Lie symmetries and conserved quantities of discrete nonholonomic Hamiltonian systems*

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This paper focuses on studying Lie symmetries and conserved quantities of discrete nonholonomic Hamiltonian systems. Firstly, the discrete generalized Hamiltonian canonical equations and discrete energy equation of nonholonomic Hamiltonian systems are derived from discrete Hamiltonian action. Secondly, the determining equations and structure equation of Lie symmetry of the system are obtained. Thirdly, the Lie theorems and the conservation quantities are given for the discrete nonholonomic Hamiltonian systems. Finally, an example is discussed to illustrate the application of the results.

Keywords: Lie symmetry, nonholonomic constraint, discrete Hamiltonian system, conserved quantity

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1. Introduction

In symmetric theory, the infinitesimal transformation of a Lie group which leaves the differential equations of motion invariant is called the Lie symmetrical transformation. The Lie group theory has been developed as a powerful tool to solve differential equations, to classify them and to establish properties of their solution space. The symmetric theory has been widely used in mathematics, physics, mechanics, and geometrical analysis of dynamical systems.[1−15]

In recent years, the Lie group theory has been applied to discrete equations, such as difference equations, differential-difference equations or q-difference equations.[16−30] The Noether's type constructions for difference functional, difference equations, and lattices have been reviewed. In 2009, Dorodnitsyn and Kozlov[31] provided a Hamiltonian version of Noether's theorem based on the Hamiltonian identity which was an analog of the well-known Noether's identity in the Lagrangian framework. Recently Fu *et al.* investigated the Noether's symmetries of nonconservative dynamical systems and mechanic-electrical systems with the discrete functional of the Lagrangian framework.[32−34] This approach can be easily applied to discrete Hamiltonian equations to establish the discrete Hamiltonian identity and to obtain discrete conservation properties of the mechanical systems. In this paper, we study the Lie symmetries of the discrete nonholonomic Hamiltonian system and find the Noether's conservations caused by Lie symmetrical transformation.

The rest of this paper is arranged as follows: in Section 2, the canonical Hamiltonian equations and energy equation of discrete nonholonomic Hamiltonian system are obtained. In Section 3, the prolongation of the Lie symmetry operator is given for the discrete case. In Section 4, the determining equations are obtained. In Section 5, the Lie theorems of nonholonomic Hamiltonian systems are given. In Section 6, a practical example is discussed to illustrate the results. In Section 7, some conclusions are given.

2. Transformation operators in the space of discrete variables

We consider the space of discrete variables. Using the simplest invariant mesh h in the t direction, we form a pair of operators which we call, respectively, the operators of discrete transformation to the right and left:

$$\underset{+h}{s} = \mathrm{e}^{hD} = \sum_{i=0}^{\infty} \frac{h^i}{i!} D^i,$$

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$$\underset{-h}{s} = \mathrm{e}^{-hD} = \sum_{i=0}^{\infty} \frac{(-h)^i}{i!} D^i, \tag{1}$$

where

$$D = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i} + \ddot{q}^i \frac{\partial}{\partial \dot{q}^i} + \ddot{p}^i \frac{\partial}{\partial \dot{p}_i} + \cdots,$$

which is a full differentiation operator, and the operators $\underset{+h}{s}$ and $\underset{-h}{s}$ can commute with each other and satisfy $\underset{+h}{s} \cdot \underset{-h}{s} = \underset{-h}{s} \cdot \underset{+h}{s} = 1$.

Using the simplest invariant mesh h in the t direction, a pair of linear difference operators $\underset{+h}{s}$ and $\underset{-h}{s}$ are defined by

$$\begin{aligned} \underset{+h}{s}\, q^i &= q_i^+, \quad \underset{-h}{s}\, q^i = q_i^-, \\ \underset{+h}{s}\, p_i &= p_i^+, \quad \underset{-h}{s}\, p_i = p_i^-, \end{aligned} \tag{2}$$

and $\underset{+h}{s}$ and $\underset{-h}{s}$ are the corresponding shift operators. Moreover, with the help of $\underset{+h}{s}$ and $\underset{-h}{s}$ we define a pair of operators of discrete (finite-difference) differentiation to the right and left:

$$\begin{aligned} \underset{+h}{D} &= \frac{\underset{+h}{s} - 1}{h} = \sum_{i=1}^{\infty} \frac{h^{(i-1)}}{i!} D^i, \\ \underset{-h}{D} &= \frac{1 - \underset{-h}{s}}{h} = \sum_{i=1}^{\infty} \frac{(-h)^{(i-1)}}{i!} D^i. \end{aligned} \tag{3}$$

The operators $\underset{+h}{s}$, $\underset{-h}{s}$, $\underset{+h}{D}$, and $\underset{-h}{D}$ commute in any combination, while $\underset{+h}{D} = \underset{-h}{D}\, \underset{+h}{s}$ and $\underset{-h}{D} = \underset{+h}{D}\, \underset{-h}{s}$ and they possess corresponding finite-difference Leibniz rule as[18,35]

$$\begin{aligned} \underset{+h}{D}(FG) &= \underset{+h}{D}(F)G + F\, \underset{+h}{D}(G) + h\, \underset{+h}{D}(F)\, \underset{+h}{D}(G), \\ \underset{-h}{D}(FG) &= \underset{-h}{D}(F)G + F\, \underset{-h}{D}(G) - h\, \underset{-h}{D}(F)\, \underset{-h}{D}(G). \end{aligned} \tag{4}$$

3. Hamiltonian canonical equations and energy equation of a discrete nonholonomic Hamiltonian system

In discrete nonholonomic Hamiltonian systems, we consider an n-dimensional configuration manifold with local coordinates $\boldsymbol{q} = \{q^1, \ldots, q^n\}$, $\boldsymbol{p} = \{p_1, \ldots, p_n\}$, and Hamiltonian function $H = H(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+)$ which can be obtained by discrete Legendre transform,[36] subject to g ($g < n$) ideal bilateral discrete constraints of the form[28]

$$f_\beta = f_\beta(t, t_+, \boldsymbol{q}, \boldsymbol{p}, \underset{h}{D}(q^i), \underset{h}{D}(p_i)),$$

$$\beta = 1, \ldots, g,\ i = 1, \ldots, n, \tag{5}$$

which satisfies Appell–Chetaev's condition

$$\frac{\partial f_\beta}{\partial\, \underset{h}{D}(q^i)} \delta q^i = 0, \quad i = 1, \ldots, n. \tag{6}$$

Hamiltonian canonical equations can be obtained by the variational principle from the discrete Hamiltonian action

$$\begin{aligned} \Delta H_h = -\sum_{\Omega} \Bigg(& Q^i(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+) \\ & + \lambda_\beta \frac{\partial f_\beta}{\partial\, \underset{h}{D}(q^i)} \Bigg) h_+ \delta q^i, \end{aligned} \tag{7}$$

where λ_β is the constraint multiplier, $Q = Q(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+)$ is the discrete nonconservative force, and δq^i is the discrete virtual placement which satisfies[37]

$$\delta q^i = \Delta q^i - \frac{q_+^i - q^i}{h_+} \Delta t. \tag{8}$$

Indeed, a variation of this functional along a curve $q^i = \varphi_i(t)$, $p_i = \psi_i(t)$, at some point $(t, \boldsymbol{q}, \boldsymbol{p})$ will affect only two terms of the sum (7):

$$\begin{aligned} H_h = \cdots & + p_i(q^i - q_-^i) - H(t_-, t, \boldsymbol{q}_-, \boldsymbol{p}) h_- \\ & + p_i^+(q_+^i - q^i) - H(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+) h_+ + \cdots. \end{aligned}$$

Therefore, we obtain the following expression for the variation:

$$\begin{aligned} \Delta H_h &= \frac{\partial H_h}{\partial p_i} \Delta p_i + \frac{\partial H_h}{\partial q^i} \Delta q^i + \frac{\partial H_h}{\partial t} \Delta t \\ &= -\left(Q^i(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+) + \lambda_\beta \frac{\partial f_\beta}{\partial\, \underset{h}{D}(q^i)} \right) h_+ \delta q^i. \end{aligned} \tag{9}$$

According to Eq. (8), equation (9) can be rewritten as

$$\begin{aligned} & \frac{\partial H_h}{\partial p_i} \Delta p_i + \Bigg(\frac{\partial H_h}{\partial q^i} + \Bigg(Q^i(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+) \\ & + \lambda_\beta \frac{\partial f_\beta}{\partial\, \underset{h}{D}(q^i)} \Bigg) h_+ \Bigg) \Delta q^i \\ & + \Bigg(\frac{\partial H_h}{\partial t} - (q_+^i - q^i) \Bigg(Q^i(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+) \\ & + \lambda_\beta \frac{\partial f_\beta}{\partial\, \underset{h}{D}(q^i)} \Bigg) \Bigg) \Delta t = 0. \end{aligned} \tag{10}$$

For the stationary value of discrete Hamiltonian action (7), we obtain the system of $2n + 1$ equations

$$q^i - q_-^i - h_- \frac{\partial H^-}{\partial p_i} = 0,$$

$$p_i^+ - p_i + h_+\frac{\partial H}{\partial q^i} - \Big(Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+) + \lambda_\beta \frac{\partial f_\beta}{\partial \underset{h}{D}(q^i)}\Big)h_+ = 0, \tag{11}$$

$$h_+\frac{\partial H}{\partial t} - H + h_-\frac{\partial H^-}{\partial t} + H^- - (q_+^i - q^i) \times \Big(Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+) + \lambda_\beta \frac{\partial f_\beta}{\partial \underset{h}{D}(q^i)}\Big) = 0. \tag{12}$$

For convenience, we use the left and the right shift operators in Eq. (1) and the corresponding discrete differentiation operators in Eq. (3). Equations (11) and (12) can be written as

$$\begin{aligned}
&\underset{h}{D}(q^i) = \frac{\partial H}{\partial p_i^+},\\
&\underset{h}{D}(p_i) = -\frac{\partial H}{\partial q^i} + Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)\\
&\qquad + \Lambda_i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+),
\end{aligned} \tag{13}$$

$$h_+\frac{\partial H}{\partial t} - H + h_-\frac{\partial H^-}{\partial t} + H^- - \underset{h}{D}(q^i) \times (Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+) + \Lambda_i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+))h_+ = 0, \tag{14}$$

where

$$\Lambda_i = \lambda_\beta \frac{\partial f_\beta}{\partial \underset{h}{D}(q^i)} = \Lambda_i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+).$$

Equations in expression (13) are the generalized discrete Hamiltonian canonical equations, and equation (14) is the energy equation for nonholonomic system which can define the lattice on which the canonical Hamiltonian equations are discredited.

4. Determining equations of Lie symmetry for the discrete nonholonomic Hamiltonian system

We consider finite-difference equations and discrete Hamiltonians at some point $(t,\boldsymbol{q},\boldsymbol{p})$ of a lattice. Generally, the lattice is not regular. The notations are clear from the following figure (See Fig. 1)

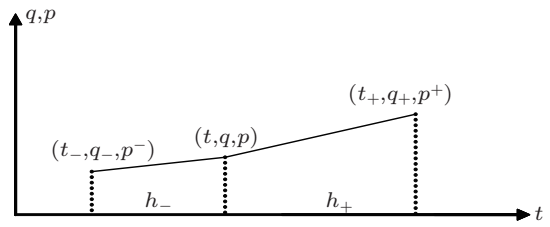


Fig. 1. Elementary stencil for a three-point difference equation.

We introduce the infinitesimal transformations with respect to time, generalized coordinates and generalized momentum, respectively, as

$$\begin{aligned}
&t^* = t + \varepsilon\xi(t,\boldsymbol{q},\boldsymbol{p}),\quad q^{i*} = q^i + \varepsilon\eta^i(t,\boldsymbol{q},\boldsymbol{p}),\\
&p_i^* = p_i + \varepsilon\zeta_i(t,\boldsymbol{q},\boldsymbol{p}),
\end{aligned} \tag{15}$$

where ε is a small parameter, ξ, η^i, and ζ_i are the infinitesimal generators. It is convenient to express the quasi-invariance condition for the Hamiltonian action in terms of a formalism involving vector fields. The infinitesimal generators ξ, η^i, and ζ_i are determined by the vector field of the form

$$\begin{aligned}
X^{(0)} =\ & \xi(t,\boldsymbol{q},\boldsymbol{p})\frac{\partial}{\partial t} + \eta^i(t,\boldsymbol{q},\boldsymbol{p})\frac{\partial}{\partial q^i}\\
&+ \zeta_i(t,\boldsymbol{q},\boldsymbol{p})\frac{\partial}{\partial p_i}.
\end{aligned} \tag{16}$$

We can extend Lie symmetry operator (16) into the discrete Hamiltonian systems as follows:

$$\begin{aligned}
X^{(1)} =\ & \xi\frac{\partial}{\partial t} + \eta^i\frac{\partial}{\partial q^i} + \zeta_i\frac{\partial}{\partial p_i} + \xi_-\frac{\partial}{\partial t_-}\\
&+ \eta_-^i\frac{\partial}{\partial q_-^i} + \zeta_i^-\frac{\partial}{\partial p_i^-} + \xi_+\frac{\partial}{\partial t_+}\\
&+ \eta_+^i\frac{\partial}{\partial q_+^i} + \zeta_i^+\frac{\partial}{\partial p_i^+}\\
&+ (\xi_+ - \xi)\frac{\partial}{\partial h_+} + (\xi - \xi_-)\frac{\partial}{\partial h_-},
\end{aligned} \tag{17}$$

where $\xi_- = \xi(t_-,\boldsymbol{q}_-,\boldsymbol{p}^-)$, $\xi_+ = \xi(t_+,\boldsymbol{q}_+,\boldsymbol{p}^+)$, $\eta_-^i = \eta^i(t_-,\boldsymbol{q}_-,\boldsymbol{p}^-)$, $\eta_+^i = \eta^i(t_+,\boldsymbol{q}_+,\boldsymbol{p}^+)$, $\zeta_i^- = \zeta_i(t_-,\boldsymbol{q}_-,\boldsymbol{p}^-)$, and $\zeta_i^+ = \zeta_i(t_+,\boldsymbol{q}_+,\boldsymbol{p}^+)$, $i = 1,\ldots,n$.

Definition 1 If infinitesimal transformations (15) leave the discrete Hamiltonian canonical equations (13) and energy equation (14) invariant, they are called Lie symmetry transformations of the discrete nonholonomic Hamiltonian system corresponding to the holonomic one.

Criterion If the generators ξ, η^i, and ζ_i satisfy the following conditions:

$$\begin{aligned}
&\underset{h}{D}(\eta^i) - \underset{h}{D}(q^i)\,\underset{h}{D}(\xi) = X^{(1)}\Big(\frac{\partial H}{\partial p_i^+}\Big),\\
&\underset{h}{D}(\zeta_i) - \underset{h}{D}(p_i)\,\underset{h}{D}(\xi)\\
&= X^{(1)}\Big(-\frac{\partial H}{\partial q^i} + Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+) + \Lambda_i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)\Big),\\
&(\underset{h}{D}(\eta^i) - \underset{h}{D}(q^i)\,\underset{h}{D}(\xi))(Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)\\
&+ \Lambda_i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+))\\
&= X^{(1)}\Big[h_+\frac{\partial H}{\partial t} - H + h_-\frac{\partial H^-}{\partial t} + H^- - \underset{h}{D}(q^i)\\
&\times (Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+) + \Lambda_i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+))\Big],
\end{aligned} \tag{18}$$

infinitesimal transformations (15) are called Lie symmetry transformations of the discrete nonholonomic Hamiltonian system corresponding to the holonomic one.

Proof The invariance of canonical Hamiltonian equations (13) and energy equation (14) can be represented as

$$\begin{aligned}
&X^{(1)}\left(\underset{h}{D}(q^i)-\frac{\partial H}{\partial p_i^+}\right)=0,\\
&X^{(1)}\Bigg(\underset{h}{D}(p_i)+\frac{\partial H}{\partial q^i}-Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)\\
&-\Lambda_i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)\Bigg)=0,\\
&X^{(1)}\Bigg[h_+\frac{\partial H}{\partial t}-H+h_-\frac{\partial H^-}{\partial t}+H^-\underset{h}{D}(q^i)\\
&\times(Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)+\Lambda_i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+))\Bigg]=0. \quad (19)
\end{aligned}$$

Using Eq. (17), we have

$$\begin{aligned}
&X^{(1)}\left(\underset{h}{D}(q^i)-\frac{\partial H}{\partial p_i^+}\right)\\
&=\underset{h}{D}(\eta^i)-\underset{h}{D}(q^i)\underset{h}{D}(\xi)-X^{(1)}\left(\frac{\partial H}{\partial p_i^+}\right)=0. \quad (20)
\end{aligned}$$

Hence, we obtain the first equation of expression (18). Similarly, we have the second and third equations of expression (18)

$$\begin{aligned}
&X^{(1)}\Bigg(\underset{h}{D}(p_i)+\frac{\partial H}{\partial q^i}-Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)\\
&-\Lambda_i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)\Bigg)\\
&=\underset{h}{D}(\zeta_i)-\underset{h}{D}(p_i)\underset{h}{D}(\xi)+X^{(1)}\Bigg(\frac{\partial H}{\partial q^i}-Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)\\
&-\Lambda_i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)\Bigg)=0, \quad (21)\\
&X^{(1)}\Bigg[h_+\frac{\partial H}{\partial t}-H+h_-\frac{\partial H^-}{\partial t}+H^-\\
&-\underset{h}{D}(q^i)(Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)+\Lambda_i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+))\Bigg]\\
&=X^{(1)}\Bigg[h_+\frac{\partial H}{\partial t}-H+h_-\frac{\partial H^-}{\partial t}+H^-\\
&-\underset{h}{D}(q^i)(Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)+\Lambda_i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+))\Bigg]\\
&-(\underset{h}{D}(\eta^i)-\underset{h}{D}(q^i)\underset{h}{D}(\xi))(Q^i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)\\
&+\Lambda_i(t,t_+,\boldsymbol{q},\boldsymbol{p}^+))=0. \quad (22)
\end{aligned}$$

Equations in expression (18) are called the determining equations of Lie symmetry transformations of the discrete nonholonomic Hamiltonian system.

The invariant of constraint equation (5) under the Lie symmetry transformations can be represented in the form

$$X^{(1)}[f_\beta(t,t_+,\boldsymbol{q},\boldsymbol{p},\underset{h}{D}(q^i),\underset{h}{D}(p_i))]=0. \quad (23)$$

Equation (23) is called discrete restrict equation of the nonholonomic Hamiltonian system, which places restrictions on the constraint conditions under the infinitesimal transformations.

Definition 2 If generators ξ, η^i, and ζ_i satisfy the determining equations (18) and discrete restrict equation (23), the infinitesimal trandsformations (15) are called weak Lie symmetry transformations of the discrete nonholonomic Hamiltonian system.

According to Eqs. (6), (8), and (15), we can obtain

$$\delta q^i=\Delta q^i-\underset{h}{D}(q^i)\Delta t=\varepsilon(\eta^i-\underset{h}{D}(q^i)\xi)=\varepsilon\tilde{\eta}^i, \quad (24)$$

where $\tilde{\eta}^i=\eta^i-\underset{h}{D}(q^i)\xi$.

Now equation (5) and (6) under the infinitesimal transformations are written as

$$\frac{\partial f_\beta}{\partial\,\underset{h}{D}(q^i)}\tilde{\eta}^i=0,\quad \beta=0,\ldots,g. \quad (25)$$

Equation (25) is called discrete additional restrict equation, which is the discrete analogue of Appell–Chetaev's conditions with respect to discrete nonholonomic constraints restricting generators of infinitesimal Lie symmetry transformations.

Definition 3 If generators ξ, η^i, and ζ_i satisfy determining equation (18), discrete restrict equation (23), and discrete additional restrict equation (25), the infinitesimal transformations are called strong Lie symmetry transformations of the discrete nonholonomic Hamiltonian system.

5. Lie theorems of discrete nonholonomic Hamiltonian systems

Theorem 1 If the infinitesimal transformations are Lie symmetry transformations of the discrete nonholonomic Hamiltonian system corresponding to the holonomic one, and there is a gauge function $G=G(t,t_+,\boldsymbol{q},\boldsymbol{p}^+)$, which satisfies the following equation:

$$\begin{aligned}
&\zeta_i^+\underset{h}{D}(q^i)+p_i^+\underset{h}{D}(\eta^i)-X^{(1)}(H)-H\underset{h}{D}(\xi)\\
&+\underset{h}{D}(G_i)+(Q^i+\Lambda_i)[\eta^i-\underset{h}{D}(q^i)\xi]=0, \quad (26)
\end{aligned}$$

then we can obtain the following conservation relationship:

$$I = \eta^i p_i - \xi\left(H^- + h_- \frac{\partial H^-}{\partial t}\right) + G_i = \text{const.} \quad (27)$$

Proof The discrete derivative of Eq. (27) with respect to t can be written as

$$\begin{aligned} \underset{h}{D}(I) &= \underset{h}{D}\left(\eta^i p_i - \xi\left(H^- + h_- \frac{\partial H^-}{\partial t}\right) + G_i\right) \\ &= \underset{h}{D}(\eta^i p_i) - \underset{h}{D}(\xi H^-) \\ &\quad - \underset{h}{D}\left(\xi h_- \frac{\partial H^-}{\partial t}\right) + \underset{h}{D}(G_i). \end{aligned} \quad (28)$$

According to the discrete Leibniz rule (4), we can obtain

$$\begin{aligned} &\underset{h}{D}(\eta^i p_i) = \eta^i \underset{h}{D}(p_i) + p_i^+ \underset{h}{D}(\eta^i), \\ &\underset{h}{D}(\xi H^-) = H \underset{h}{D}(\xi) + \xi \underset{h}{D}(H^-), \\ &\underset{h}{D}\left(h_- \frac{\partial H^-}{\partial t}\right) = h_- \underset{h}{D}\left(\frac{\partial H^-}{\partial t}\right) + \frac{\partial H^-}{\partial t} \underset{h}{D}(h_-) \\ &+ \left(\frac{\partial H}{\partial t^+} - \frac{\partial H^-}{\partial t}\right) \underset{h}{D}(h_-) \\ &= \frac{h_-}{h_+}\left(\frac{\partial H}{\partial t^+} - \frac{\partial H^-}{\partial t}\right) + \frac{h_+ - h_-}{h_+} \frac{\partial H}{\partial t^+} \\ &= \frac{\partial H}{\partial t_+} - \frac{h_-}{h_+} \frac{\partial H^-}{\partial t}, \\ &\underset{h}{D}\left(\xi h_- \frac{\partial H^-}{\partial t}\right) = \xi \underset{h}{D}(h_- \frac{\partial H^-}{\partial t}) + h_- \frac{\partial H^-}{\partial t} \underset{h}{D}(\xi) \\ &+ (\xi_+ - \xi) \underset{h}{D}\left(h_- \frac{\partial H^-}{\partial t}\right) \\ &= \xi_+\left(\frac{\partial H}{\partial t_+} - \frac{h_-}{h_+} \frac{\partial H^-}{\partial t}\right) + \frac{h_-}{h_+} \frac{\partial H^-}{\partial t}(\xi_+ - \xi) \\ &= \xi_+ \frac{\partial H}{\partial t_+} - \xi \frac{h_-}{h_+} \frac{\partial H^-}{\partial t}, \end{aligned} \quad (29)$$

by Eqs. (13), (14), and (29), and we can find that $\underset{h}{D}(I) = 0$. Equation (26) is called Lie symmetry structure equation of the discrete nonholonomic Hamiltonian systems.

Theorem 2 If infinitesimal transformations in expression (15) are the weak Lie symmetry transformations, and there is a gauge function $G = G(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+)$, which satisfies Eq. (26), we can obtain conservation relationship (27).

Theorem 3 If the infinitesimal transformations in expression (15) are the strong Lie symmetry transformations, and there is a gauge function $G = G(t, t_+, \boldsymbol{q}, \boldsymbol{p}^+)$, which satisfies the equation (26), we can obtain conservation relationship (27).

6. Applications

Considering a dynamical system with Hamiltonian function

$$H = \frac{1}{2} p^{+2} + q, \quad (30)$$

subject to the nonholonomic constraint

$$f = \frac{h_+}{\underset{h}{D}(q)} - \underset{h}{D}(q) - h_+ + p = 0. \quad (31)$$

According to Eq. (14), we can obtain the discrete generalized Hamiltonian canonical equations

$$\underset{h}{D}(q) = p^+, \quad \underset{h}{D}(p) = -1 + \lambda \frac{\partial f}{\partial \underset{h}{D}(q)}. \quad (32)$$

And we can calculate the constraint multiplier λ from Eqs. (5), (13), and (14) as

$$\lambda = -\frac{p^+}{(p^+)^2 + h_+}, \quad (33)$$

so Hamiltonian equations in expression (32) can be rewritten as

$$\underset{h}{D}(q) = p^+, \quad \underset{h}{D}(p) = -1 + \frac{1}{p^+}, \quad (34)$$

on the lattice there exists

$$\frac{1}{2} p^{+2} - \frac{1}{2} p^2 + q - q_- + h_+ = 0, \quad (35)$$

which can be obtained by the energy equation.

Using determining equations in expression (18), we can have the following equations:

$$\begin{aligned} &\underset{h}{D}(\eta) - \underset{h}{D}(q) \underset{h}{D}(\xi) = \underset{h}{D}(\eta) - p^+ \underset{h}{D}(\xi) \\ &= X^{(1)}\left(\frac{\partial H}{\partial p^+}\right) = X^{(1)}(p^+) = \zeta_+, \\ &\underset{h}{D}(\zeta) - \underset{h}{D}(p) \underset{h}{D}(\xi) = \underset{h}{D}(\zeta) - \left(-1 + \frac{1}{p^+}\right) \underset{h}{D}(\xi) \\ &= X^{(1)}\left(-\frac{\partial H}{\partial q} + \frac{1}{p^+}\right) = -\frac{1}{p^{+2}} \zeta_+, \\ &p^+ \zeta_+ - p\zeta + \eta - \eta_- + \xi_+ - \xi = 0. \end{aligned} \quad (36)$$

Then we can obtain three special solutions of Eq. (36):

$$\xi_1 = 1, \quad \eta_1 = 0, \quad \zeta_1 = 0, \quad (37)$$

$$\xi_2 = 0, \quad \eta_2 = 1, \quad \zeta_2 = 0, \quad (38)$$

$$\xi_3 = 1, \quad \eta_3 = 1, \quad \zeta_3 = 0. \quad (39)$$

Substituting the solutions of ξ, η, and ζ into the structure equation (26), we can obtain

$$G_1 = t, \quad G_2 = -p, \quad G_3 = -p + t. \quad (40)$$

According to Theorem 1, we have the first integral:

$$I_1 = -\frac{1}{2}p^2 - q_- + t, \quad I_2 = 0,$$
$$I_3 = -\frac{1}{2}p^2 - q_- + t. \tag{41}$$

Note that $I_1 = I_3$, so solutions (37) and (39) of the infinitesimal transformations result in the same conservation relationships. Because of $I_2 = 0$, solution (38) is called a trivial one which does not result in the conservation of the system. Hence, for the discrete nonholonomic Hamiltonian systems, Lie symmetries do not certainly result in invariance.

Discrete restrict equation (23) can be written as

$$(\underset{h}{D}(\eta) - p^+ \underset{h}{D}(\xi))\left(- \frac{h_+}{p^{+2}} - 1\right)$$
$$+ (\xi_+ - \xi)\left(- 1 + \frac{1}{p^+}\right) + \zeta = 0. \tag{42}$$

We can find that solutions (37) and (39) both satisfy Eq. (42), so the infinitesimal transformations are weak Lie symmetry transformations, and we can obtain the conservation relationships I_1 and I_3 by Theorem 2.

Using discrete canonical Hamiltonian equation (34), the constraint equation can be rewritten as

$$\frac{h_+}{p^+} - p^+ - h_+ + p = 0. \tag{43}$$

Then we can find that solutions (37) and (39) satisfy discrete additional restrict equation (25), so the infinitesimal transformations are strong generalized Lie symmetry transformations, then we can obtain the conservation relationships I_1 and I_3 by Theorem 3.

7. Conclusion

In this paper, the symmetry theory and the conservation theory are both extended to discrete nonholonomic Hamiltonian system, and developed further consummately, which will be convenient and useful for us to solve problems in different cases.

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