New WKB method in supersymmetry quantum mechanics*

Tian Gui-Hua(田贵花)†

School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, *China*

(Received 20 September 2011; revised manuscript received 16 November 2011)

In this paper, we combine the perturbation method in supersymmetric quantum mechanics with the WKB method to restudy an angular equation coming from the wave equations for a Schwarzschild black hole with a straight string passing through it. This angular equation serves as a naive model for our investigation of the combination of supersymmetric quantum mechanics and the WKB method, and will provide valuable insight for our further study of the WKB approximation in real problems, like the one in spheroidal equations, etc.

Keywords: new WKB method, super-potential, shape-invariance, eigenvalues and eigenfunctions

PACS: 03.65.Ge, 02.30.Gp, 11.30.Pb **DOI:** 10.1088/1674-1056/21/4/040301

The semiclassical WKB method is one of the most useful approximations for computing the eigenvalues of the differential equations.[1] It belongs to singular perturbations and has a wider range of applicability than regular perturbation theory, which is restricted to potentials with small coupling constants. Usually, the differential equations will be written in the Schrödinger form

$$-\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi(x) = E\Psi(x). \tag{1}$$

The usual WKB method will give the eigenvalues E_n by the formulae

$$\int_a^b \sqrt{E_n - V(x)}\,\mathrm{d}x = \left(n + \frac{1}{2}\right)\hbar\pi,$$
$$V(a) = V(b) = E_n. \tag{2}$$

Inspired by supersymmetric quantum mechanics, SWKB methods combine both the WKB and the supersymmetric WKB methods to study the differential equation and could give more precise results than those found by using the usual WKB method for the integrable equations.[2–4] The SWKB methods in supersymmetric quantum mechanics (SUSYQM) differ from the usual WKB in that

$$\int_a^b \sqrt{E_n - W^2(x)}\,\mathrm{d}x = n\hbar\pi,$$
$$W^2(a) = W^2(b) = E_n, \tag{3}$$

where the super-potential $W(x)$ is related to the potential $V(x)$ and ground eigenvalue E_0 by[4]

$$W^2(x) - W'(x) = V(x) - E_0. \tag{4}$$

However, the WKB method has some shortcoming concerning the singular Sturm–Liouville problem. For example, the associated Lengdre equations

$$\left[\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right) - \frac{m^2}{\sin^2\theta} + E\right]\Theta(\theta) = 0, \tag{5}$$

have the boundary conditions that $\Theta(0)$, $\Theta(\pi)$ are finite at $\theta = 0,\ \pi$. This is a singular Sturm–Liouville problem. If they are solved by the WKB method, their large eigenvalues will be $(l+1/2)^2$ with l being the integer, which will be different from the correct results by 1/4. This problem is also difficult to treat by the SKWB method.

In this paper, we will provide a new WKB method in SUSYQM to deal primarily with the singular Sturm–Liouville problem. In the new method, we will not use Eq. (3) to determine the eigenvalues for the equation. Instead, we will make full use of the shape-invariance properties. At first, the WKB method is applied to solve only the ground eigenvalue problem. Then the solutions to the excited eigenvalue problems will be obtained by the recurrence relations derived from the shape-invariance properties of the relevant equation.

From the viewpoint of perturbation, WKB methods belong to the singular perturbation. Usually, the

*Project supported by the National Natural Science Foundation of China (Grant No. 10875018).

†Corresponding author. E-mail: hua2007@126.com

© 2012 Chinese Physical Society and IOP Publishing Ltd

relevant equations have some parameters, which approach zero when the equations are solved. It is different from the regular perturbation in that the order of the differential equations will reduce when the relevant parameters become zero, whereas the order of differential equations remains unchanged with the parameters of the regular perturbation changing to zero.

In this new WKB method, the first eigenvalue is determined neither by Eq. (2) nor by Eq. (3). Because the new methods are confined to treat the singular Sturm–Liouville eigenvalue problem, its first eigenvalue and eigenfunction, the ground eigenvalue and eigenfunction, will be obtained by the use of the boundary conditions related with the problem and the properties of the ground-eigenfunction. Whereas the other eigenvalues, the excited eigenvalues, are obtained by the application of the shape-invariance property, which is in its wide meaning[5−13] (also see below for the meaning of the wide shape-invariance).

Here, we will use the equation

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) + \left[\alpha\cos^2\theta - \frac{(m/b)^2}{\sin^2\theta} + E\right]\Theta = 0 \tag{6}$$

as an example to elucidate the new method. This equation is the separated angular equation of the KG equation in a Schwarzschild black hole pierced by a string. The metric for such an object is[14−19]

$$\mathrm{d}s^2 = -\left(1-\frac{2M}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2M}{r}\right)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + b^2\sin^2\theta\mathrm{d}\phi^2\right). \tag{7}$$

The parameter M is related to the black hole energy, and b is defined as $b = 1 - 4\rho_{\mathrm{s}}$ where ρ_{s} is the linear mass density of the string. From physical viewpoint, it is easy to see $\rho_{\mathrm{s}} \ll 1$. Of course, b lies in the range $(0, 1]$. The metric (7) is locally identical to the Schwarzschild metric, so the motions of test particles are locally unchanged by the presence of the string. For the physical properties of such a metric, see Refs. [14]–[19]. As stated before, the linear mass density of the string ρ_{s} satisfies the condition $\rho_{\mathrm{s}} \ll 1$, as does $1 - b \ll 1$. However, as our naive model for the new WKB method, we will study Eq. (6) under the non-physical condition $b \to 0$, which belongs to the perturbation of WKB.

Equation (6) is the generalized associated Legendre equation. The corresponding boundary conditions become Θ and are finite at $\theta = 0,\ \pi$. The solutions to the singular Sturm–Liouville eigenvalue problem are the generalized associated Legendre functions.

The reason for Eq. (6) being our naive model lies in the fact that the solutions to its singular eigenvalue problems are exact and known. This just meets our requirement of the singular Sturm–Liouville eigenvalue problem and we can also check the results of the new WKB method with its exact solutions.

Equation (6) is transformed into

$$\left[\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right] + E - \frac{m^2/b^2}{1-x^2}\right]\Theta = 0, \quad x \in (-1, +1), \tag{8}$$

by $x = \cos\theta$, which were treated in Refs. [18] and [19]. We here give brief results in Refs. [18] and [19].

For convenience, we will assume that $m/b > 0$ in the following. The solutions could be written as

$$\Theta(x) = (1-x^2)^{m/2b}\psi(x), \tag{9}$$

and the differential equation for ψ is

$$(1-x^2)\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \left(\frac{2m}{b}+1\right)x\psi + \left[E + \frac{m}{b}(\frac{m}{b}+1)\right]\psi = 0, \quad x \in (-1, +1). \tag{10}$$

The functions ψ are analytic at $x = 0$ and could be written as $\psi = \sum_{n=0}^{\infty} a_n x^n$. The relations among a_n are

$$(n+2)(n+1)a_{n+2} = -\left[E - \frac{m}{b}\left(\frac{m}{b}+1\right) - 2n\frac{m}{b} - n(n+1)\right]a_n. \tag{11}$$

In order to satisfy the boundary condition, ψ could not be infinite series. ψ will be polynomials if the eigenvalues E take

$$E_n = \frac{m}{b}\left(\frac{m}{b}+1\right) + 2n\frac{m}{b} + n(n+1), \quad n = 0, 1, 2, 3, \ldots, \tag{12}$$

and the corresponding functions $\psi_n(x)$ become the polynomials of variable x of power of n. Note that there is an error in Refs. [18] and [19] in that ψ is put in the form

$$\psi(x) = \frac{\mathrm{d}^{\bar{m}}}{\mathrm{d}x^{\bar{m}}}P_l(x), \quad \bar{m} = \frac{m}{b}. \tag{13}$$

The above form for $\Psi(x)$ is wrong generally due to the fact that $\bar{m} = m/b$ might not be an integer. Here we present the first several forms for $\psi(x)$

$$\psi_0 = 1,\ \psi_1 = x,\ \psi_2 = 1 - (3 + 2\bar{m})x^2, \tag{14}$$

$$\psi_3 = 3x - (5 + 2\bar{m})x^3,$$

$$\psi_4 = 3 - (30 + 12\bar{m})x^2 + (5 + 2\bar{m})(7 + 2\bar{m})x^4. \tag{15}$$

In fact, equation (6) is integrable. Alternatively, one could also use the method in supersymmetry quantum mechanics to treat the problem, which will give the relations between the excited eigenvalues, eigenfunctions, and the ground ones. Of course, the results of the supersymmetry quantum mechanics will be the same as that of the above method.

As stated before, we will study the most interesting thing related to Eq. (6), the WKB solutions in the limit $b \to 0$. Actually, $b \to 0$ makes no sense in the physical ground for the study of scalar fields in black hole space time containing a string, nevertheless, it still can provide valuable mathematical information for our new method. Because the solutions to it are known, they can be used to confirm or make null some new mathematical methods. This is just what we want to do in the following.

Equation (6) contains a parameter b and is a singular Sturm–Liouville problem. From the mathematical viewpoint, when $b \to 0$, the equation corresponds to its WKB case. We want to use it to treat the singular Sturm–Liouville problem in the WKB approximation, which is somewhat difficult issue.

By the transformation[5−13]

$$\Theta = \frac{\Psi}{\sin^{1/2}\theta}, \tag{16}$$

the differential equations turn out to be

$$-\frac{\mathrm{d}^2\Psi}{\mathrm{d}\theta^2} + V(\theta, b)\Psi = E\Psi, \tag{17}$$

where the potential $V(\theta, b)$ is

$$V(\theta, b) = -\frac{1}{4} + \frac{(m/b)^2 - 1/4}{\sin^2\theta}. \tag{18}$$

The boundary conditions become

$$\Psi|_{\theta=0} = \Psi|_{\theta=\pi} = 0. \tag{19}$$

Though the usual method in SUSYQM could be used to solve the problem completely, it will not be described here. We will solve it by the new WKB method in SUSYQM. One writes Eq. (17) as

$$b^2\frac{\mathrm{d}^2\Psi}{\mathrm{d}\theta^2} + \left[\frac{1}{4}b^2 - \frac{m^2 - b^2/4}{\sin^2\theta} + Eb^2\right]\Psi = 0. \tag{20}$$

As $b \to 0$, the usual WKB methods expand the functions $\Psi(\theta)$ as

$$\Psi = \exp\left[b^{-1}\sum_{n=0}^{\infty} b^n S_n(\theta)\right], \tag{21}$$

and equation (20) is solved subsequently. In SUSYQM, the ground eigenfunction $\Psi_0(\theta)$ is related to the super-potential W by

$$\Psi_0 = N\exp\left[-\int W\,\mathrm{d}\theta\right], \tag{22}$$

hence, it is the super-potential W that will be expanded in the series form of the parameter b as

$$W = b^{-1}\sum_{n=0}^{\infty} b^n W_n(\theta). \tag{23}$$

We employ the new method to solve Eq. (17) for the ground eigen-value E_0 and eigen-function Ψ_0 as $b \to 0$. The ground eigen-value E_0 is also expanded as

$$E_0 = b^{-2}\sum_{n=0}^{\infty} b^n E_{0,n}. \tag{24}$$

The relation between the potential and super-potential is

$$\begin{aligned} & W^2 - W' \\ &= V(\theta, b) - E_0 \\ &= -\frac{1}{4} - \frac{1/4}{\sin^2\theta} + \frac{m^2}{\sin^2\theta}b^{-2} - b^{-2}\sum_{n=0}^{\infty} E_{0,n}b^n. \end{aligned} \tag{25}$$

From Eq. (23), one obtains

$$\begin{aligned} & W^2 - W' \\ &= b^{-2}W_0^2 + b^{-1}\left[2W_0W_1 - W_0'\right] \\ &\quad + \sum_{n=0}^{\infty} b^n\left[2W_{n+2}W_0 - W_{n+1}' + \sum_{k=0}^{n} W_{k+1}W_{n+1-k}\right]. \end{aligned} \tag{26}$$

From Eqs. (25) and (26), one obtains

$$W_0^2 = \frac{m^2}{\sin^2\theta} - E_{0,0}, \tag{27}$$

$$2W_0W_1 - W_0' = -E_{0,1}, \tag{28}$$

$$\begin{aligned} & 2W_0W_2 + W_1^2 - W_1' \\ &= -\frac{1}{4} - \frac{1/4}{\sin^2\theta} - E_{0,2}, \end{aligned} \tag{29}$$

$$\begin{aligned} & 2W_{n+2}W_0 - W_{n+1}' + \sum_{k=0}^{n} W_{k+1}W_{n+1-k} \\ &= -E_{0,n+2},\ n \geq 1. \end{aligned} \tag{30}$$

Equations (27)–(30) are easy to solve for W_n, $n = 0, 1, 2, \ldots$, that is

$$W_0 = \sqrt{\frac{m^2}{\sin^2\theta} - E_{0,0}}, \tag{31}$$

$$W_1 = \frac{W_0'}{2W_0} - \frac{E_{0,1}}{2W_0}, \tag{32}$$

$$W_2 = \frac{W_1'}{2W_0} - \frac{W_1^2}{2W_0} - \frac{1}{8W_0}\left[1+\frac{1}{\sin^2\theta}\right] - \frac{E_{0,2}}{2W_0}, \tag{33}$$

$$W_{n+2} = \frac{W_{n+1}'}{2W_0} - \frac{\sum_{k=0}^{n} W_{k+1}W_{n+1-k}}{2W_0} - \frac{E_{0,n+2}}{2W_0},\ n \geq 1. \tag{34}$$

On the contrary, the quantities $E_{0,n},\ n = 0, 1, 2, \ldots$ in the ground eigenvalue (24) require more effort to treat. At this point, we use the boundary conditions that $\Psi(\theta)$ is finite at points $\theta = 0, \pi$ and in the interval $(0, \pi)$. For reasons of convenience, we will suppose $\Re mb \geq 0$ from now on. The details are the following.

From Eq. (31), we first assume $E_{0,0} > m^2$, then it is easy to see that at some point θ_0, $W_0(\theta_0) = 0$. From Eqs. (32) and (22), one notices that the function $\Psi_0(\theta)$ will blow up at θ_0. Hence we must have $E_{0,0} \leq m^2$. When $E_{0,0} < m^2$, we obtain

$$\begin{aligned} W_0 &= \sqrt{\frac{m^2}{\sin^2\theta} - E_{0,0}} \\ &= \pm\frac{m}{\sin\theta}\sqrt{1-x} \\ &= \pm\frac{m}{\sin\theta}\left[1+\sum_{n=1}^{\infty} a_n x^n\right], \end{aligned} \tag{35}$$

where $x = (E_{0,0}/m^2)\sin^2\theta$. It is easy to see that $x < 1$ will be obtained under the assumption of $E_{0,0} < m^2$. The term $\pm m/\sin\theta$ in Eq. (35) will make $\Psi_0 \propto N\exp[-b^{-1}\int W_0\,\mathrm{d}\theta] \propto N\tan^{\mp m/b}(\theta/2)$ divergent either at $\theta = 0$ or at $\theta = \pi$. Therefore, the assumption $E_{0,0} < m^2$ cannot be a right choice either. That is, in order to retain the finiteness of the function Ψ_0 in the interval $[0, \pi]$, the parameter $E_{0,0}$ can neither be less than m^2 nor can be greater than m^2, the only choice for it is

$$E_{0,0} = m^2.$$

So, one obtains

$$W_0 = \pm m\cot\theta. \tag{36}$$

In order to satisfy the condition that

$$\Psi_0 \propto \exp\left[-b^{-1}\int W_0\,\mathrm{d}\theta\right]$$

approaches zero as $\theta \to 0,\ \pi$, one must make the choice

$$W_0 = -m\cot\theta. \tag{37}$$

Here, W_1 is easy to be calculated as

$$W_1 = \frac{W_0'}{2W_0} + \frac{E_{0,1}}{2m}\tan\theta, \tag{38}$$

and Ψ_0 becomes

$$\begin{aligned} \Psi_0 &\propto N\exp\left[-\int W_1\,\mathrm{d}\theta\right] \\ &= N\tan^{1/2}\theta\cos^{(E_{0,1}/2m)}\theta \\ &= \sin^{1/2}\theta\cos^{[E_{0,1}/2m-1/2]}\theta. \end{aligned} \tag{39}$$

In order for the function Ψ_0 to remain bounded in the interval $[0, \pi]$, one must select

$$\frac{E_{0,1}}{2m} - \frac{1}{2} \geq 0. \tag{40}$$

The boundary condition of the eigen-function only provides the above constraint on the parameter $E_{0,1}$, but can not determine $E_{0,1}$ uniquely. Here we will apply the properties of the ground eigenfunction to solve the problem. Except for the zeros at the boundaries $\theta = 0,\ \pi$, the ground eigenfunction should not have any zero in the interval $(0, \pi)$. This is a well-known fact in quantum theory. From Eq. (39), it is easy to see that $E_{0,1}/2m - 1/2 > 0$ will make the point $\theta = \pi/2$ a zero point for the ground eigenfunction. So one must have

$$E_{0,1} = m \tag{41}$$

to exclude that possibility. Therefore, we have

$$W_1 = -\frac{1}{2\sin\theta\cos\theta} + \frac{1}{2}\tan\theta = -\frac{1}{2}\cot\theta. \tag{42}$$

Subsequently, W_2 turns out to be

$$W_2 = \frac{E_{0,2}}{2m}\tan\theta. \tag{43}$$

We will follow the same process as before. Now, it is easy to obtain

$$\begin{aligned} \Psi_0 &\propto N\exp\left[-\int bW_2\,\mathrm{d}\theta\right] \\ &= N\cos^{E_{0,2}b/2m}\theta. \end{aligned} \tag{44}$$

So, $E_{0,2}b/2m < 0$ will make the eigenfunction blows up at the boundary and $E_{0,2}b/2m > 0$ will make the point $\theta = \pi/2$ a zero point for the ground eigenfunction. The only right choice is

$$E_{0,2} = 0.$$

Furthermore similar study will show $E_{0,n} = 0$, for $n > 2$. The detailed process will be omitted here.

In summary, we have obtained the super-potential W and the ground eigenvalue E_0 and eigenfunction Ψ_0 in the WKB form as

$$W = b^{-1}\sum_{n=0}^{\infty} b^n W_n(\theta) = -\frac{m}{b}\cot\theta - \frac{1}{2}\cot\theta, \quad (45)$$

$$E_0 = b^{-2}\sum_{n=0}^{\infty} b^n E_{0,n} = \frac{m^2}{b^2} + \frac{m}{b}, \quad (46)$$

$$\Psi_0 = N\sin^{m/b+1/2}\theta,$$

$$\Theta_0 = \frac{\Psi_0}{\sin^{1/2}\theta} = N\sin^{m/b}\theta. \quad (47)$$

The results thus obtained is identical with the exact solutions in Eqs. (12) and (15) when $n = 0$.

As to the excited eigenvalues and eigenfunctions, we will resort to the shape-invariance property in the WKB methods. Related to the super-potential W of an equation, there are two partner potentials $V^{\pm} = W^2 \pm W'$. Suppose that the super-potential is a function of both the independent variable θ and some constant parameters, for example a_1, and written as $W(\theta, a_1)$. The shape-invariance property of the equation means that its two partner potentials $V^{\pm}(\theta, a_1)$ satisfy the following equations

$$\begin{aligned} V^-(\theta, a_1) &\equiv W^2(\theta, a_1) - W', \\ V^+(\theta, a_1) &\equiv W^2(\theta, a_1) + W'(\theta, a_1), \quad (48) \\ V^+(\theta, a_1) &= V^-(\theta, a_2) + R(a_1) \\ &= W^2(\theta, a_2) - W'(\theta, a_2) + R(a_1), \quad (49) \end{aligned}$$

where a_1 is a set of parameters, a_2 is a set of functions of a_1 (say $a_2 = f(a_1)$) and the remainder $R(a_1)$ is independent of θ. Note that in some cases, a_1 actually are several parameters.

If the equation has the shape-invariance properties, it is integrable and all its excited eigenvalues and eigenfunctions could be obtained from the ground ones. Of course, this is a strict meaning of the shape-invariance.[4]

We sometimes need a wide meaning of shape-invariance. If the equation has some parameter, like b in Eq. (20), and the super-potential could be written as a series form of the parameter, so do the partner potentials $V^{\pm}$. The shape-invariance properties mean that every order term of the partner potentials $V^{\pm}$ satisfy the relations like those in Eq. (49). In the following, we will exploit the wide shape-invariance properties for Eq. (20). Actually, references [5]–[7], and [10]–[12] have already applied this wide sense of shape-invariance properties to study the spheroidal and spin-weighted spheroidal equations in the low frequency limit.

The shape-invariance property in the WKB methods means that

$$\begin{aligned} V^-(\theta, a_1) &= W^2(\theta, a_1) - W' \\ &= b^{-2}\sum_{n=0}^{\infty} b^n V_n^-(\theta, a_1), \quad (50) \\ V^+(\theta, a_1) &= W^2(\theta, a_1) + W' \\ &= b^{-2}\sum_{n=0}^{\infty} b^n V_n^-(\theta, a_1), \quad (51) \\ V_n^+(\theta, a_1) &= V_n^-(\theta, a_2) + R_n(a_1), \\ & \qquad n = 0, 1, 2, \ldots. \quad (52) \end{aligned}$$

In this way, it is easy to find the excited eigenvalues and subsequently the eigenfunctions. Now the super-potential W is written as

$$W = b^{-1}\sum_{n=0}^{\infty} b^n W_n(\theta, A_n), \quad (53)$$

$$W_0(\theta, A_0) = -A_0 \cdot \frac{m}{b}\cot\theta, \quad (54)$$

$$W_1(\theta, A_1) = -A_1 \cdot \frac{1}{2}\cot\theta, \quad (55)$$

$$W_n(\theta, A_n) = 0, \ n \geq 2. \quad (56)$$

Similar to Eq. (26), we have

$$\begin{aligned} V_0^-(\theta, A_0) &= W_0^2 = A_0^2 \cdot \frac{m^2}{b^2}\cot^2\theta, \quad (57) \\ V_1^-(\theta, A_0, A_1) &= \Big[2W_0(\theta, A_0)W_1(\theta, A_1) \\ & \qquad - W_0'(\theta, A_1)\Big], \quad (58) \\ V_2^-(\theta, A_1) &= W_1^2(\theta, A_1) - W_1'(\theta, A_1), \quad (59) \\ V_n^-(\theta, A_n) &= 0, \quad n \geq 3, \quad (60) \end{aligned}$$

and

$$\begin{aligned} V_0^+(\theta, A_0) &= W_0^2 \\ &= A_0^2\frac{m^2}{b^2}\cot^2\theta \\ &= V_0^-(\theta, A_0), \quad (61) \\ V_1^+(\theta, A_0, A_1) &= \Big[2W_0(\theta, A_0)W_1(\theta, A_1) \\ & \qquad + W_0'(\theta, A_1)\Big], \quad (62) \\ V_2^+(\theta, A_1) &= W_1^2(\theta, A_1) + W_1'(\theta, A_1), \quad (63) \\ V_n^+(\theta, A_n) &= 0, \quad n \geq 3. \quad (64) \end{aligned}$$

Here we denote $a_1 = (A_0,\ A_1)$. One sees that

$$\begin{aligned} V_0^+(\theta, A_0) &= V_0^-(\theta, B_0) \Rightarrow B_0 = A_0, \quad (65) \\ V_1^+(\theta, A_0, A_1) &= V_1^-(\theta, B_0, B_1) + R_1, \quad (66) \\ & \Rightarrow B_1 = A_1 + 2,\ R_1(a_1) \end{aligned}$$

$$= mA_0(B_1 - A_1) = 2mA_0, \quad (67)$$

$$V_2^+(\theta, A_1) = V_2^-(\theta, B_1) + R_2, \Rightarrow R_2(a_1)$$
$$= A_1 + 1, \quad (68)$$

$$V_n^+(\theta, A_n) = V_n^-(\theta, B_n) + R_n, \Rightarrow R_n$$
$$= 0, \ n \geq 3, \quad (69)$$

where we denote $a_2 = (B_0, \ B_1)$. Therefore,

$$E_{n+1} - E_0 = \sum_{k=1}^{n} \left[b^{-1} R_1(a_k) + R_2(a_k) \right]$$
$$= 2mnA_0 b^{-1} + n(A_1 - 1) + (n+1)n,$$

where we have used the fact $R_2(a_k) = A_1 + 2(k-1) + 1 = A_1 + 2k - 1$. So, the result is

$$E_{n+1} = \frac{m}{b}\left(\frac{m}{b} + 1\right) + 2mnA_0 b^{-1} + n(A_1 - 1) + (n+1)n.$$

From the fact $A_1 = 1$, it is easy to see that the above results are the same as that in Eq. (12). The excited eigenfunctions are obtained through repeatedly applying the following formula[4]

$$\Psi_n(\theta; a_1) = A^\dagger(\theta; a_1)\Psi_{n-1}^{(1)}(\theta; a_2), \quad (70)$$

where

$$A^\dagger(\theta; a_1) = -\frac{\mathrm{d}}{\mathrm{d}\theta} + W(a_1, \theta)$$
$$= -\frac{\mathrm{d}}{\mathrm{d}\theta} - A_0 \frac{m}{b} \cot\theta - A_1 \frac{1}{2} \cot\theta, \quad (71)$$

$$\Psi_0(\theta, a_2) = \exp\left[-\int W(\theta, a_2) \mathrm{d}\theta \right]. \quad (72)$$

The detailed process is omitted here.

In conclusion, the new WKB method has been applied to study a naive model. The solutions to this model are known as they have been obtained by other methods in Refs. [18] and [19] and are given at the beginning of this paper, and are used to compare with the results obtained by the new WKB method. The same results obtained in comparison show that the new WKB method is a feasible and reliable method.

The new WKB method supplies a new way to treat the singular Sturm–Liouville problem. It combines the singular boundary conditions and the properties of the ground eigenfunction to solve the Sturm–Liouville problem related to the ground state. Then it further unites the shape-invariance properties and the expansion of the relevant quantities into the wide shape-invariance to obtain the solutions to the excited eigen-value problems. This method will provide a new way to study the high frequency limit of the spheroidal equations and the spin-weighted spheroidal equations, whose low frequency limit has been investigated and applied in a series of papers.[5–13] These problems will be our further studies.

References

[1] Froman N and Froman P O 1965 *JWKB Approximation, Contributions to the Theory* (Amsterdam: North Holland Publishing Company) Chap. 1

[2] Comtet A, Bandrauk A and Campbell D K 1985 *Phys. Lett.* B **150** 159

[3] Khare A 1985 *Phys. Lett.* B **161** 131

[4] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 268

[5] Tian G H and Zhong S Q 2009 "Investigation of the Recurrence Relations for the Spheroidal Wave Functions" *arXiv*: gr-qc/0906.4687v3

[6] Tian G H and Zhong S Q 2009 "Study the Spheroidal Wave Functions by SUSY Method" *arXiv*: gr-qc/0906.4685v3

[7] Tian G H and Zhong S Q 2010 *Chin. Phys. Lett.* **27** 040305

[8] Tang W L and Tian G H 2011 *Chin. Phys.* B **20** 010304

[9] Tang W L and Tian G H 2011 *Chin. Phys.* B **20** 050301

[10] Tian G H 2010 *Chin. Phys. Lett.* **27** 030308

[11] Sun Y, Tian G H and Dong K 2011 *Chin. Phys.* B **20** 061101

[12] Dong K, Tian G H and Sun Y 2011 *Chin. Phys.* B **20** 071101

[13] Tian G H 2010 *Chin. Phys. Lett.* **27** 100306

[14] Vilenkin A 1981 *Phys. Rev.* D **23** 852

[15] Linet B 1985 *Gen. Rel. Grav.* **17** 1109

[16] Gott J R 1985 *Astrophys. J.* **288** 422

[17] Chen S B, Wang B and Su R K 2008 *arXiv*: gr-qc/0701088v2

[18] Geusa de A Marques and Bezerra V B 2002 *Class. Quantum Grav.* **19** 985

[19] Geusa de A Marques and Bezerra V B 2001 *arXiv*: gr-qc0111019 v1