Exact scaling for the mean first-passage time of random walks on a generalized Koch network with a trap*

Zhang Jing-Yuan(张静远)[a], Sun Wei-Gang(孙伟刚)[a]†, and Chen Guan-Rong(陈关荣)[b]

[a] *School of Science, Hangzhou Dianzi University, Hangzhou* 310018, *China*

[b] *Department of Electronic Engineering, City University of Hong Kong, Hong Kong SAR, China*

(Received 13 November 2011; revised manuscript received 9 December 2011)

In this paper, we study the scaling for the mean first-passage time (MFPT) of the random walks on a generalized Koch network with a trap. Through the network construction, where the initial state is transformed from a triangle to a polygon, we obtain the exact scaling for the MFPT. We show that the MFPT grows linearly with the number of nodes and the dimensions of the polygon in the large limit of the network order. In addition, we determine the exponents of scaling efficiency characterizing the random walks. Our results are the generalizations of those derived for the Koch network, which shed light on the analysis of random walks over various fractal networks.

Keywords: mean first-passage time, random walks, Koch networks

PACS: 89.75.Hc, 05.40.Fb, 05.10.–a **DOI:** 10.1088/1674-1056/21/3/038901

1. Introduction

Recently, deterministically growing networks, especially deterministic small-world and scale-free network models, have attracted considerable attention.[1,2] The main reason is that they provide good analytical results for topological properties and dynamical processes.[3−8] In particular, fractal networks constructed by some fractals, such as Koch fractals, Sierpinski fractals, and Vicsek fractals, have been widely studied.[9−14] Research shows that they have small-world[15] and scale-free[16] characteristics, which show a profound impact on the dynamical processes taking place on the networks.[17,18]

Among various dynamical processes, random walks on networks are the most fundamental,[19,20] and are related to a plethora of other dynamics, such as particle transport, disease spreading, neuron firing, and target search, to name just a few. Random walks are useful for studying the topological structures of complex networks, such as average path length, betweenness, and community detection. In addition, the rich topology of a generic network can have dramatic effects on the properties of a random walk. In particular, trapping is an integral major scenario of a random walk, which is relevant to a wide range of applications.[21] The primarily interesting quantity closely related to the trapping problem is the mean first-passage time (MFPT),[22,23] i.e., the average of the first-passage time defined as the expected time for a walker to first reach a given destination starting from a source point. A central issue on the random walk is how the MFPT scales with the network size. The intrinsic relationships between the scaling behaviors of the random walks and the topological structures of the networks are also very important.[24]

In Ref. [21], the Koch network was proposed. In our previous work,[9] we put forward generalized Koch networks and transformed the initial state of the Koch network from a triangle to an m-polygon. The Koch network is a special case of our generalized networks (with $m = 3$). So far, the relation between the network topology and the network dynamics has been widely studied, however those studies focus on some simple network structures. However, for the generalized Koch networks, the exact scaling and the relation between MFPT and m are rarely studied and reported. In this article, we study the random walks and the trapping on the generalized Koch networks and obtain the exact solution of the MFPT, which shows that the MFPT grows linearly with the increasing order of the networks, and the MFPT increases with network

*Project supported by the Research Foundation of Hangzhou Dianzi University, China (Grant Nos. KYF075610032 and zx100204004-7) and the Hong Kong Research Grants Council, China (Grant No. CityU 1114/11E).

†Corresponding author. E-mail: wgsun999@yahoo.com.cn

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parameter m. Our study provides theoretical analyses of the physical characteristics of deterministically growing networks.

2. Generalized Koch networks

The generalized Koch networks introduced in Ref. [9] are the generalization of the Koch network.[21] The algorithm generating the generalized Koch networks, denoted as $G_{\rm K}(g)$ at step g ($g \geq 0$), is described as follows. At the initial step $g = 0$, $G_{\rm K}(0)$ consists of m ($m \geq 3$) nodes forming an m-polygon. At step $g \geq 1$, $G_{\rm K}(g)$ is obtained from $G_{\rm K}(g-1)$, every existing node in $G_{\rm K}(g)$ produces $(m-1)$ new nodes, and these $(m-1)$ new nodes and their mother node are linked together to create a new m-polygon. Figure 1 shows the generating process of $G_{\rm K}(g)$ at the first two steps.

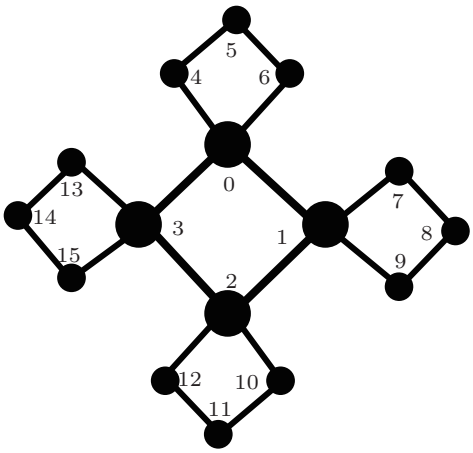


Fig. 1. $G_{\rm K}(1)$ with $m = 4$, trap $n_{\rm T}$ is located at node 0.

According to the network construction, the total node number $V \equiv V_g$ in $G_{\rm K}(g)$ is

$$V_g = (m-1)(m+1)^g + 1, \tag{1}$$

and the degree $k_i(g)$ of node i at step g produced at step g_0 ($g_0 \geq 0$) is

$$k_i(g) = 2^{g+1-g_0} = 2k_i(g-1). \tag{2}$$

Compared to the Koch network, the generalized Koch networks have some typical characteristics of real-life networks with the power-law degree distribution $P(k) \sim k^{-\gamma}$, where $\gamma = 1 + \ln(m+1)/\ln 2$. Moreover, the average path length increases logarithmically with the network order.[9]

3. First-passage properties

To obtain an analytic formulation of the MFPT for $G_{\rm K}(g)$, we first label all the nodes belonging to $G_{\rm K}(g)$, where the initial m nodes in $G_{\rm K}(0)$ are labeled as $0, 1, \ldots, (m-1)$ respectively. Then, in each step, only the new nodes created at the current step are labeled, while the labels of all old nodes remain unchanged, i.e., the new nodes are labeled as $M+0, M+1, \ldots, M+\Delta(M-1)$, where M is the total number of the pre-existing nodes, and $\Delta(M-1)$ is the number of the newly created nodes. At time g, all nodes are labeled from 0 to $V_g = (m-1)(m+1)^g$, as shown by Fig. 1. Meanwhile, we locate the trap $n_{\rm T}$ at node 0.

A typical problem of random walks with a trap is as follows. At each step, the walker, starting from any node of $G_{\rm K}(g)$ except the trap, jumps to any of its nearest neighbors with an equal probability. The jumping probability from node i to node j denoted by p_{ij} is defined as

$$p_{ij} = (d_i)^{-1} a_{ij} = (\boldsymbol{D}^{-1}\boldsymbol{A})_{ij}, \tag{3}$$

where $\boldsymbol{A} = (a_{ij})_{V\times V}$ is the adjacency matrix of $G_{\rm K}(g)$, and $\boldsymbol{D}$ is the diagonal degree matrix of $\boldsymbol{A}$. As a matter of fact, $\boldsymbol{P} = (p_{ij})_{V\times V}$ is the state transition matrix of the jumping process. The first-passage time T_i is defined as the expected time steps for the walker located at node i ($i \neq n_{\rm T}$) to first reach the trap. Then, T_i obeys the following recurrence equation:

$$T_i = \sum_j p_{ij} T_j + 1. \tag{4}$$

Equation (4) can be recast in a matrix form as

$$\boldsymbol{T} = \boldsymbol{C}\boldsymbol{e}, \tag{5}$$

where $\boldsymbol{C} = (\boldsymbol{I} - \boldsymbol{P}')^{-1}$, $\boldsymbol{I}$ is an identity matrix of order $(V_g - 1) \times (V_g - 1)$, matrix $\boldsymbol{P}'$ is obtained from the transition matrix by deleting the first row and the first column that correspond to the trap node, and $\boldsymbol{e}$ is the $(V_g - 1)$-dimensional unit vector $(1, 1, \ldots, 1)^{\rm T}$.

Given the trap $n_{\rm T}$ at node 0, the walker will be inevitably absorbed.[17] The MFPT $\langle T \rangle_g$ to be absorbed by the trap, namely the average of T_i over all nodes in $G_{\rm K}(g)$ except the trap, is given by

$$\langle T \rangle_g = \sum_{i=2}^{V} T_i = \sum_{i=1}^{V-1}\sum_{j=1}^{V-1} C_{ij}. \tag{6}$$

It is noted that the order of $\boldsymbol{C}$ is $(V_g-1)\times(V_g-1)$, and V_g increases exponentially with step g (see Eq. (1)).

Hence, for a large g, it is difficult to obtain $\langle T\rangle_g$ through a direct calculation using Eq. (5). The MFPT measures the efficiency of the trapping process, the smaller the MFPT, the higher the efficiency.

4. An exact solution for MFPT

In this section, we establish the first-passage time to be absorbed by the trap of $G_{\rm K}(g)$, which dominates the evolution scaling of T_i^g at step g, where T_i^g is the first-passage time to be absorbed by the trap for a walker originating at node i in $G_{\rm K}(g)$.

4.1. First-passage time at the first step

It is obvious that $T_0^g = 0$ for all $g \geq 0$. Since $G_{\rm K}(0)$ is an m-polygon, the first-passage time $T_n^0 = y_n$ of node n at the initial state is

$$y_n = \frac{1}{2}(y_{n+1}+1) + \frac{1}{2}(y_{n-1}+1), \tag{7}$$

with the initial conditions of $y_0 = 0$ and $y_m = 0$ for $1 \leq n \leq m-1$. The solution of Eq. (7) is

$$y_n = mn - n^2, \quad 0 \leq n \leq m-1. \tag{8}$$

When $m = 3$, the values of T_1^0 and T_2^0 are 2, which is the same as those obtained in Ref. [21].

4.2. Evolution scaling for the first-passage time

Consider an arbitrary node i in $G_{\rm K}(g)$. The mean transmit time for a random walker going from node i to any of its k_i old neighbors is denoted as X. Similarly, the mean transmit time for going from any of its k_i new neighbors to one of the k_i old ones is Y, and that for going from node A_n to one of the k_i old neighbors is a_n, where $0 \leq n \leq (m-1)$, nodes $A_0, A_1, \ldots, A_{m-1}$ form an m-polygon, node A_0 is node i, and nodes A_1 and A_{m-1} are the two new neighbors of node i, as shown by Fig. 2.

Hence, we can establish the following underlying backward equations:

$$X = \frac{1}{2} + \frac{1}{2}(1+Y), \tag{9}$$

$$Y = \frac{1}{2}(1+X) + \frac{1}{2}(1+a_{m-2}), \tag{10}$$

$$a_n = \frac{1}{2}(a_{n+1}+1) + \frac{1}{2}(a_{n-1}+1), \tag{11}$$

with the initial conditions of $a_1 = Y$ and $a_{m-1} = Y$ for $2 \leq n \leq m-2$. From Eq. (11), we obtain

$$\begin{aligned} a_{m-2} &= m(m-2) + \frac{Y + (m-1)(m-2-Y)}{2-m} \\ &\quad - (m-2)^2 \\ &= Y + m - 3. \end{aligned} \tag{12}$$

Inserting Eq. (12) into Eq. (11) yields $X = m+1$. Therefore, the passage time from any node i to any node j increases by a factor of $m+1$, i.e.,

$$T_i^{g+1} = (m+1)T_i^g, \tag{13}$$

which is useful for deriving an analytic formula for the MFPT below.

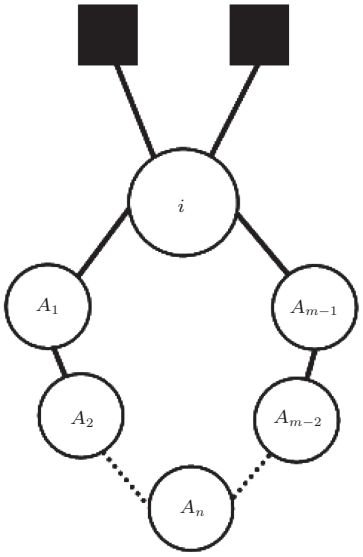


Fig. 2. Node $i \in G_{\rm K}(g)$ has k_i neighbor nodes (■) at step g and k_i new neighbor nodes (◯) at step $g+1$. Nodes A_0 (node i), A_1, …, A_{m-1} form an m-polygon.

4.3. Analytic formula for the MFPT

The set of nodes in $G_{\rm K}(g)$ is denoted as Ω_g, and the set of nodes created at step g is denoted as $\bar{\Omega}_g$. Obviously, $\Omega_g = \Omega_{g-1} \cup \bar{\Omega}_g$. Define the following quantities for $n \leq g$:

$$T_{n,\rm sum}^g = \sum_{i\in\Omega_n} T_i^g, \tag{14}$$

$$\bar{T}_{n,\rm sum}^g = \sum_{i\in\bar{\Omega}_n} T_i^g, \tag{15}$$

then

$$T_{g,\rm sum}^g = T_{g-1,\rm sum}^g + \bar{T}_{g,\rm sum}^g. \tag{16}$$

The average of the first-passage time is

$$\langle T\rangle_g = \frac{1}{V_g - 1} T_{g,\rm sum}^g, \tag{17}$$

where $T_0^g = 0$. To this end, we first calculate $\bar{T}_{g,\rm sum}^g$. At step $g = 1$, $\bar{\Omega}_1 = \{m, m+1, \ldots, 2m-2, 2m-1, \ldots, 3m-3, \ldots, m(m-1)+m-1\}$. Given nodes

$0, m, m+1, \ldots, 2m-2$ form an m-polygon. The $T_0, T_m, T_{m+1}, \ldots, T_{2m-2}$ are exactly the m solutions of the following difference equations:

$$x_n = \frac{1}{2}(x_{n+1}+1) + \frac{1}{2}(x_{n-1}+1), \tag{18}$$

with the initial conditions of $x_0 = T_0^1$ and $x_m = T_0^1$ for $0 \le n \le (m-1)$. From Eq. (18), we get $x_n = mn - n^2 + T_0^1$ for $0 \le n \le m$. Thus,

$$\begin{aligned}
&(T_m + T_{m+1} + \cdots + T_{2m-2}) \\
&= \sum_{i=1}^{m-1} x_i \\
&= \sum_{i=1}^{m-1} (mi - i^2 + T_0^1) \\
&= \frac{1}{6}m(m^2-1) + (m-1)T_0^1.
\end{aligned} \tag{19}$$

Proceeding analogously, we obtain

$$\begin{aligned}
&T_{i(m-1)+m-1+1} + T_{i(m-1)+m-1+2} + \cdots \\
&+ T_{(i+1)(m-1)+m-1} \\
&= \frac{1}{6}m(m^2-1) + (m-1)T_i^1, \\
&\quad 1 \le i \le (m-1).
\end{aligned} \tag{20}$$

Through the construction of $G_{\mathrm{K}}(1)$, we find that

$$\begin{aligned}
\bar{T}_{1,\mathrm{sum}}^1 &= \sum_{i\in\bar{\Omega}_1} T_i^1 \\
&= (T_m + T_{m+1} + \cdots + T_{2m-2}) \\
&\quad + (T_{2m-1} + \cdots + T_{3m-3}) \\
&\quad + \cdots + (T_{(m-1)(m-1)+m-1+1} + \cdots \\
&\quad + T_{m(m-1)+m-1}) \\
&= \frac{1}{6}m^2(m^2-1) + (m-1)\sum_{i=0}^{m-1} T_i^1 \\
&= \frac{1}{6}m^2(m^2-1) + (m-1)\bar{T}_{0,\mathrm{sum}}^1.
\end{aligned} \tag{21}$$

At step $g = 2$, we have $\Omega_2 = \bar{\Omega}_0 \cup \bar{\Omega}_1 \cup \bar{\Omega}_2$. For node $i \in \bar{\Omega}_0$, its degree is $k_i(2) = 8$. And there are $k_i(2)/2 = 4$ m-polygons attached to it, two of which are new m-polygons (an m-polygon is new if it is created at the current step). From Eqs. (19) and (20), we find that each of the new m-polygons attaching to node i contributes to the quantities of $\frac{1}{6}m(m^2-1)$ and $(m-1)T_i^2$ for $i \in \bar{\Omega}_0$. Then, the two new m-polygons attaching to node i contribute to the quantities of $2\times\frac{1}{6}m(m^2-1)$ and $2\times(m-1)T_i^2$ for $i \in \bar{\Omega}_0$. For $i \in \bar{\Omega}_1$, it is the same. Thus, we have

$$\begin{aligned}
\bar{T}_{2,\mathrm{sum}}^2 &= (m-1)\times\sum_{i\in\bar{\Omega}_1} T_i^2 + 2(m-1)\times\sum_{i\in\bar{\Omega}_0} T_i^2 \\
&\quad + [(m+1)^2 - (m+1)] \\
&\quad \times \frac{1}{6}(m-1)m(m+1) \\
&= (m-1)\bar{T}_{1,\mathrm{sum}}^2 + 2(m-1) \\
&\quad \times \bar{T}_{0,\mathrm{sum}}^2 + \frac{1}{6}(m-1)m^2(m+1)^2.
\end{aligned} \tag{22}$$

Furthermore, we have

$$\begin{aligned}
\bar{T}_{g,\mathrm{sum}}^g &= (m-1)\times\bar{T}_{g-1,\mathrm{sum}}^g + 2(m-1)\times\bar{T}_{g-2,\mathrm{sum}}^g \\
&\quad + 2^2(m-1)\times\bar{T}_{g-3,\mathrm{sum}}^g + \cdots \\
&\quad + 2^{g-1}(m-1)\times\bar{T}_{0,\mathrm{sum}}^g \\
&\quad + \left[(m+1)^g - (m+1)^{g-1}\right] \\
&\quad \times \frac{1}{6}m(m^2-1)
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
\bar{T}_{g+1,\mathrm{sum}}^{g+1} &= (m-1)\bar{T}_{g,\mathrm{sum}}^{g+1} + 2(m-1)\bar{T}_{g-1,\mathrm{sum}}^{g+1} \\
&\quad + 2^2(m-1)\times\bar{T}_{g-2,\mathrm{sum}}^{g+1} + \cdots \\
&\quad + 2^g(m-1)\bar{T}_{0,\mathrm{sum}}^{g+1} \\
&\quad + \left[(m+1)^{g+1} - (m+1)^g\right] \\
&\quad \times \frac{1}{6}m(m^2-1),
\end{aligned} \tag{24}$$

where $(m+1)^g - (m+1)^{g-1}$ and $(m+1)^{g+1} - (m+1)^g$ are the numbers of m-polygons generated at steps g and $g+1$, respectively. Using Eq. (24) to subtract Eq. (23) and then multiplied by $2(m+1)$, and making use of Eq. (13), we obtain

$$\begin{aligned}
&\bar{T}_{g+1,\mathrm{sum}}^{g+1} - 2(m+1)\bar{T}_{g,\mathrm{sum}}^g \\
&= (m-1)\bar{T}_{g,\mathrm{sum}}^{g+1} \\
&\quad + \frac{1}{6}m(m^2-1)[m(m+1)^g - 2(m+1) \\
&\quad \times m(m+1)^{g-1}],
\end{aligned} \tag{25}$$

which can be rewritten as

$$\begin{aligned}
\bar{T}_{g+1,\mathrm{sum}}^{g+1} &= (m+1)^2\bar{T}_{g,\mathrm{sum}}^g \\
&\quad - \frac{1}{6}(m-1)m^2(m+1)^{g+1}.
\end{aligned} \tag{26}$$

Using $\bar{T}_{1,\mathrm{sum}}^1 = \frac{1}{6}m(m^2-1)(m^2+m-1)$, we can write the solution of Eq. (25) as

$$\bar{T}_{g,\mathrm{sum}}^g = c(m+1)^{2g} + \frac{m(m-1)}{6}(m+1)^g, \tag{27}$$

where $c = \frac{m(m-1)(m^2+m-2)}{6(m+1)}$. Substituting Eq. (27) into Eq. (16) gives

$$\begin{aligned}
T_{g,\mathrm{sum}}^g &= T_{g-1,\mathrm{sum}}^g + C(m+1)^{2g} \\
&\quad + \frac{1}{6}m(m-1)(m+1)^g \\
&= (m+1)T_{g-1,\mathrm{sum}}^{g-1} + C(m+1)^{2g}
\end{aligned}$$

$$+\frac{1}{6}m(m-1)(m+1)^g. \quad (28)$$

Using the initial condition of $T^0_{0,\mathrm{sum}} = \frac{1}{6}m(m^2-1)$, we obtain

$$T^g_{g,\mathrm{sum}} = \frac{(m-1)(m+1)^g}{6} \times [2+(m^2+m-2)(m+1)^g+mg]. \quad (29)$$

Plugging Eq. (29) into Eq. (17), we arrive at the accurate formula

$$\langle T\rangle_g = \frac{1}{V_g-1}T^g_{g,\mathrm{sum}} = \frac{1}{3}+\frac{1}{6}(m^2+m-2)(m+1)^g+\frac{1}{6}mg \quad (30)$$

for the average of the mean time to be absorbed by the trap located at node 0 on the generalized Koch network $G_{\mathrm{K}}(g)$. When $m=3$, $\langle T\rangle_g = \frac{1}{3}+\frac{g}{2}+\frac{5}{3}\times 4^g$, which is the same as that given in Ref. [21].

Next, we express the MFPT as a function of the network order to obtain the scaling between these two quantities. From Eq. (1), we have $g = \log_{m+1}\left(\frac{V_g-1}{m-1}\right)$. Hence, Eq. (30) can be written as

$$\langle T\rangle_g = \frac{1}{3}+\frac{1}{6}(m+2)(V_g-1) + \frac{1}{6}m\log_{m+1}\left(\frac{V_g-1}{m-1}\right). \quad (31)$$

When $V_g \to \infty$, the mean trapping time $\langle T\rangle_g$ becomes

$$\langle T\rangle_g \approx \frac{1}{6}(m+2)V_g. \quad (32)$$

Thus, the mean trapping time grows linearly with the increasing order of the network and parameter m. Numerical results are shown in Fig. 3.

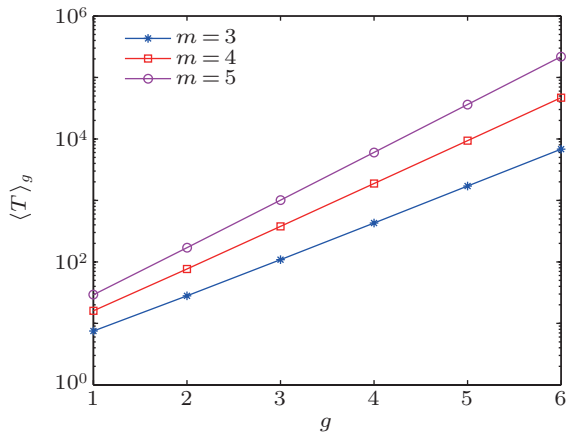


Fig. 3. (colour online) Mean trapping time $\langle T\rangle_g$ versus step g.

5. Conclusion

In the present study, we have investigated the random walks on a generalized Koch network with a trap located at a hub node and derived the exact scaling of the mean first-passage time. We have also shown that the MFPT increases linearly with the number of the network nodes and parameter m. The obtained results are the generalizations of those in Ref. [21]. Our findings are insightful for the study of the random walks on various deterministically growing networks.

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