=\sum_{l=-1}^{[\frac{n}{2}]}T_{n,l}\sin^{2m+2l}\theta,\end{aligned} \tag{54}$$

where

$$\begin{aligned}R_{n,-1} =& -\frac{4E_{0,n;m}}{m-1},\quad R_{n,0}=\frac{2E_{0,n;m}}{m}+\frac{-8h_{n,2}+8g_{n,2}}{2m},\\ R_{n,1} =& \frac{(4h_{n,2}-8g_{n,2})+(-8h_{n,3}+8g_{n,3})}{2m+2},\\ R_{n,p} =& \frac{g_{n,p}+(4h_{n,p+1}-8g_{n,p+1})+(-8h_{n,p+2}+8g_{n,p+2})}{2m+2p}\quad (p>1),\end{aligned} \tag{55}$$

and

$$\begin{aligned}T_{n,-1} =& \frac{2(7m+4)E_{0,n;m}}{(2m-1)(2m+1)}+\sum_{p=2}^{[\frac{n}{2}]+1}\left[\frac{8(-h_{n,p}+g_{n,p})}{2m+2p-4}I(2m+2p-5,p-2)\right]\\ &+\sum_{p=2}^{[\frac{n}{2}]+1}\left[\frac{(8h_{n,p}-12g_{n,p})}{2m+2p-2}I(2m+2p-3,p-1)\right]+\sum_{p=2}^{[\frac{n}{2}]+1}\left[\frac{(-h_{n,p}+4g_{n,p})}{2m+2p}I(2m+2p-1,p)\right]\end{aligned}$$

$$T_{n,0} = -\frac{E_{0,n;m}}{2m+1} + \sum_{p=3}^{[\frac{n}{2}]+1}\left[\frac{8(-h_{n,p}+g_{n,p})}{2m+2p-4}I(2m+2p-5,p-3)\right] + \sum_{p=2}^{[\frac{n}{2}]+1}\left[\frac{(8h_{n,p}-12g_{n,p})}{2m+2p-2}I(2m+2p-3,p-2)\right] + \sum_{p=2}^{[\frac{n}{2}]+1}\left[\frac{(-h_{n,p}+4g_{n,p})}{2m+2p}I(2m+2p-1,p-1)\right]; \quad (56)$$

and when $>= 1$, $T_{n,l}$ can be abbreviated as

$$T_{n,l} = \sum_{p=l+3}^{[\frac{n}{2}]+1}\frac{8(-h_{n,p}+g_{n,p})}{2m+2p-4}I(2m+2p-5,p-3-l) + \sum_{p=l+2}^{[\frac{n}{2}]+1}\frac{(8h_{n,p}-12g_{n,p})}{2m+2p-2}I(2m+2p-3,p-2-l) + \sum_{p=l+1}^{[\frac{n}{2}]+1}\frac{(-h_{n,p}+4g_{n,p})}{2m+2p}I(2m+2p-1,p-1-l). \quad (57)$$

From Eq. (42), one can obtain

$$\begin{aligned} &R_{n,p} = 0, \text{ if } p < -1 \text{ or } p > [\tfrac{n+1}{2}]; \\ &T_{n,p} = 0, \text{ if } p < -1 \text{ or } p > [\tfrac{n}{2}]. \end{aligned} \quad (58)$$

According to Eq. (18),

$$\begin{aligned} W_n(\theta) &= [R_n(\theta) + \cos\theta T_n(\theta)] \times (1+\cos\theta)^4 \sin^{-2m-5}\theta \\ &= [(8R_n+8T_n) - (8R_n+12T_n)\sin^2\theta + (R_n+4T_n)\sin^4\theta]\sin^{-2m-5}\theta \\ &\quad + \cos\theta[(8R_n+8T_n) - (4R_n+8T_n) \times \sin^2\theta + T_n\sin^4\theta]\sin^{-2m-5}\theta \\ &= X_n + \cos\theta Y_n, \end{aligned} \quad (59)$$

where

$$X_n = \sum_{l=-3}^{[\frac{n}{2}]+1} X_{n,l}\sin^{2l-1}\theta, \quad (60)$$

and

$$\begin{aligned} X_{n,l} &= 8R_{n,l+2} + 8T_{n,l+2} - 8R_{n,l+1} - 12T_{n,l+1} + R_{n,l} + 4T_{n,l} \\ &= -\sum_{p=l+1}^{[\frac{n}{2}]+1}\frac{4(2m+2l-2)I(2m+2p-1,p-1-l)}{(2m+2l+2)(2m+2l+4)}A_{n,p} + \frac{g_{n,l}}{2m+2l} \end{aligned} \quad (61)$$

with

$$A_{n,p} = -\frac{(m+p+1)[h_{n,p}(m+p)-2g_{n,p}]}{2(m+p-2)(m+p-1)(m+p)}. \quad (62)$$

And Y_n can be expressed in the same way,

$$Y_n = \sum_{l=-3}^{[\frac{n}{2}]+1} Y_{n,l}\sin^{2l-1}\theta, \quad (63)$$

where

$$\begin{aligned} Y_{n,l} &= 8R_{n,l+2} + 8T_{n,l+2} - 4R_{n,l+1} - 8T_{n,l+1} + T_{n,l} \\ &= \sum_{p=l+1}^{[\frac{n}{2}]+1}\frac{(2m+2l)(2m+2l-2)I(2m+2p-1,p-1-l)}{(2m+2l+2)(2m+2l+4)}A_{n,p}. \end{aligned} \quad (64)$$

Obviously, these results contain some items that may cause $W_n(\theta)$ to be divergent. The choice of $E_{0,n;m}$ in Eq. (52) guarantees the boundedness of $W_n(\theta)$ at $\theta = 0, \pi$. The following analysis mainly aims at the divergent terms in $X_{n,l}$ and $Y_{n,l}$, with $l = -3, -2, -1, 0$. After an accurate calculation, when $l = -3, -2, -1, 0$

$$X_{n,-3} = 8R_{n,-1} + 8T_{n,-1} = 0, \quad (65)$$

$$X_{n,-2} = 8R_{n,0} + 8T_{n,0} - 8R_{n,-1} - 12T_{n,-1} = 0, \tag{66}$$

$$X_{n,-1} = 8R_{n,1} + 8T_{n,1} - 8R_{n,0} - 12T_{n,0} + R_{n,-1} + 4T_{n,-1} = 0, \tag{67}$$

$$X_{n,0} = 8R_{n,2} + 8T_{n,2} - 8R_{n,1} - 12T_{n,1} + R_{n,0} + 4T_{n,0} = 0. \tag{68}$$

Similarly, we have $Y_{n,-3} = Y_{n,-2} = Y_{n,-1} = Y_{n,0} = 0$. Hence, all the coefficients of negative indexes of $\sin\theta$ in $W_n(\theta) = X_n + \cos\theta Y_n$ are zero, and W_n could be convergent in the boundary region. $W_n(\theta)$ now becomes

$$W_n(\theta) = \sum_{l=1}^{[\frac{n}{2}]+1} X_{n,l}\sin^{2l-1}\theta + \cos\theta \sum_{l=1}^{[\frac{n}{2}]+1} Y_{n,l}\sin^{2l-1}\theta = \sum_{l=1}^{[\frac{n}{2}]+1} b_{n,l}\sin^{2l-1}\theta + \cos\theta \sum_{l=1}^{[\frac{n}{2}]+1} a_{n,l}\sin^{2l-1}\theta \tag{69}$$

with

$$a_{n,l} = Y_{n,l} = \sum_{p=l+1}^{[\frac{n}{2}]+1} \frac{(2m+2l)(2m+2l-2)I(2m+2p-1,p-1-l)}{(2m+2l+2)(2m+2l+4)} A_{n,p}, \tag{70}$$

$$b_{n,l} = X_{n,l} = -\sum_{p=l+1}^{[\frac{n}{2}]+1} \frac{4(2m+2l-2)I(2m+2p-1,p-1-l)}{(2m+2l+2)(2m+2l+4)} A_{n,p} + \frac{g_{n,l}}{2m+2l}. \tag{71}$$

It is easy to observe that equation (69) is slightly different in the maximum limit with the general formula Eq. (41). Because of Eq. (58): if $p < -1$ or $p > [\frac{n+1}{2}]$, $R_{n,p} = 0$, and if $p < -1$ or $p > [\frac{n}{2}]$, $T_{n,p} = 0$. Then we discuss the upper limit of l.[14,15]

When n is even and $l = [\frac{n}{2}] + 1$, we have

$$a_{n,[\frac{n}{2}]+1} = 0, \quad b_{n,[\frac{n}{2}]+1} = 0; \tag{72}$$

and when $l = [\frac{n}{2}]$,

$$a_{n,[\frac{n}{2}]} \neq 0, \quad b_{n,[\frac{n}{2}]} \neq 0. \tag{73}$$

So under the condition that n is even, $W_n(\theta)$ becomes

$$W_n(\theta) = \cos\theta \sum_{l=1}^{[\frac{n}{2}]} a_{n,l}\sin^{2l-1}\theta + \sum_{l=1}^{[\frac{n}{2}]} b_{n,l}\sin^{2l-1}\theta. \tag{74}$$

On the other hand, when n is odd and $l = [\frac{n}{2}] + 1$, then we have

$$a_{n,[\frac{n}{2}]+1} = 0, \quad b_{n,[\frac{n}{2}]+1} \neq 0; \tag{75}$$

and when $l = [\frac{n}{2}]$,

$$a_{n,[\frac{n}{2}]} \neq 0, \quad b_{n,[\frac{n}{2}]} \neq 0. \tag{76}$$

While n is odd, $W_n(\theta)$ has the new form

$$W_n(\theta) = \cos\theta \sum_{l=1}^{[\frac{n}{2}]} a_{n,l}\sin^{2l-1}\theta + \sum_{l=1}^{[\frac{n+1}{2}]} b_{n,l}\sin^{2l-1}\theta, \tag{77}$$

with the help of $[\frac{n}{2}] + 1 = [\frac{n+1}{2}]$. Through comparing Eq. (74) and Eq. (77), the super-potential $W_n(\theta)$ can be rewritten as

$$W_n(\theta) = \cos\theta \sum_{l=1}^{[\frac{n}{2}]} a_{n,l}\sin^{2l-1}\theta + \sum_{l=1}^{[\frac{n+1}{2}]} b_{n,l}\sin^{2l-1}\theta. \tag{78}$$

Therefore, we have proved the correctness of our induction about the general formula W_n. That is to say, for any $n \geq 1$, W_n satisfy the form of Eq. (41).

4. The eigenvalue and eigenfunction of the ground state

According to the form of W_n in Eq. (41), the super-potential can be written as

$$W = W_0 + \sum_{n=1}^{\infty} \left[\cos\theta \sum_{k=1}^{[\frac{n}{2}]} a_{n,k}\sin^{2k-1}\theta + \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k}\sin^{2k-1}\theta \right] \beta^n. \tag{79}$$

Because of Eq. (13) the ground eigenfunction becomes

$$\begin{aligned}\Psi_0 &= N\exp\left[-\int W_0\,\mathrm{d}\theta - \sum_{n=1}^{\infty}\int W_n\,\mathrm{d}\theta\right] \\ &= N(1-\cos\theta)^2\sin^{m-3/2}\theta \\ &\quad \times \exp\left[-\sum_{n=1}^{\infty}\beta^n\left(\sum_{k=1}^{[\frac{n}{2}]} a_{n,k}\frac{\sin^{2k}\theta}{2k}\right.\right.\end{aligned}$$

$$+\sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k}P(2k-1,\theta)\Big)\Big], \quad (80)$$

with $E_{0,n;m}$ having been obtained in Eq. (52) and the ground eigenvalue can be written as

$$\begin{aligned} E_{0;m} &= E_{0,0;m} + \sum_{n=1}^{\infty} \beta^n E_{0,n;m} \\ &= m^2 + m - 6 + \sum_{n=1}^{\infty} E_{0,n;m}\beta^n. \end{aligned} \quad (81)$$

5. The eigenvalue and eigenfunction of the excited state

In this section, we will prove the spheroidal equation has the property of shape-invariance, hence the bound state wave functions Ψ_n for any shape-invariant potential can be easily obtained from the ground state wave function Ψ_0, which in turn is known in terms of the superpotential by the recurrence relations derived from shape-invariance. The potentials $V^{\mp}$ are determined by super-potential W as[18]

$$V^{\mp}(\theta) = W^2(\theta) \mp W'(\theta). \quad (82)$$

If the pair of SUSY partner potentials $V^{\pm}(\theta)$ are similar in shape and differ only in the parameters that they are called have the property of the shape-invariance.

According to the value of W obtained before,[10−17]

$$\begin{aligned} W(A_{n,j}, B_{n,j}, \theta) &= -A_{0,0}(m+\frac{1}{2})\cot\theta - 2B_{0,0}\csc\theta \\ &\quad + \sum_{n=1}^{\infty} \beta^n W_n(A_{n,j}, B_{n,j}, \theta), \end{aligned} \quad (83)$$

where

$$\begin{aligned} W_n(A_{n,j}, B_{n,j}, \theta) &= \cos\theta \sum_{j=1}^{[\frac{n}{2}]} \bar{a}_{n,j} \sin^{2j-1}\theta \\ &= \sum_{j=1}^{[\frac{n+1}{2}]} \bar{b}_{n,j} \sin^{2j-1}\theta, \end{aligned} \quad (84)$$

with $\bar{a}_{n,j} = A_{n,j}a_{n,j}$, $\bar{b}_{n,j} = B_{n,j}b_{n,j}$. Then, $V^{\pm}(A_{n,j}, B_{n,j}, \theta)$ can be expanded in β as

$$\begin{aligned} V^{\pm}(A_{n,j}, B_{n,j}, \theta) &= W^2(A_{n,j}, B_{n,j}, \theta) \pm W' \\ &= \sum_{n=0}^{\infty} \beta^n V_n^{\pm}(A_{i,j}, B_{n,j}, \theta), \end{aligned} \quad (85)$$

where the parameters $A_{i,j}$ and $B_{i,j}$ require $i < n$ and $j < n$. Note that, we need to find some quantities $C_{i,j}, D_{i,j}$ meeting the relations

$$\begin{aligned} &V_n^{+}(A_{i,j}, B_{n,j}, \theta) \\ &= V_n^{-}(C_{i,j}, D_{n,j}, \theta) + R_{n;m}(A_{i,j}, B_{n,j}). \end{aligned} \quad (86)$$

It holds with $R_{n;m}(A_{i,j}, B_{n,j}) = R_{n;m}$ which are pure quantities. After some complex calculations we could prove that the property of the shape-invariance is true for the special cases $n = 0,\ 1,\ 2$, that is,

$$V_0^{+}(A_{0,0}, B_{0,0}, \theta) = V_0^{-}(C_{0,0}, D_{0,0}, \theta) + R_{0;m}, \quad (87)$$

$$V_1^{+}(A_{1,1}, B_{1,1}, \theta) = V_1^{-}(C_{1,1}, D_{1,1}, \theta) + R_{1;m}, \quad (88)$$

$$V_2^{+}(B_{2,1}, A_{2,1}, \theta) = V_2^{-}(C_{2,1}, D_{2,1}, \theta) + R_{2;m}. \quad (89)$$

In order to make the above equations hold, one must have

$$C_{0,0} = A_{0,0} + \frac{2}{2m+1}, \quad (90)$$

$$D_{0,0} = 4B_{0,0}, \quad (91)$$

$$D_{1,1} = \frac{(2m+1)A_{0,0}-1}{(2m+1)A_{0,0}+3}B_{1,1}, \quad (92)$$

$$\begin{aligned} D_{2,1} &= \frac{(2m+1)A_{0,0}-1}{(2m+1)A_{0,0}+3}B_{2,1} \\ &\quad + \frac{24B_{0,0}B_{2,1}}{[(2m+1)A_{0,0}+3][(2m+1)A_{0,0}+4]} \\ &\quad - \frac{32[(2m+1)A_{0,0}+1]B_{0,0}B_{1,1}^2}{[(2m+1)A_{0,0}+3]^3[(2m+1)A_{0,0}+4]}, \end{aligned} \quad (93)$$

$$\begin{aligned} C_{2,1} &= \frac{8[(2m+1)A_{0,0}+1]B_{1,1}^2}{[(2m+1)A_{0,0}+3]^3[(2m+1)A_{0,0}+4]} \\ &\quad + \frac{(2m+1)A_{0,0}-2}{(2m+1)A_{0,0}+4}A_{2,1}, \end{aligned} \quad (94)$$

with

$$\begin{aligned} &R_{0;m}(A_{0,0}) \\ &= (2m+1)A_{0,0}+1, \end{aligned} \quad (95)$$

$$\begin{aligned} &R_{1;m}(A_{0,0}, B_{0,0}, B_{1,1}) \\ &= -\frac{16B_{0,0}B_{1,1}}{(2m+1)A_{0,0}+3}, \end{aligned} \quad (96)$$

$$\begin{aligned} &R_{2;m}(A_{0,0}, B_{0,0}, B_{1,1}, B_{2,1}, A_{2,1}) \\ &= -\frac{16B_{0,0}B_{2,1}}{(2m+1)A_{0,0}+3} + AB_{1,1}^2 + BA_{2,1}, \end{aligned} \quad (97)$$

where

$$A = \frac{128B_{0,0}^2 - 8[(2m+1)A_{0,0}-1][(2m+1)A_{0,0}+3]}{[(2m+1)A_{0,0}+3]^3[(2m+1)A_{0,0}+4]},$$

$$B=\frac{96B_{0,0}^2-2[(2m+1)A_{0,0}-1][(2m+1)A_{0,0}+3]}{[(2m+1)A_{0,0}+3][(2m+1)A_{0,0}+4]}. \tag{98}$$

Using the induction method we could prove Eq. (85) for the general n that the shape-invariance holds for it too. Therefore, the shape-invariance property for the spin-weighted equation with $s=2$ holds too as in the case of $s=\frac{1}{2}$[14] and $s=1$.[15] Now the results are given in the following without further details

$$D_{n,p} = \frac{D_{0,0}a_{n,p}}{\alpha_p b_{n,p}}C_{n,p}-\frac{U_{n,p}}{\alpha_p b_{n,p}}, \tag{99}$$

$$C_{n,p-1} = \frac{\left(\alpha_p+\frac{D_{0,0}^2}{\alpha_p}\right)a_{n,p}}{(\alpha_p-1)a_{n,p-1}}C_{n,p}+\frac{\check{U}_{n,p}-\frac{D_{0,0}}{\alpha_p}U_{n,p}}{(\alpha_p-1)a_{n,p-1}}, \tag{100}$$

where

$$G_{n,p}(A_{i,j},B_{i,j})=\sum_{k=1}^{n-1}\sum_{j=1}^{p-1}[\bar{b}_{k,p-j}\bar{a}_{n-k,j}+\bar{a}_{k,p-j}\bar{b}_{n-k,j}], \tag{101}$$

$$H_{n,p}(A_{i,j},B_{i,j})=\sum_{k=1}^{n-1}\sum_{j=1}^{p-1}[\bar{b}_{k,p-j}\bar{b}_{n-k,j}+\bar{a}_{k,p-j}\bar{a}_{n-k,j}-\bar{a}_{k,p-1-j}\bar{a}_{n-k,j}], \tag{102}$$

$$\bar{a}_{i,j}=A_{i,j}a_{i,j},\ \bar{b}_{i,j}=B_{i,j}b_{i,j}, \tag{103}$$

$$U_{n,p}=G_{n,p}(A_{i,j},B_{i,j})-G_{n,p}(C_{i,j},D_{i,j})+P_{n,p}^{+}(A_{n,j},B_{n,j}), \tag{104}$$

$$\check{U}_{n,p}=H_{n,p}(A_{i,j},B_{i,j})-H_{n,p}(C_{i,j},D_{i,j})+Q_{n,p}^{+}(A_{n,j},B_{n,j}), \tag{105}$$

$$\alpha_p=[(2m+1)C_{0,0}+(2p-1)] \tag{106}$$

for $p=2,3,\ldots$. Considering the n's parity, we note that

(i) $D_{n,[\frac{n+1}{2}]+1}=C_{n,[\frac{n}{2}]+1}=0$;

(ii) When n is odd, $C_{n,[\frac{n+1}{2}]}=0$ and one can obtain $D_{n,[\frac{n+1}{2}]}$ from Eq. (99) and $C_{n,[\frac{n}{2}]}$ from Eq. (100) under the condition $p=[\frac{n+1}{2}]$;

(iii) When n is even, one needs to calculate $C_{n,\frac{n}{2}}$ from Eq. (100) under the condition $p=[\frac{n+1}{2}]$ and $C_{n,[\frac{n+1}{2}]}=0$. Then, it is easy to calculate the others subsequently.

Solving the above equations recursively we obtain the excited eigenvalues $E_{l;m}^{-}$ as follows:[18]

$$\begin{aligned}E_{l;m}^{-} &= E_{0;m}+\sum_{k=1}^{l}R(a_k,b_k)\\ &= m(m+1)-2+\sum_{n=1}^{\infty}E_{0,n;m}\beta^n\\ &\quad+\sum_{k=1}^{l}R(a_k,b_k),\end{aligned} \tag{107}$$

with

$$R(a_k,b_k)=R_{0;m}+\sum_{n=1}^{\infty}\beta^n R_{n;m}(a_k,b_k), \tag{108}$$

$$a_1=(A_{i,j},B_{i,j}),\ a_2=(C_{i,j},D_{i,j}),\ldots, \tag{109}$$

and it is often convenient to have explicit expressions for the wave functions, the corresponding eigenfunction Ψ_n^{-} is obtained by the following recurrence relation,[18]

$$\Psi_0\propto\exp\left[-\int^{\theta}W(A_{n,j},B_{n,j},\theta)\mathrm{d}\theta\right], \tag{110}$$

$$\mathcal{A}^{\dagger}=-\frac{\mathrm{d}}{\mathrm{d}\theta}+W(A_{n,j},B_{n,j},\theta); \tag{111}$$

also, the shape-invariance condition yields

$$\begin{aligned}&\Psi_n^{-}=\mathcal{A}^{\dagger}(A_{n,j},B_{n,j},\theta)\Psi_{n-1}^{-}(C_{n,j},D_{n,j},\theta),\\ &n=1,2,3,\ldots,\end{aligned} \tag{112}$$

it is simpler to use to identify, which is clearly a generalization of the operator method of constructing the energy eigenfunction.

6. Conclusion

In this work we have performed a comprehensive analysis of the spin-weighted spheroidal wave function in the case of $s=2$. Not only can one easily obtain the ground eigenvalue and eigenfunctions by Eqs. (81) and (80), but it is also convenient to have explicit expressions for the n-th state eigenvalue and eigenfunctions by the recurrence relations as Eqs. (108)

and (113). These results are derived through theoretical calculations. An immediate future work is to revisit the spin-weighted spheroidal wave function by numerical analysis, which can obtain more characteristics of the equation. Another direction along which we are currently investigating is to research the spin-weighted spheroidal wave function with the high-frequency method.

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