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Compact implicit integration factor methods for some complex-valued nonlinear equations

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The compact implicit integration factor (cIIF) method is an efficient time discretization scheme for stiff nonlinear diffusion equations in two and three spatial dimensions. In the current work, we apply the cIIF method to some complex-valued nonlinear evolutionary equations such as the nonlinear Schrödinger (NLS) equation and the complex Ginzburg–Landau (GL) equation. Detailed algorithm formulation and practical implementation of cIIF method are performed. The numerical results indicate that this method is very accurate and efficient.

Keywords: compact implicit integration factor method, finite difference, nonlinear Schrödinger equation, complex Ginzburg–Landau equation

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1. Introduction

The complex-valued nonlinear diffusion equation is an important nonlinear evolutionary equation and has various applications in many fields. Numerical methods always attract research interest because in general there are no analytical solutions for nonlinear complex-valued equations. The nonlinear partial differential equation (PDE) is discretized in space and we obtain the stiff ordianry differential equations (ODEs). Devising an efficient and highly accurate numerical scheme become important because of the stiffness. Some high order time discretization methods have been developed to solve the ODEs, such as the total variation diminishing (TVD) Runge–Kutta (RK) schemes;[1] the high order implicit–explicit (IMEX) Runge–Kutta methods,[2] the spectral deferred correction (SDC) methods,[3] and the exponential time differencing (ETD) methods.[4]

The implicit integration factor (IIF) method is a class of efficient time discretization scheme introduced recently for stiff reaction–diffusions.[5] In particular, its second-order scheme is unconditionally linearly stable with respect to diffusion and reaction. The IIF schemes require the calculation of exponentials of the matrices that result from spatial discretization of the diffusion term. The main difficulty in implementing the IIF method for high dimensional problems lies in the significantly expensive storage and calculation of exponentials of the matrices. Although the discretization matries are sparse, their exponentials matrices will be dense.

To overcome this difficulty, the compact integration factor method[6] was introduced on rectangular meshes. This approach applies exponential operations in every spatial direction by introducing a compact representation of the discretization matrices. As a result, cIIF has the same stability properties as the IIF,[6] but with a significant improvement on storage and CPU saving. For example, in a two-dimensional (2D) case, if we have N grid points in every direction, then we just need to consider several $N \times N$ matrices and their exponentials other than the full $N^2 \times N^2$ matrix and its exponential. See Ref. [6] for more details.

The cIIF method has a nice property that the implicit terms are free of the exponential operation of the linear terms. So we can decouple the exact evaluation of the linear part from the implicit treatment of the nonlinear source terms. As a result, only a local nonlinear equation needs to be solved at each spatial grid point. Classical iterations methods such as Newton iteration and direct iteration can be used to solve the nonlinear equation.

The rest of the paper is organized as follows: in Section 2, we apply the cIIF method combined with the finite difference method to the 2D nonlinear complex-valued diffusion equation. In Section 3, we test the method for the nonlinear Schrödinger (NLS) equation and Ginzburg–Landao (GL) equation in two space dimensions. Finally, we summarize our results

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in Section 4.

2. The cIIF method

We consider the following complex-valued nonlinear evolutionary equation with periodic boundary conditions

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(u), \quad (x, y) \in [a, b] \times [c, d\,], \tag{1}$$

where $u(x, y)$ is a complex function, D is a complex constant, and $f(u)$ is the nonlinear term. The computation domain is divided into N_x+2 and N_y+2 discrete points, $(x_j, y_k) = (a+j\Delta x, c+k\Delta y), j = 0, 1, \ldots, N_x+1$; $k = 0, 1, \ldots, N_y + 1$ with $\Delta x = (b-a)/(N_x+1)$, $\Delta y = (d-c)/(N_y+1)$. The time interval $[0, +\infty)$ is discretized by $t_n = n\Delta t, n = 0, 1, \ldots$ with Δt being time step. The numerical solution of $u(x, y)$ at point (x_j, y_k) is denoted by u_{jk}. Applying the second order central difference discretization to the diffusion, we obtain the system of the nonlinear ODEs

$$\frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = \boldsymbol{A}\boldsymbol{U} + \boldsymbol{U}\boldsymbol{B} + \boldsymbol{F}(\boldsymbol{U}), \tag{2}$$

where

$$\boldsymbol{U} = \begin{pmatrix} u_{00} & u_{01} & \cdots & u_{0,N_y} \\ u_{10} & u_{11} & \cdots & u_{1,N_y} \\ \vdots & \vdots & \vdots & \vdots \\ u_{N_x,0} & u_{N_x,1} & \cdots & u_{N_x,N_y} \end{pmatrix},$$

$$\boldsymbol{A} = \frac{D}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & 1 & -2 & 1 \\ 1 & \cdots & 0 & 1 & -2 \end{pmatrix}, \quad \boldsymbol{B} = \frac{\Delta x^2}{\Delta y^2}\boldsymbol{A}.$$

We multiply Eq. (2) by exponential matrix $\mathrm{e}^{-\boldsymbol{A}t}$ from the left and $\mathrm{e}^{-\boldsymbol{B}t}$ from the right to obtain

$$\frac{\mathrm{d}(\mathrm{e}^{-\boldsymbol{A}t}\boldsymbol{U}\,\mathrm{e}^{-\boldsymbol{B}t})}{\mathrm{d}t} = \mathrm{e}^{-\boldsymbol{A}t}\boldsymbol{F}(\boldsymbol{U})\,\mathrm{e}^{-\boldsymbol{B}t}. \tag{3}$$

Integration of Eq. (3) over one time step from t_n to t_{n+1} results in

$$\boldsymbol{U}^{n+1} = \mathrm{e}^{\boldsymbol{A}\Delta t}\boldsymbol{U}^n\,\mathrm{e}^{\boldsymbol{B}\Delta t} + \Delta t(\alpha_1\boldsymbol{F}(\boldsymbol{U}^{n+1}) + \sum_{j=0}^{r-2}\alpha_{-j}\,\mathrm{e}^{(j+1)\boldsymbol{A}\Delta t}\boldsymbol{F}(\boldsymbol{U}^{n-j})\,\mathrm{e}^{(j+1)\boldsymbol{B}\Delta t}), \tag{4}$$

where $\alpha_1, \alpha_0, \ldots, \alpha_{-r+2}$ are taken as in Ref. [6]. In our numerical experiment, we apply the second-order cIIF method to achieve consistent spatial-temporal accuracy orders. The second-order scheme of the cIIF method is in the following form

$$\boldsymbol{U}^{n+1} = \mathrm{e}^{\boldsymbol{A}\Delta t}\left(\boldsymbol{U}^n + \frac{\Delta t}{2}\boldsymbol{F}(\boldsymbol{U}^n)\right)\mathrm{e}^{\boldsymbol{B}\Delta t} + \frac{\Delta t}{2}\boldsymbol{F}(\boldsymbol{U}^{n+1}). \tag{5}$$

We use direct iteration to solve the nonlinear equation on every grid point

$$\boldsymbol{U}^{n+1,k} = \mathrm{e}^{\boldsymbol{A}\Delta t}\left(\boldsymbol{U}^n + \frac{\Delta t}{2}\boldsymbol{F}(\boldsymbol{U}^n)\right)\mathrm{e}^{\boldsymbol{B}\Delta t} + \frac{\Delta t}{2}\boldsymbol{F}(\boldsymbol{U}^{n+1,k-1}), \quad k = 1, 2, \ldots \tag{6}$$

The nonlinear terms are evaluated at the previous nonlinear iteration level, $k-1$. The initial guess at the time level t_{n+1}, $U^{n+1,0}$ in the nonlinear iteration is set to be U^n. The threshold value for judging nonlinear iteration convergence is set to be 10^{-13}.

3. Numerical tests

In this section, we apply the cIIF method to the 2D nonlinear Schrödinger equations and complex Ginzburg–Landau equations.

3.1. Nonlinear Schrödinger equation

We consider the 2D cubic nonlinear Schrödinger equation

$$\frac{\partial u}{\partial t} = \mathrm{i}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \mathrm{i}\,q|u|^2 u. \tag{7}$$

The NLS equation is one of the most important mathematical models describing many physical phenomenon such as nonlinear optics, quantum mechanics, plasma physics, etc. There have been numerous investigations based on different numerical methods.[7–9]

It is well known that the NLS equation (7) has two conservation laws: the charge conservation

$$Q(t) = \int_\Omega |u(x, y, t)|^2\,\mathrm{d}x\,\mathrm{d}y = Q(0), \tag{8}$$

and energy conservation

$$E(t) = \int_\Omega \left(|\frac{\partial}{\partial x}u(x, y, t)|^2 + \frac{\partial}{\partial y}u(x, y, t)|^2 - \frac{q}{2}|u(x, y, t)|^4\right)\mathrm{d}x\,\mathrm{d}y = E(0). \tag{9}$$

To test the charge and energy conservation properties of the present scheme (4) at time $t_n = n\Delta t$, we define the discretized expressions for the global norms (8) and (9) as

$$\begin{aligned} Q^n &= \sum_{j=0}^{N_x}\sum_{k=0}^{N_y} |u_{jk}^n|^2 \Delta x \Delta y, \\ E^n &= \sum_{j=0}^{N_x}\sum_{k=0}^{N_y} \Big(|\partial_x u_{jk}^n|^2 + |\partial_y u_{jk}^n|^2 \\ &\quad - \frac{q}{2}|u_{jk}^n|^4 \Big) \Delta x \Delta y, \end{aligned} \tag{10}$$

where $u_{jk}^n = u(x_j, t^n)$, $\partial_x u_{jk}^n = (u_{j+1,k}^n - u_{jk}^n)/\Delta x$, and $\partial_y u_{jk}^n = (u_{j,k+1}^n - u_{jk}^n)/\Delta y$. We will measure the accuracy of the proposed schemes using the error in L_∞ norm $\|\mathrm{Err}\|_\infty = \max\limits_j(\max\limits_k |u_{jk}^n - u(x_j, y_k, t^n)|)$.

Example 1 In our first numerical test, we give an accuracy test of NLS equation (7) on domain $[0, 2\pi]^2$ until $t = 1.0$. The exact solution is $u = \exp(\mathrm{i}(x+y-t))$ with $q = 1$. We give the L_∞ error, convergence order, and CPU time in Table 1 with different mesh sizes. It is demonstrated in Table 1 that our scheme obtains second-order accuracy. Moreover, the time step $\Delta t = 0.5\Delta x$ saves calculation time considerably. The discrete charge and energy are shown in Fig. 1, which preserves the conservation laws.

Table 1. L_∞ norm error of the difference solutions at $t = 1.0$.

$N_x \times N_y$	L_∞ error	Order	CPU/s
40×40	3.6090×10^{-3}	–	0.1562
80×80	9.0123×10^{-4}	2.0016	1.0156
160×160	2.2496×10^{-4}	2.0022	7.9687
320×320	5.6240×10^{-5}	2.0000	72.687

For NLS equation (7) with $q > 0$, when $\|u(x, y, 0)\|_2$ is above a threshold value, the solution can self-focus and yield a blow-up phenomenon in finite time. Next we demonstrate the blow-up by two numerical examples.

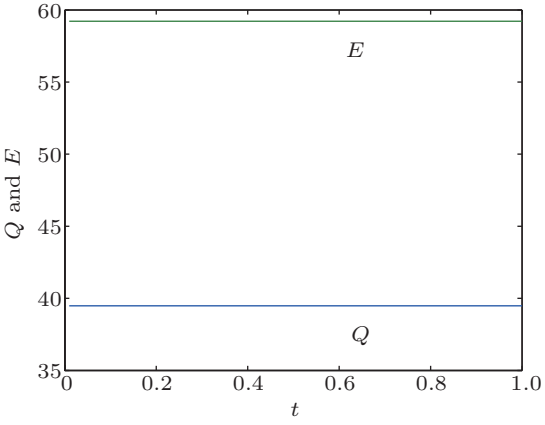


Fig. 1. (colour online) The discrete charge and energy for Example 1.

Example 2 We consider Eq. (7) with $q = 1$ and the initial condition $u(x, y, 0) = 2 + 0.01\sin(x + \pi/4)\sin(y + \pi/4)$ in $[-\pi, \pi]^2$. The L_2 norm of initial value is $\|u(x, y, 0)\|_2 = 12.56$ above the threshold value and we observe the blow-up phenomenon at $t = 1.5813$ in Fig. 2.

Example 3 We consider NLS equation (7) as in Example 2 but with different initial condition $u(x, y, 0) = (1 + \sin x)(2 + \sin y)$ and domain $[0, 2\pi]^2$. The L_2 norm of initial value is $\|u(x, y, 0)\|_2 = 16.32$. Figure 3 shows the initial value and the strong singular solution obtained at $t = 0.108$.

We plot the numerical charges of Example 2 and Example 3 in Fig. 4 and observe that the numerical charges are still unchanged although there exists a strong singular solution.

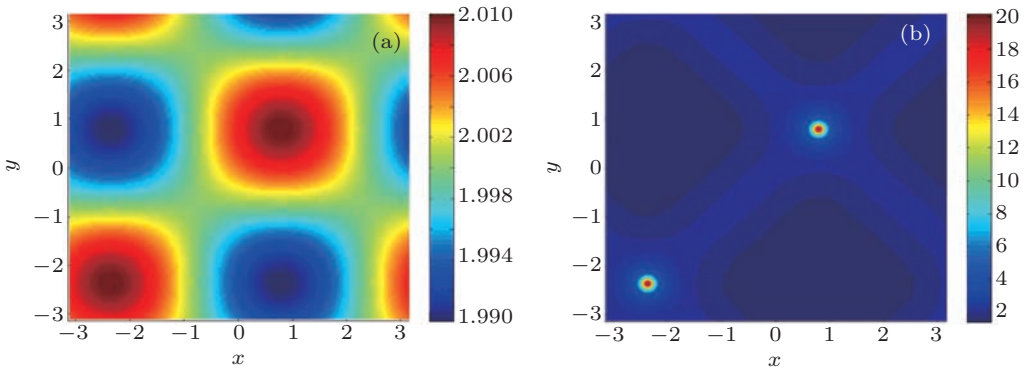


Fig. 2. (colour online) The solutions of Example 2 at (a) $t = 0.0$, (b) $t = 1.5813$.

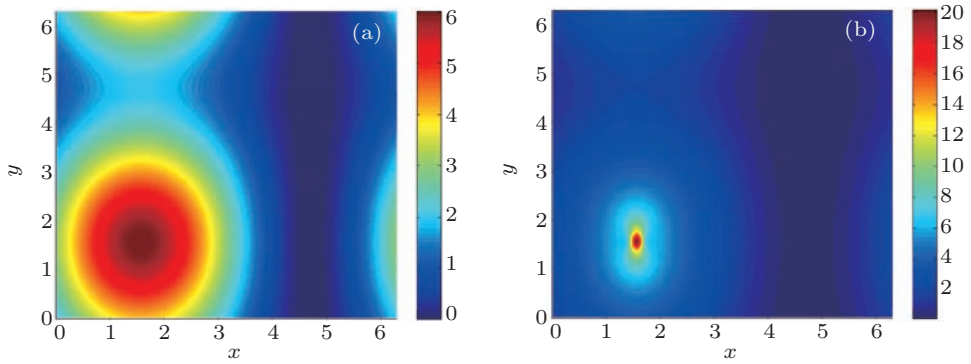


Fig. 3. (colour online) The solutions of the Example 3 at (a) $t = 0.0$, (b) $t = 0.108$.

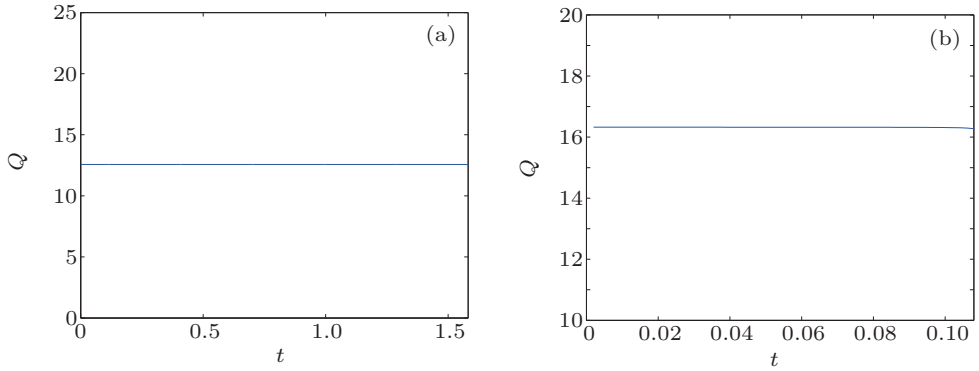


Fig. 4. (colour online) The numerical charges of (a) Example 2, (b) Example 3.

3.2. Ginzburg–Landau equation

The Ginzburg–Landau-type equation, which has been extensively applied to different physical, chemical, and biological systems, is a kind of important nonlinear evolution equation and has various forms.[10–13] In this paper we consider the following complex Ginzburg–Landau equation in two spatial dimensions

$$\frac{\partial u}{\partial t} = (1 + \mathrm{i}\,\alpha)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \mu(x, y)u - (1 + \mathrm{i}\,\beta)|u|^2 u, \tag{11}$$

where $\mu(x, y) = 1 - \frac{1}{2}\Omega^2(x^2 + y^2)$, Ω is the Einstein frequency. There have been many numerical schemes on the Ginsburg–Landau equations of the type given in Eq. (11).[14–16] Now we apply our scheme (4) to solving the Ginzburg–Landau equation (11) with the periodic boundary condition.

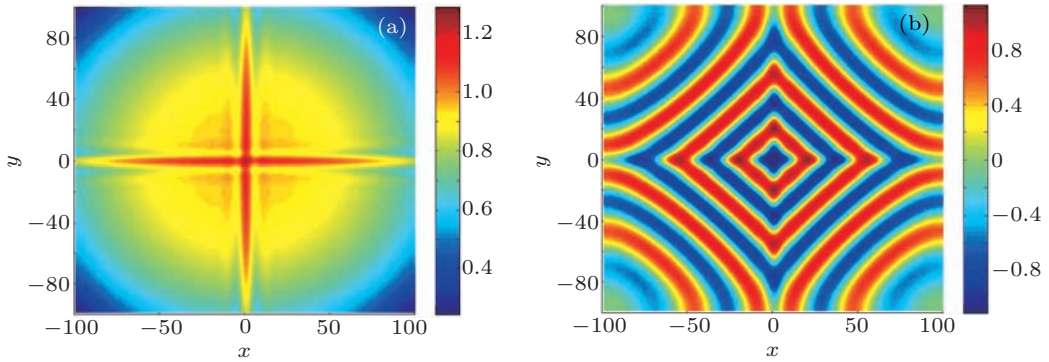


Fig. 5. (colour online) The numerical solution of Example 4 for $\Omega = 0.01$: (a) $|u|$, (b) $\mathrm{Re}(u)$.

Example 4 We consider the Ginzburg–Landau equation (11) with the random initial condition: $\mathrm{Re}(u(x, y, 0)) = 10^{-6}\ \mathrm{rand}(\cdot, \cdot)$, $\mathrm{Im}(u(x, y, 0)) = 10^{-6}\ \mathrm{rand}(\cdot, \cdot)$, where $\mathrm{rand}(\cdot, \cdot)$ is a random function in Matlab. We take the

parameters $\alpha = 3.5$ and $\beta = -0.34$. The computations are performed on $[-100, 100]^2$ with $N_x = N_y = 199$ and $\Delta t = 0.1$. It has been found that $\Omega = 0.01, 0.02$ corresponding to the X and target waves, respectively. We plot the real part and the absolute values of the numerical results in Figs. 5 and 6, which reproduce the X and target waves as in Ref. [11]. It should be noted that our scheme allows for a much larger time step than that in Ref. [11] and the total CPU time is 451 s.

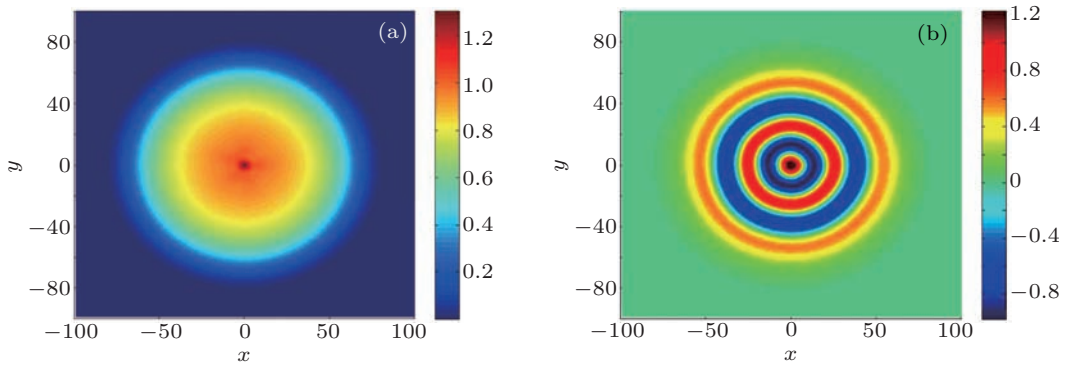


Fig. 6. (colour online) The numerical solution of Example 4 for $\Omega = 0.02$: (a) $|u|$, (b) Re(u).

4. Conclusion

In this paper the cIIF scheme (4) for complex-valued nonlinear evolutionary equations was proposed. The test examples show that this method reduces the computational cost significantly in both storage and CPU, and it makes the cIIF method efficient and attractive for 2D nonlinear problems. Moreover, the new compact IIF (cIIF) preserves the stability property of the IIF method. In general, the cIIF method provides us with a simple, efficient, stable tool for solving complex-valued nonlinear evolutionary equations.

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