Rotational symmetry of classical orbits, arbitrary quantization of angular momentum and the role of the gauge field in two-dimensional space*

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We study quantum–classical correspondence in terms of the coherent wave functions of a charged particle in two-dimensional central-scalar potentials as well as the gauge field of a magnetic flux in the sense that the probability clouds of wave functions are well localized on classical orbits. For both closed and open classical orbits, the non-integer angular-momentum quantization with the level space of angular momentum being greater or less than $\hbar$ is determined uniquely by the same rotational symmetry of classical orbits and probability clouds of coherent wave functions, which is not necessarily 2π-periodic. The gauge potential of a magnetic flux impenetrable to the particle cannot change the quantization rule but is able to shift the spectrum of canonical angular momentum by a flux-dependent value, which results in a common topological phase for all wave functions in the given model. The well-known quantum mechanical anyon model becomes a special case of the arbitrary quantization, where the classical orbits are 2π-periodic.

Keywords: quantum–classical correspondence, anyon, rotational symmetry, arbitrary quantization of angular momentum

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1. Introduction

Recently, renewal of interest has been evoked in the fractional angular momentum (FAM) in two-dimensional (2D) space in relation with the correspondence between quantum mechanical wave-functions and classical periodic orbits.[1,2] The FAM is only possible in 2D multiply connected space, since in three- or higher dimensional space, the angular momentum, being integer or half-integer, is completely determined by the commutation relation of angular-momentum operators. In 2D space, the angular momentum operator has only one component, which does not give rise to a unique determination of the angular-momentum eigenvalue, and the common belief of integer quantization is based on the 2π periodic boundary condition of wave functions which, however, is not justified.

In his pioneering work, Wilczek proposed a quantum mechanical model consisting of a charged particle and magnetic flux in 2D space to demonstrate the fractional eigenvalues of angular momentum known as an anyon.[3−12] In the existence of a gauge field, however, we have both kinetic angular momentum (KAM) and canonical angular momentum (CAM), which are different because of the velocity-dependent forces.[13−16] It is true that "the generator of rotation should be the CAM and is prescribed by Noether's theorem as a conserved quantity",[17] which is gauge-dependent and integer-quantized.[15−17] In particular, the KAM is a gauge-invariant dynamic quantity and generally fractional because of the gauge field.[15−17] Although the consistency of fractional CAM with Aharonov–Bohm (AB) phase-interference was shown long ago,[15] it remains a long- standing open question whether or not the fractional CAM plays a role in quantum physics. It is worthwhile to remark that the gauge potential of an AB flux can only shift the angular momentum eigenvalues by a common fractional number and cannot change the integer quantization rule, namely the eigenvalue-space of angular momentum is still $\hbar$.[15,18]

The existence of FAM in a wide class of 2D central potentials without a gauge field has been discussed more recently[1,2] by the localization of coherent wave functions on classical orbits, which imposes a special boundary condition leading to the unusual angular

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phase of wave functions. Following the interesting studies of Refs. [1] and [2], which provide an exactly solvable model both quantum mechanically and classically, we consider a charged particle in the gauge field of a magnetic flux string perpendicular to the 2D plane with the central scalar-potentials and study the non-integer quantization of angular momentum (NIQAM) in terms of quantum–classical correspondence. Since the coherent state is constructed by the superposition of angular momentum eigenstates and thus the probability clouds have to possess the same rotational symmetry as that of classical orbits, which is not necessarily 2π periodic. As a consequence NIQAM with level space being greater or less than $\hbar$ is uniquely determined by the correspondence principle. Moreover the gauge field of an AB flux cannot change the quantization rule but is able to shift the angular momentum spectrum by a flux-dependent value. By the explicit calculation it is shown that only the shift of CAM, which gives rise to a common topological phase for all wave functions, is in agreement with the correspondence principle. The validity of wave functions with a topological phase is further confirmed by the expectation values of the KAM operator, which coincide with the classical values of KAM $\pounds^k$. The long-standing problem whether or not the FAM is related to the CAM is resolved.

More specifically, we refer here to the particular zero-energy states, which can be obtained analytically for both classical and quantum solutions.[19,20] In addition, the zero energy states are of importance in fields such as cold-atom collisions,[21,22] the construction of some vortex lattices,[23] and quantum cosmology.[24]

This paper is organized as follows. In Section 2, classical orbits in the presence of the gauge field of magnetic flux are derived and rotational symmetry is explained. In Section 3, NIQAM is obtained by consistence with the rotational symmetry of classical orbits and we show the difference between the CAM and KAM associated with the flux. In Section 4, the shift of CAM by the gauge field is demonstrated to be in exact agreement with the quantum–classical correspondence principle[25] but not the KAM. The shift of CAM is confirmed by the expectation values of angular momentum operators. Finally, a conclusion and discussion are given in Section 5.

2. Classical orbits and rotational symmetry

We consider a charged particle with charge e and mass m in a gauge field of an infinitely long magnetic flux line of total flux Φ located at the origin of 2D space, and the central scalar-potential is of the form[1,2]

$$A_0(r) = -\frac{\gamma_\upsilon}{r^{2\mu+2}}, \quad \gamma_\upsilon > 0, \quad -\infty < \upsilon < \infty, \tag{1}$$

where $r = \sqrt{x^2+y^2}$ and $\upsilon = 2\mu + 2$. Outside the flux-line ($r > 0$), the Lorentz force on the charged particle is always zero because of the vanishing magnetic field $\boldsymbol{B} = 0$, while the vector potential in the polar coordinate is seen to be

$$\boldsymbol{A} = \frac{\Phi}{2\pi r}\boldsymbol{e}_\varphi$$

with $\boldsymbol{e}_\varphi$ being the unit-vector of angular direction. In order to establish the quantum–classical correspondence, we ought to evoke the canonical variables. In the polar coordinates (r, φ), the Lagrangian of the system is

$$L = \frac{1}{2}m[\dot{r}^2 + (r\dot{\varphi})^2] - eA_0(r) + L_{\rm WZ}, \tag{2}$$

where $L_{\rm WZ} = \alpha\hbar\dot{\varphi}$ is the so-called Wess–Zumino topological interaction term with the parameter $\alpha = \Phi/\Phi_0$ being the dimensionless magnetic flux in the quantum unit $\Phi_0 = ch/e$. The Wess–Zumino term does not affect the equation of motion but the initial value of angular momentum. Canonical momenta corresponding to the coordinate variables r, φ are defined by

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \tag{3}$$

$$\pounds^c = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} + \alpha\hbar. \tag{4}$$

Here $\pounds^c$ is CAM, while $\pounds^k = mr^2\dot{\varphi}$ is the KAM. Then the Hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{(\pounds^c - \alpha\hbar)^2}{2mr^2} + eA_0(r). \tag{5}$$

From canonical equations we find that both the CAM and KAM are conserved quantities

$$\frac{{\rm d}\pounds^c}{{\rm d}t} = -\frac{\partial H}{\partial \varphi} = 0, \quad \pounds^k = \pounds^c - \alpha\hbar.$$

For the case of zero-energy and nonvanishing initial KAM $\pounds^k = \pounds^c - \alpha\hbar \neq 0$ we have

$$\frac{1}{2}m\left(\frac{\pounds^c - \alpha\hbar}{mr^2}\right)^2\left[\left(\frac{{\rm d}r}{{\rm d}\varphi}\right)^2 + r^2\right] - e\frac{\gamma_\upsilon}{r^\upsilon} = 0.$$

We assume an initial value that

$$\pounds^k = \xi_k\hbar, \tag{6}$$

where ξ_k is an arbitrary dimensionless quantity. Thus in the considered case only the CAM is shifted by the flux. Introducing a dimensionless variable

$$u = r/\tilde{a}_c, \quad \tilde{a}_c = \frac{(2me\gamma_\upsilon)^{1/2\mu}}{[\xi_k\hbar]^{1/\mu}}, \tag{7}$$

we obtain the equation of particle trajectories such as[18,26]

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2 = u^{4-\upsilon} = u^{2-2\mu}. \tag{8}$$

The general solution of Eq. (8) is given as[18,26]

$$r^\mu = \tilde{a}_c^\mu \cos[\,\mu(\varphi - \varphi_0)]. \tag{9}$$

Using atomic unit $me_s^2/\hbar^2$ with $e_s = e^2/(4\pi\varepsilon_0)$, which is the dimension of length, we set $\sqrt{2m\gamma_\upsilon e/\hbar^2} = 1$ in the numerical evaluation. The classical orbits with the initial angle set to zero $\varphi_0 = 0$ are shown in Figs. 1–5 (solid white curves), which depend on the initial angular momentum ξ_k only for given scalar potential. In general, we have closed orbits (Figs. 1–4) for $\mu > 0$ ($\upsilon > 2$), while open trajectories for $\mu < -2$ are shown in Fig. 5. From Eq. (9) it is obvious that the rotation symmetry of classical orbits depends on the power index μ such that the orbits are invariant under a rotation angle $2\pi/|\mu|$, and 2π-symmetry holds only for $|\mu| = 1$. We demonstrate in this paper that the angular momentum quantization can be determined only by the rotation symmetry of classical orbits based on the requirement of quantum–classical correspondence. The rotation angles of symmetry for closed orbits in Figs. 1–4 are 2π, 6π, $4\pi/5$, and $2\pi/5$ respectively.

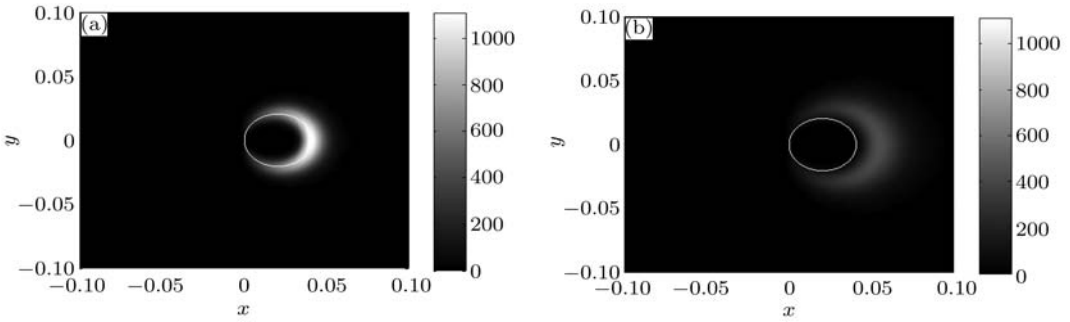


Fig. 1. Probability density image $\left|\Psi_{\mu,N}(r,\varphi)\right|^2$ and the closed classical orbit (solid white curve) for $\mu = 1$ ($\upsilon = 4$), $\alpha = 6\frac{1}{2}$, $\xi_k = 23$, (a) shift of CAM, (b) shift of KAM.

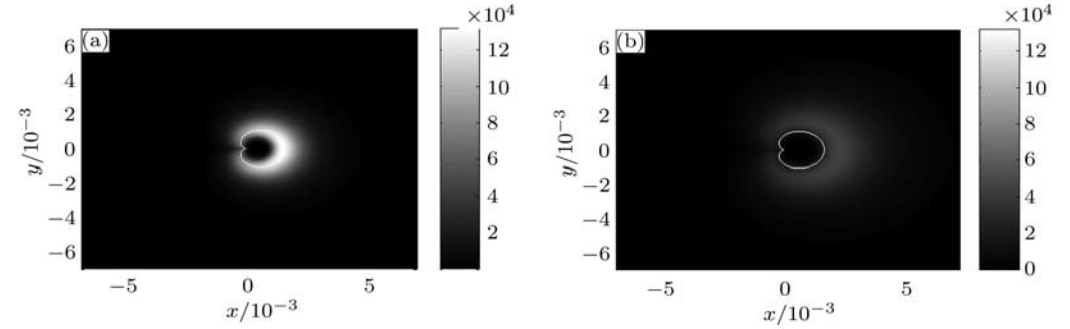


Fig. 2. Probability density image $\left|\Psi_{\mu,N}(r,\varphi)\right|^2$ and the closed classical orbit (solid white curve) for $\mu = 1/3$ ($\upsilon = 8/3$), $\alpha = 5/4$, $\xi_k = 8$, (a) shift of CAM, (b) shift of KAM.

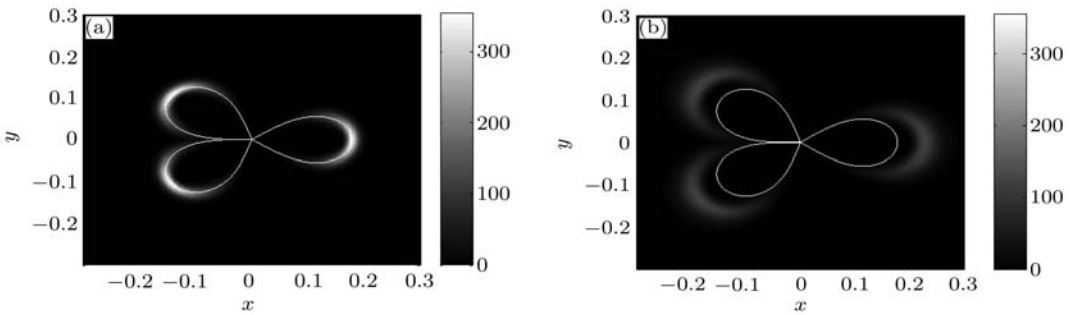


Fig. 3. Probability density image $\left|\Psi_{\mu,N}(r,\varphi)\right|^2$ and the closed classical orbit (solid white curve) for $\mu = 5/2$ ($\upsilon = 7$), $\alpha = 31\frac{1}{3}$, $\xi_k = 72.5$, (a) shift of CAM, (b) shift of KAM.

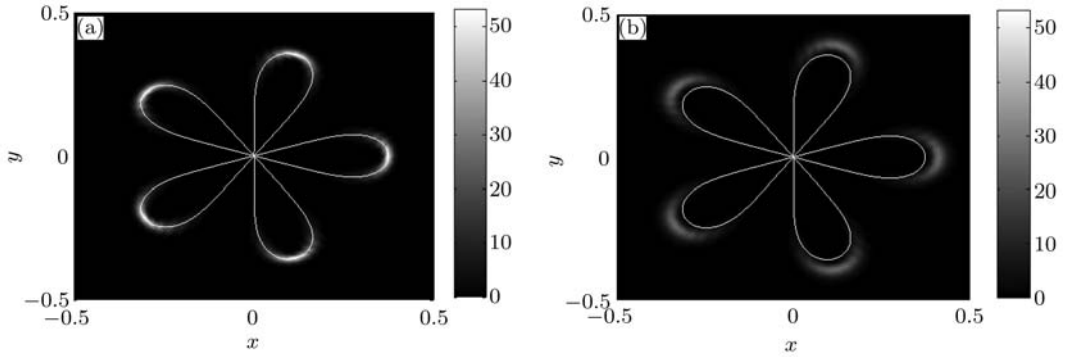


Fig. 4. Probability density image $\left|\Psi_{\mu,N}(r,\varphi)\right|^2$ and the closed classical orbit (solid white curve) for $\mu = 5$ ($\upsilon = 12$), $\alpha = 55\frac{3}{5}$, $\xi_k = 135$, (a) shift of CAM, (b) shift of KAM.

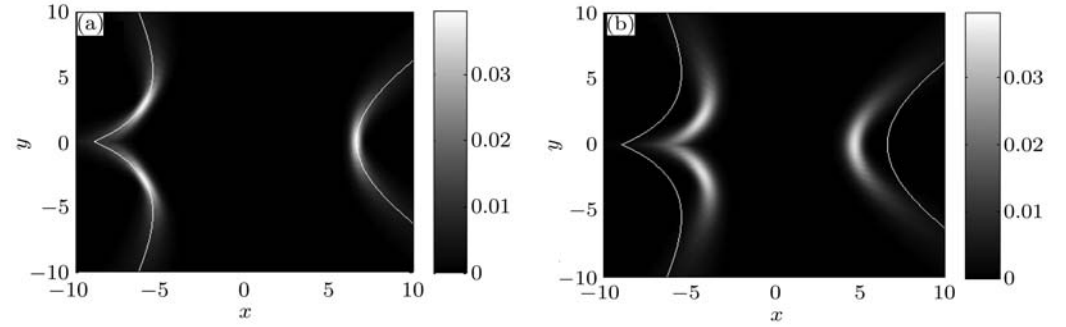


Fig. 5. Probability density image $\left|\Psi_{\mu,N}(r,\varphi)\right|^2$ and the open classical orbit (solid white curve) for $\mu = -7/3$ ($\upsilon = -8/3$), $\alpha = 42\frac{7}{10}$, $\xi_k = 238/3$, (a) shift of CAM, (b) shift of KAM.

3. Angular momentum quantization

In the polar coordinates (r,φ), the zero-energy Schrödinger equation

$$\frac{-\hbar^2}{2m}\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left(\frac{\partial}{\partial\varphi}-\mathrm{i}\alpha\right)^2\right]\psi+eA_0\psi=0 \tag{10}$$

becomes

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{\lambda^2}{r^2}\right)R(r)+\frac{2me}{\hbar^2}\frac{\gamma_\upsilon}{r^\upsilon}R(r)=0,$$
$$\left(\frac{\partial}{\partial\varphi}-\mathrm{i}\alpha\right)^2\Theta(\varphi)=-\lambda^2\Theta(\varphi)$$

by the separation of variables $\psi = R(r)\Theta(\varphi)$. The eigenvalue solution of angular parts of Eq. (10) is

$$\Theta_{l^c}(\varphi)=N_\varphi\,\mathrm{e}^{\mathrm{i}l^c\varphi} \tag{11}$$

with N_φ being the normalization constant and the CAM eigenvalue l^c to be determined. For the usual requirement of 2π-periodic boundary condition, $\Theta(\varphi) = \Theta(\varphi+2\pi)$, one can obtain the integer angular momentum quantization and the normalization constant $N_\varphi = 1/\sqrt{2\pi}$. However, the 2π-periodic boundary condition is not justified in the 2D space, although the potential $A_0(r) = -\gamma_\upsilon/r^{2\mu+2}$ is symmetric under rotation. It has been demonstrated that a macroscopic quantum state, here $SU(2)$ coherent state,[27] can be constructed with a probability density of wave functions that is well localized on the classical orbits for quantum–classical correspondence, which results in a special boundary condition of the angular momentum eigenstates such that the rotational period of the wave function is not necessarily 2π but should be the same as that of classical orbits $\Theta(\varphi) = \Theta(\varphi+2\pi/|\mu|)$.[1] Thus the CAM eigenvalue is no longer an integer but should be set as

$$l_n^c = n\,|\mu|\,, \tag{12}$$

where n is an integer. The angular momentum now is quantized with an eigenvalue space

$$\Delta l = |\mu|. \tag{13}$$

The integer quantization is only a special case of $|\mu| = 1$. The normalization constant becomes

$$N_\varphi = \sqrt{\frac{|\mu|}{2\pi}}. \tag{14}$$

The radial equation is

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{(l_n^c-\alpha)^2}{r^2}\right)R(r)+\frac{2me}{\hbar^2}\frac{\gamma_\upsilon}{r^\upsilon}R(r)=0. \tag{15}$$

In the choice of Eq. (12) for l^c, the n-th KAM eigenvalue is shifted by the flux number

$$l_n^k=l_n^c-\alpha=n\,|\mu|-\alpha, \tag{16}$$

indicating the dynamic effect of the gauge potential in contradiction with the classical solution in which the gauge field does not apply a torque on the particle. We do have the other choice of l_n^c such that

$$l_n^c=n\,|\mu|+\alpha, \tag{17}$$

while the KAM eigenvalue

$$l_n^k=n\,|\mu| \tag{18}$$

does not depend on the flux in consistence with classical solution. The common phase $\mathrm{e}^{\mathrm{i}\alpha\varphi}$ factor called the topological phase[15,19] does not change the angular momentum quantization Eq. (13) nor the normalization constant Eq. (14) since the whole eigenfunctions have the same additional angular phase. It is the main goal of the present paper to show that only the choice of CAM Eq. (17) gives rise to the exact quantum–classical correspondence.

4. Topological phase of the gauge field and exact quantum–classical correspondence

We demonstrate in this section that only the CAM eigenvalues Eq. (17) are consistent with the quantum–classical correspondence, such that the probability densities of coherent wave functions are well localized on classical orbits. Introducing the dimensionless radius $\chi=r/\tilde{a}_q$ and $y=1/\chi$, where $\tilde{a}_q$ is a quantity with dimension of length, we obtain

$$\left(y^2\frac{\mathrm{d}^2}{\mathrm{d}y^2}+y\frac{\mathrm{d}}{\mathrm{d}y}-(l_n^k)^2+B^2y^{\upsilon-2}\right)R_\tau(y)=0, \tag{19}$$

where the parameter B is defined as

$$B^2\equiv\frac{2me\gamma_\upsilon}{\hbar^2\tilde{a}_q^{\upsilon-2}},$$

and the corresponding KAM eigenvalues l_n^k given by Eq. (18) do not depend on the flux number. Square-integrable solutions of Eq. (19) are found in terms of Bessel functions of the first kind[25,28–30]

$$R_{l_n^k}(y)=N_{l_n^k}\mathrm{J}_{l_n^k/|\mu|}\left(\frac{1}{|\mu|\,r^\mu}\right) \tag{20}$$

with the normalization constant given by

$$N_{l_n^k}=\sqrt{2\sqrt{\pi}\,|\mu|^{\frac{2}{\mu}+1}\,\frac{\Gamma(1+1/\mu)\Gamma[1+l_n^k/\,|\mu|+1/\mu]}{\Gamma(1/2+1/\mu)\Gamma[l_n^k/\,|\mu|-1/\mu]}}, \tag{21}$$

where we have set the parameter $\sqrt{2m\gamma_\upsilon e/\hbar^2}=1$, and thus $B\tilde{a}_q^\mu=1$, $\tilde{a}_q^2B^{2/\mu}=1$ according to the definition of B. If the following conditions are satisfied

$$\mathrm{Re}\left(\frac{2l_n^k}{|\mu|}+1\right)>\mathrm{Re}\left(\frac{2}{\mu}+1\right)>0, \tag{22}$$

we have bound states corresponding to the classical closed-orbits as demonstrated in Ref. [26]. For the zero-energy $E=0$ there exist classical solutions of closed orbits with any non-zero angular momentum when $\upsilon>2$, while the condition of bound states in quantum mechanics is $l_n^k>1$. It has been shown[29] that the normalizable quantum solutions can be classified into two classes: (i) bound states ($l_n^k>1$) for $\mu>0$ ($\upsilon>2$) corresponding to the classical closed-orbits (Figs. 1–4, solid white curves); and (ii) scattering states ($l_n^k\geq 0$) for $\mu<-2$ ($\upsilon<-2$), which are normalizable, corresponding to classical open-orbits (Fig. 5, solid white curves).

In the region $-2\leq\upsilon\leq 2$ the solutions of wave functions are not square-integrable,[28,29] which we do not discuss in the present paper. The complete eigenfunctions of the Schrödinger equation can be written in an explicit form as[25,28–30]

$$\psi_{\mu,l_n^k}(r,\varphi)=N_\varphi\,\mathrm{e}^{\mathrm{i}(l_n^k+\alpha)\varphi}N_{l_n^k}\mathrm{J}_{l_n^k/|\mu|}\left(\frac{1}{|\mu|\,r^\mu}\right). \tag{23}$$

Following Refs. [1] and [2] we construct the stationary $SU(2)$ coherent state for the central-scalar potentials in the standard way as[27,31–33]

$$\begin{aligned}\Psi_{\mu,N}(r,\varphi)\;=\;&\frac{1}{(2)^{N/2}}\sum_{n=0}^{N}\binom{N}{n}^{1/2}N_\varphi\,\mathrm{e}^{\mathrm{i}(l_n^k+c+\alpha)\varphi}\\&\times N_{l_n^k+c}\,\mathrm{J}_{(l_n^k+c)/|\mu|}\left(\frac{1}{|\mu|\,r^\mu,}\right),\end{aligned} \tag{24}$$

with l_n^c and l_n^k given in Eqs. (17) and (18), respectively. An additional angular momentum number

$$c=n_c\,|\mu|\,, \tag{25}$$

with n_c being an integer parameter, is introduced to adjust the probability density, which does not change the rotational symmetry of wave-function density but the spatial expansion of it. In the following section

we will see that the parameter n_c is not arbitrary but related to the classical KAM $\pounds^k$ such that the expectation value of the KAM operator coincides with the classical one. The wave functions are normalized to unity in the angular range of φ from $-\pi$ to π. From Eq. (24) we can see that the common topological phase factor resulting from the gauge potential does not affect the probability density of wave functions, which is in agreement with the classical assumption of no torque applied on the charged particle. In Figs. 1–5 the practical values of parameters are given by $\mu = 1$, $\alpha = 6\frac{1}{2}$, $n_c = 8$; $\mu = 1/3$, $\alpha = 5/4$, $n_c = 9$; $\mu = 5/2$, $\alpha = 31\frac{1}{3}$, $n_c = 14$; $\mu = 5, \alpha = 55\frac{3}{5}$, $n_c = 12$, and $\mu = -7/3$, $\alpha = 42\frac{7}{10}$, $n_c = 19$, respectively. In all of the figures the total number of eigenfunctions is $N = 30$. The density $|\Psi_{\mu,N}(r,\varphi)|^2$ of the coherent-state wave function possesses the same rotation symmetry of the classical orbits. Figures 1(a)–5(a) show that the probability densities of coherent wave functions are well localized on the classical orbits in each case, indicating the exact quantum–classical correspondence. We thus obtain the angular-momentum quantization with the level spaces 1, 1/3, 5/2, 5, and 7/3 (of $\hbar$) respectively. The CAM spectrum is shifted by the corresponding flux number α. Besides the AB interference,[14] the fractional spins have many important physical applications, for example, the two-vortex system in a superfluid film,[34] necklace-ring,[35,36] chiral-wave superconductor[37] or quantum billiard[38] inside a boundary defined by the wedge-shaped section of a circle.

Shift of the KAM eigenvalues by the gauge potential and break down of the correspondence

We, of course, can use the other choice of CAM eigenvalues i.e. Eq. (12), which does not depend on the flux and as a consequence the spectrum of KAM is shifted by the gauge potential as shown in Eq. (16). In this case we do not have the topological phase of the wave functions but the flux-dependent KAM eigenvalues instead. The quantization of angular momentum, namely the level space of eigenvalues, is not changed by the gauge potential. Replacing the KAM eigenvalues l_n^k in Eqs. (23) and (24) by Eq. (16) the corresponding probability densities of coherent states are depicted in Figs. 1(b)–5(b), from which we see that probability densities are no longer localized on the classical orbits, indicating that additional torques are applied on the particle, which is in contradiction with the classical assumption.

Expectation values of angular momentum

We now calculate the expectation values of the KAM operator

$$\hat{\pounds}^k = -\mathrm{i}\hbar\left[\frac{\partial}{\partial\varphi} - \mathrm{i}\alpha\right]$$

in the $SU(2)$ coherent-state to find the explicit relation between the adjusting parameter c and the initial classical KAM $\pounds^k = \xi_k\hbar$ in order to confirm further the quantum–classical correspondence. The average of KAM operator in the $SU(2)$ coherent state, Eq. (24), with the topological phase is evaluated as

$$\left\langle \hat{\pounds}^k \right\rangle = \left\langle \Psi_{\mu,N} \left| \hat{\pounds}^k \right| \Psi_{\mu,N} \right\rangle = \hbar\,|\mu| \left[n_c + \frac{N}{2} \right]. \quad (26)$$

Substituting the corresponding parameter values of μ, n_c, and N into the above formula, one can find the exact agreement with the initial values of classical KAM such that

$$\left\langle \hat{\pounds}^k \right\rangle = \xi_k \quad (27)$$

for $\xi_k = 23, 8, 72.5, 135, 238/3$ respectively. However, for the CAM eigenvalue choice of Eq. (12), namely the shift of KAM eigenvalue by the gauge potential, the average of the KAM operator becomes

$$\left\langle \hat{\pounds}^k \right\rangle = \hbar\left[(n_c + \frac{N}{2})\,|\mu| - \alpha \right], \quad (28)$$

which, of course, disagrees with classical values.

5. Conclusion

In summary, the NIQAM in two-dimensional space is demonstrated in terms of quantum–classical correspondence with an exactly solvable model, such that the probability clouds of macroscopic quantum states, here the $SU(2)$ coherent wave functions, are well localized on the classical orbits. The eigenfunctions of angular momentum have to possess the same rotational symmetry of classical orbits, in which the rotation period can be greater or less than 2π depending on the power index μ of the central potential only. As a consequence the level space of the angular momentum spectrum is less or greater than $\hbar$ and the integer quantization (level space $\hbar$) is possible only if $\mu = 1$. The gauge potential of AB flux does not affect the angular momentum quantization but can shift the spectrum of angular momentum by a common value. By explicit calculations it is shown that the quantum–classical correspondence results in the unambiguous determination of CAM eigenvalues with a common topological phase in the wave functions, the

probability densities of which coincide with the classical orbits for any power index of potentials. The quantum mechanical model of an anyon,[3,4] later clarified as the fractional CAM in Ref. [14], becomes a special case of the present model with $\mu = 1$.

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