Approximate solution of the magneto-hydrodynamic flow over a nonlinear stretching sheet*

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The approximate solution of the magneto-hydrodynamic (MHD) boundary layer flow over a nonlinear stretching sheet is obtained by combining the Lie symmetry method with the homotopy perturbation method. The approximate solution is tabulated, plotted for the values of various parameters and compared with the known solutions. It is found that the approximate solution agrees very well with the known numerical solutions, showing the reliability and validity of the present work.

Keywords: magneto-hydrodynamic (MHD) boundary layer flow, Lie symmetry method, homotopy perturbation method, approximate solution

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1. Introduction

Nonlinear phenomena, which appear in many scientific fields, such as plasma physics, solid state physics, chemical kinetics and fluid mechanics, can be modeled by solving nonlinear boundary layer differential equations. The boundary layer flow of an incompressible viscous fluid over a continuously stretching sheet is often encountered in many engineering and industrial processes. In particular, the magneto-hydrodynamic (MHD) boundary layer flow of viscous fluids has been a subject of great interest due to its widespread applications in designing cooling systems with liquid metals, accelerators, MHD generators, flow meters and MHD pumps. Many nonlinear models have been proposed to explain the behaviours of the viscous MHD flow under different boundary conditions.[1,2]

We consider the MHD flow of an incompressible viscous fluid over a stretching sheet at $y = 0$.[1,3,4] The fluid is electrically conducting under the influence of an applied magnetic field $B(x)$ normal to the stretching sheet. The induced magnetic field is neglected. The resulting boundary layer governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0,$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{\sigma B^2(x)}{\rho}u, \tag{1}$$

where u and v are the velocity components in the x and y directions respectively, ν is the kinematic viscosity, ρ is the fluid density and σ is the electrical conductivity of the fluid. In Eq. (1), the external electric field and the polarization effects are negligible and

$$B(x) = B_0 x^{(m-1)/2}. \tag{2}$$

The boundary conditions corresponding to the nonlinear stretching of a sheet are

$$u(x,0) = \nu x^m, \quad v(x,0) = 0,$$
$$u(x,y) \to 0 \quad (y \to \infty). \tag{3}$$

Recently, Hayat *et al.*[1] investigated the numerical solution of the MHD boundary layer flow (1)–(3) by employing the modified Adomian decomposition method and the Padé approximation. Later, Ghotbi[3] found the approximate solution for the problem by using the homotopy analysis method. Ganji *et al.*[4] obtained the analytical solution of the MHD boundary layer flow by applying the homotopy perturbation method (HPM) and the Padé approximation. In the past, many researchers have employed complex transformations to reduce the boundary values for partial

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differential equations into initial values for ordinary differential equations, which are usually solved by using various methods.[5−12] In the present article, based on the Lie symmetry method and the HPM, we focus on the approximate analytical solution of a boundary values problem (BVP) of the MHD flow over a nonlinear stretching sheet. We will apply the Lie symmetry group method to reduce the MHD boundary layer problem (1)–(3) into an initial values problem (IVP) of an ordinary differential equation. Then, a highly accurate approximate solution of the IVP is obtained by using the HPM.

2. Symmetry reduction of the governing equations

We introduce the non-dimensional stream function[13] $\psi(x,y)$, which is related to the components of the velocity field by the equations

$$u=\nu\frac{\partial\psi}{\partial y},\quad v=-\nu\frac{\partial\psi}{\partial x}. \tag{4}$$

Inserting Eqs. (4) and (2) into the boundary layer Eq. (1) for the MHD flow, we obtain

$$\psi_{yyy}+\psi_{yy}\psi_x-\psi_y\psi_{xy}-\frac{\sigma B_0^2x^{m-1}}{\rho\nu}\psi_y=0. \tag{5}$$

The associated boundary conditions can be written as

$$\begin{aligned}&y=0:\psi_y=x^m,\quad \psi_x=0,\\ &y\to\infty:\psi_y\to 0.\end{aligned} \tag{6}$$

In order to convert BVP (5), (6) into an IVP of an ordinary differential equation, we will look for any possible symmetry group of the boundary value problem described by Eq. (5) and boundary conditions (6). The infinitesimal generator associated with the possible symmetry for the system (5), (6) has the general form[14]

$$\begin{aligned}X=&\xi(x,y,\psi)\frac{\partial}{\partial x}+\tau(x,y,\psi)\frac{\partial}{\partial y}\\ &+\eta(x,y,\psi)\frac{\partial}{\partial\psi}.\end{aligned} \tag{7}$$

Then, we obtain the following infinitesimal functions for Eq. (5) by using the method given in Refs. [15]–[17]:

$$\xi=C_1x,\quad \tau=\frac{1}{2}C_1(1-m)y+A(x),$$

$$\eta=\frac{1}{2}C_1(1+m)\psi+C_2, \tag{8}$$

where C_1 and C_2 are two arbitrary symmetry parameters and $A(x)$ is an arbitrary function. To obtain the symmetry of the boundary conditions (6), starting with the invariance condition[13,14] to the boundary surface $y=0$, we can write the boundary surfaces of the problem under study as follows:

$$\theta_1(x,y)=y,\quad \theta_2(x,y)=y-K,\ K\in\mathbb{R}^+. \tag{9}$$

Therefore, the invariance requirement for the former takes the form

$$X(\theta_1)=0\mid_{\theta_1=0}\quad\Rightarrow\quad\tau(x,0)=0. \tag{10}$$

We easily obtain $A(x)=0$ from (8) and (10). For simplicity, we set $C_1=-1$ and $C_2=0$, which leads to the following scaling symmetry for Eqs. (5) and (6):

$$\begin{aligned}&\xi=-x,\quad \tau=\frac{1}{2}(m-1)y,\\ &\eta=-\frac{1}{2}(m+1)\psi,\end{aligned} \tag{11}$$

and the infinitesimal generator

$$X=-x\frac{\partial}{\partial x}+\frac{1}{2}(m-1)y\frac{\partial}{\partial y}-\frac{1}{2}(m+1)\psi\frac{\partial}{\partial\psi}. \tag{12}$$

The invariant solution of (5) corresponding to the symmetry X is

$$\psi(x,y)=x^{(m+1)/2}\omega(\zeta), \tag{13}$$

where $\omega(\zeta)$ is an arbitrary function of the following similarity variable:

$$\zeta=x^{(m-1)/2}y. \tag{14}$$

After substituting the invariant form (13) into Eq. (5), we find that $\omega(\zeta)$ satisfies the ordinary differential equation

$$\begin{aligned}\omega'''(\zeta)+&\frac{m+1}{2}\omega(\zeta)\omega''(\zeta)-m\omega'^2(\zeta)\\ &-\frac{\sigma B_0^2}{\rho\nu}\omega'(\zeta)=0,\end{aligned} \tag{15}$$

with the following boundary conditions:

$$\omega(0)=0,\quad \omega'(0)=1,\quad \omega'(\infty)=0. \tag{16}$$

Under the transformation

$$\omega(\zeta)=\frac{2}{m+1}f(\zeta),\ m\neq-1, \tag{17}$$

Eq. (15) and its boundary conditions (16) become

$$f'''(\zeta)+f(\zeta)f''(\zeta)-\beta f'^2(\zeta)-Mf'(\zeta)=0, \tag{18}$$

and

$$f(0)=0,\quad f'(0)=1,\quad f'(\infty)=0, \tag{19}$$

where $\beta = 2m/(m+1)$ and $M = \sigma B_0^2/(\rho\nu)$. The function $f(\zeta)$ denotes the dimensionless stream function, and $f'(\zeta)$ is the dimensionless tangential velocity. Equation (18) describes the accelerated flow while $\beta > 0$ and the decelerated flow with separation while $\beta < 0$.

3. Solution by using the HPM

The principles of the HPM and its applicability to various kinds of differential equations are given in Refs. [8], [9], and [18]–[23]. Recently, the natural convection flow of a third grade fluid through a circular tube has been successfully computed using a homotopy perturbation formulation by Ariel.[24] We find that the solution of Eqs. (18) and (19) found through the HPM is in agreement with the ideas of Refs. [18] and [23] and Ariel's homotopy perturbation formulation. We suppose a simple homotopy perturbation formula by rewriting IVP (18), (19) as follows:

$$f'''(\zeta) - Mf'(\zeta) + \theta[f(\zeta)f''(\zeta) - \beta f'^2(\zeta)] = 0,\quad \theta\in[0,1], \tag{20}$$

with the initial conditions

$$f(0)=0,\quad f'(0)=1,\quad f'(\infty)=0. \tag{21}$$

When $\theta = 0$, Eq. (20) becomes a linearized equation, $f'''(\zeta) - Mf'(\zeta) = 0$; when $\theta = 1$, it becomes the original nonlinear one. In light of the HPM, we assume that the solution to Eq. (20) may be written as a power series in θ as

$$f(\zeta) = \sum_{i=0}^{\infty} \theta^i f_i(\zeta). \tag{22}$$

Substituting Eq. (22) into Eq. (20) and equating the terms with the identical powers of θ, we obtain the following differential equations:

$$\theta^0 : f_0'''(\zeta) - Mf_0'(\zeta) = 0,\quad f_0(0)=0,\quad f_0'(0)=1,\quad f_0'(\infty)=0, \tag{23}$$

$$\theta^1 : f_1'''(\zeta) - Mf_1'(\zeta) + \theta(f_0(\zeta)f_0''(\zeta) - \beta f_0'^2(\zeta)) = 0,\quad f_1(0)=0,\ f_1'(0)=0,\ f_1'(\infty)=0. \tag{24}$$

The solution of Eq. (23) can be directly derived as follows:

$$f_0(\zeta) = \frac{1-\mathrm{e}^{-\sqrt{M}\zeta}}{\sqrt{M}}, \tag{25}$$

where $M > 0$.

Substituting Eq. (25) into Eq. (24), we have

$$f_1'''(\zeta) - Mf_1'(\zeta) = (\beta-1)\,\mathrm{e}^{-2\sqrt{M}\zeta} + \mathrm{e}^{-\sqrt{M}\zeta}. \tag{26}$$

We can solve f_1 from Eq. (26) under the initial conditions $f_1(0) = 0, f_1'(0) = 0$ and the boundary condition $f_1'(\infty) = 0$ and obtain the following solution:

$$f_1(\zeta) = \frac{\mathrm{e}^{-2\sqrt{M}\zeta}}{6M^{3/2}}\big(-(\beta+2)\,\mathrm{e}^{2\sqrt{M}\zeta} + \mathrm{e}^{\sqrt{M}\zeta} \times (3\sqrt{M}\zeta + 2\beta + 1) - \beta + 1\big). \tag{27}$$

Consequently, we obtain the first-order approximate solution for $M > 0$, which reads

$$\begin{aligned} f(\zeta) &= f_0(\zeta) + f_1(\zeta) \\ &= \frac{1-\mathrm{e}^{-\sqrt{M}\zeta}}{\sqrt{M}} + \frac{\mathrm{e}^{-2\sqrt{M}\zeta}}{6M^{3/2}}\big(-(\beta+2)\,\mathrm{e}^{2\sqrt{M}\zeta} \\ &\quad + (3\sqrt{M}\zeta + 2\beta + 1)\,\mathrm{e}^{\sqrt{M}\zeta} - \beta + 1\big). \end{aligned} \tag{28}$$

It is obvious that

$$f''(\zeta) = \frac{\mathrm{e}^{-2\sqrt{M}\zeta}}{6\sqrt{M}}\big((3\sqrt{M}\zeta - 6M + 2\beta - 5) \times \mathrm{e}^{\sqrt{M}\zeta} - 4\beta + 4\big),\quad M > 0. \tag{29}$$

Therefore, we can easily obtain the initial slope which reads $f''(0) = (-6M - 2\beta - 1)/(6\sqrt{M})$ for $M > 0$.

For the special case of $\beta = 1$, the exact analytical solution[25] of Eqs. (18) and (19) is

$$f(\zeta) = \frac{1-\exp(-\sqrt{1+M}\zeta)}{\sqrt{1+M}},\quad f''(0) = -\sqrt{1+M}. \tag{30}$$

To test the accuracy of the approximate analytic solution (18), the result obtained by using the HPM is compared with those computed by numerical methods.[1,4,25] Comparisons of the solutions are plotted graphically in Figs. 1(a) and 1(b). The numerical comparisons of the initial slope $f''(0)$ obtained by using the HPM with the numerical solutions given in [1] and [4] and the exact solution in Ref. [25] are presented in Tables 1 and 2. It can be seen that the result computed by using the HPM are in excellent agreement with those obtained by using the numerical method for various values of the parameters of β and M. Moreover, Tables 1 and 2 reveal that the values of the initial slope $f''(0)$ increase when the values of the magnetic parameter M increase. On the other hand, Figs. 1(a) and 1(b) show the effect of the magnetic parameter M on the MHD boundary layer flow and depict that an increase in the magnetic parameter M will increase the velocity gradient of the fluid in the boundary layer.

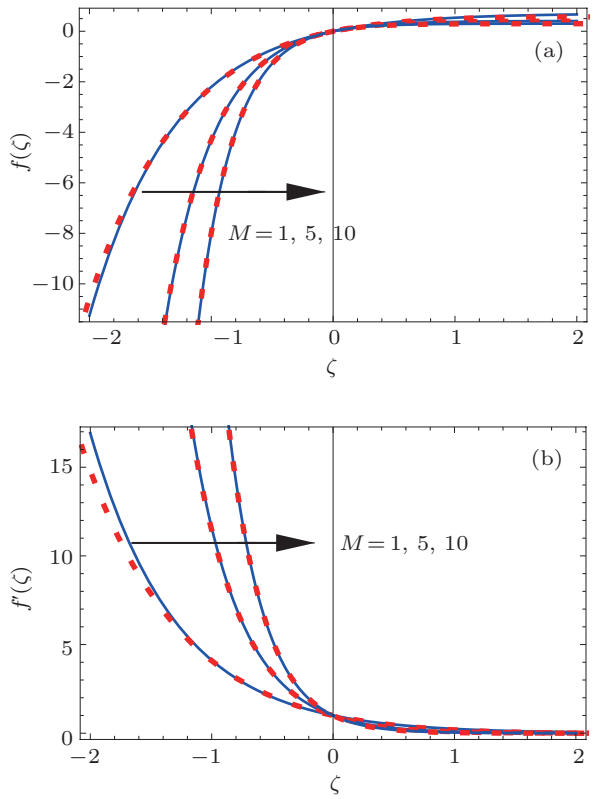


Fig. 1. (colour online) (a) Comparison plot of the dimensionless stream function $f(\zeta)$ obtained by using the HPM (dashed line) and the exact solution (continuous line) for $\beta = 1$ and different values of M; (b) Comparison plot of the dimensionless tangential velocity $f'(\zeta)$ obtained by HPM (dashed line) and the exact solution (continuous line) for $\beta = 1$ and different values of M.

Table 1. Comparison of the initial slope $f''(0)$ for $\beta = 1$ obtained by using the HPM and other known solutions.

M	HPM	Ref. [4] ([5/5])	Exact solution[25]
1	−1.50000	−1.41349	−1.41421
5	−2.45967	−2.44019	−2.44948
10	−3.32039	−3.31636	−3.31662
50	−7.14178	−7.14098	−7.14142
100	−10.05000	−10.04926	−10.04987
500	−22.38304	−22.38147	−22.38302
1000	−31.63859	−31.63672	−31.63858

Table 2. Comparison of the initial slope $f''(0)$ for $\beta = 1.5$ obtained by using the HPM and other known solutions.

M	HPM	Ref. [4] ([5/5])	Numerical solution[1]
1	−1.6667	−1.5163	−1.5252
5	−2.53421	−2.5167	−2.5161
10	−3.3731	−3.3662	−3.3663
50	−7.16535	−7.1643	−7.1647
100	−10.0667	−10.0658	−10.0776
500	−22.3905	−22.38916	−22.3904
1000	−31.6439	−31.6420	−31.6438

4. Conclusion

In the present work, the approximate analytical solution of the MHD boundary layer flow of an incompressible viscous fluid over a nonlinear stretching sheet has been obtained by employing the Lie symmetry method and the HPM. Firstly, the BVP of the MHD flow problem was reduced to the IVP for an ordinary differential equation by using the Lie symmetry method. Secondly, an approximate analytical solution of the IVP for the ordinary differential equation was computed by using the HPM. Finally, the efficiency of the results calculated by using the HPM was verified by the results obtained with the numerical method offered in the literature. The comparisons show the reliability and validity of the HPM and demonstrate that this method can be widely applied to other BVPs of partial differential equations.

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