The Fourier slice transformation of the Wigner operator and the quantum tomogram of the density operator*

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(Received 27 July 2011; revised manuscript received 31 August 2011)

Using the Weyl quantization scheme and based on the Fourier slice transformation (FST) of the Wigner operator, we construct a new expansion formula of the density operator ρ, with the expansion coefficient being the FST of ρ's classical Weyl correspondence, and the latter the Fourier transformation of ρ's quantum tomogram. The coordinate–momentum intermediate representation is used as the Radon transformation of the Wigner operator.

Keywords: quantum tomography, Fourier slice transformation, density operator

PACS: 42.50.–p **DOI:** 10.1088/1674-1056/21/3/034203

1. Introduction

Quantum states can be expressed in the conventional Heisenberg and Schrödinger representations.[1,2] For example, pure quantum states can be expressed by wave functions or vectors in the Hilbert space. Mixed quantum states are associated with density operator ρ. However, in quantum mechanics it is not possible to directly observe a quantum state. Instead, one is usually interested in reconstructing density matrix ρ[3–5] by using the quantum tomography method.[6] The tomogram of a quantum state can be considered as a primary object, which contains complete information of the quantum state and can serve as an alternative to the density operator. The quantum tomogram is also an important component of quantum phase space theory.

In this paper, we pay attention to the Weyl quantization scheme in the quantum phase space and find a route from it to the quantum tomogram theory. In other words, we shall point out that the formalism of the quantum tomogram can be deduced directly from the Weyl quantization rule. This will deepen our knowledge about the essence of the quantum tomogram.

To be concrete, and based on the Weyl correspondence of the density operator, we shall derive an expansion formula of ρ in terms of the Fourier slice transformation (FST) of the Wigner operator, with the expansion coefficient being the FST of ρ's classical Weyl correspondence. The latter is also the Fourier transformation of ρ's quantum tomogram.

Let us briefly review the Fourier slice theorem of an arbitrary classical function.[5] It states that the resulting $F(g,\varphi)$ produced by the following transformation of $f(p,x)$

$$F(g,\varphi) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}x\,\mathrm{d}p f(p,x)\ \mathrm{e}^{-\mathrm{i}g(x\cos\varphi+p\sin\varphi)} \quad (1)$$

can be the Fourier transform of $P_\varphi(t)$

$$F(g,\varphi)=\int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{-\mathrm{i}gt}P_\varphi(t), \quad (2)$$

where g is a real parameter, φ indicates the direction of a projection line, and $P_\varphi(t)$ is a projection of $f(p,x)$ along the line determined by function $\delta(t-x\cos\varphi-p\sin\varphi)$

$$P_\varphi(t)=\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}p f(p,x)\,\delta(t-x\cos\varphi-p\sin\varphi). \quad (3)$$

In our previous work,[7] we extended the above type of transformation to the Wigner operator $\Delta(p,x)$, i.e. we proposed the following Fourier slice transform:

$$\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{d}p\Delta(p,x)\ \mathrm{e}^{-\mathrm{i}g(\lambda x+\nu p)}=\mathrm{e}^{-\mathrm{i}g(\lambda X+\nu P)}, \quad (4)$$

*Project supported by the Natural Science Foundation of Huangshi Institute of Technology, China (Grant No. 10yjz03R) and the National Natural Science Foundation of China (Grant No. 10874174).

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where X and P are the coordinate and the momentum operators, respectively, $[X,P]=\mathrm{i}$, $\hbar=1$. This equality holds because $\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}p\Delta(p,x)=1$, and $\mathrm{e}^{-\mathrm{i}\lambda(x\mu+p\nu)}$ is only the Weyl correspondence[8] of $\mathrm{e}^{-\mathrm{i}\lambda(X\mu+P\nu)}$. The Wigner operator is expressed as[9]

$$\Delta(x,p)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|x-\frac{v}{2}\right\rangle\left\langle x+\frac{v}{2}\right|\mathrm{e}^{-\mathrm{i}pv}\mathrm{d}v, \quad (5)$$

where $|x\rangle$ is the eigenvector of coordinate operator X. Similar to the form of Eq. (2), we know that the right-hand side of Eq. (4) can be expressed as

$$\mathrm{e}^{-\mathrm{i}g(X\lambda+P\nu)}=\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{-\mathrm{i}gx}\left|x\right\rangle_{\lambda,\nu\ \lambda,\nu}\left\langle x\right|, \quad (6)$$

where $|x\rangle_{\lambda,\nu\ \lambda,\nu}\langle x|$ is a projection operator similar to $P_{\varphi}(t)$ in Eq. (2), and by analogy with Eqs. (2) and (3), we have

$$\begin{aligned}&\left|x\right\rangle_{\lambda,\nu\ \lambda,\nu}\left\langle x\right|\\ =&\int_{-\infty}^{\infty}\mathrm{d}x'\mathrm{d}p'\Delta(p',x')\,\delta(x-\lambda x'-\nu p'). \end{aligned}\quad (7)$$

Substituting the normally ordered form of the Wigner operator[10−12]

$$\Delta(p,x)=\frac{1}{\pi}\colon\mathrm{e}^{-(x-X)^2-(p-P)^2}\colon \quad (8)$$

into Eq. (7), we can perform the integration using the integration technique within an ordered product (IWOP) of operators[13−16] and obtain

$$\begin{aligned}&\left|x\right\rangle_{\lambda,\nu\ \lambda,\nu}\left\langle x\right|\\ =&\frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}x'\mathrm{d}p'\colon\mathrm{e}^{-\left(x'-X\right)^2-\left(p'-P\right)^2}\colon\\ &\times\delta(x-\lambda x'-\nu p')\\ =&[\pi(\lambda^2+\nu^2)]^{-1/2}\colon\\ &\times\exp\left[\frac{-1}{\lambda^2+\nu^2}(x-\lambda X-\nu P)^2\right]\colon. \end{aligned}\quad (9)$$

Using the normally ordered form of the vacuum projector

$$\left|0\right\rangle\left\langle 0\right|=\colon\mathrm{e}^{-a^{\dagger}a}\colon \quad (10)$$

and $X=(a+a^{\dagger})/\sqrt{2}, P=(a-a^{\dagger})/\sqrt{2}\mathrm{i}$, we know the explicit form of $|x\rangle_{\lambda,\nu}$,

$$\begin{aligned}\left|x\right\rangle_{\lambda,\nu}=&[\pi(\lambda^2+\nu^2)]^{-1/4}\exp\left[-\frac{x^2}{2(\lambda^2+\nu^2)}\right.\\ &\left.+\sqrt{2}a^{\dagger}\frac{x}{\lambda-\mathrm{i}\nu}-\frac{\lambda+\mathrm{i}\nu}{2(\lambda-\mathrm{i}\nu)}a^{\dagger2}\right]\left|0\right\rangle. \end{aligned}\quad (11)$$

The above state obeys the eigenvector equation

$$(\lambda X+\nu P)\left|x\right\rangle_{\lambda,\nu}=x\left|x\right\rangle_{\lambda,\nu}, \quad (12)$$

and the completeness relation

$$\int_{-\infty}^{\infty}\mathrm{d}x\left|x\right\rangle_{\lambda,\nu\ \lambda,\nu}\left\langle x\right|=1, \quad (13)$$

thus $|x\rangle_{\lambda,\nu}$ is named the coordinate–momentum intermediate representation, and $|x\rangle_{\lambda,\nu\ \lambda,\nu}\langle x|$ is the Radon transformation of the Wigner operator.

Now the following questions arise: in the Weyl quantization scheme,

$$\rho=\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}p\Delta(p,x)\,h(p,x), \quad (14)$$

$h(p,x)$ is named the classical Weyl correspondence of density operator ρ, when $h(p,x)$ is replaced by its Fourier slice transformation function

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}ph(p,x)\,\mathrm{e}^{\mathrm{i}g(x\cos\varphi+p\sin\varphi)}\equiv S(g,\varphi), \quad (15)$$

then what is the expansion formula of ρ in terms of $S(g,\varphi)$? and what is the physical interpretation of $S(g,\varphi)$ and the expansion?

In the following, we shall answer these two questions, which will lead to the conclusion that this new expansion of density operator is based on the Wigner operator's Fourier slice transformation and the expanding coefficient is the Fourier slice transformation of ρ's classical Weyl correspondence. The latter is just the Fourier transformation of the quantum tomogram of ρ.

2. The Weyl correspondence of the density operator in terms of $S(g,\varphi)$

Since the Weyl correspondence of $\mathrm{e}^{-\mathrm{i}g(\lambda X+\nu P)}$ is $\mathrm{e}^{-\mathrm{i}g(\lambda x+\nu p)}$, by introducing $g'/\sqrt{\lambda^2+\nu^2}=g$, we can express Eq. (4) as

$$\begin{aligned}&\mathrm{e}^{-\mathrm{i}g(\lambda X+\nu P)}\\ =&\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}p\Delta(p,x)\,\mathrm{e}^{-\mathrm{i}g'\left(\frac{\lambda x+\nu p}{\sqrt{\lambda^2+\nu^2}}\right)}\\ =&\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}p\Delta(p,x)\,\mathrm{e}^{-\mathrm{i}g'(x\cos\varphi+p\sin\varphi)}, \end{aligned}\quad (16)$$

where

$$\cos\varphi=\frac{\lambda}{\sqrt{\lambda^2+\nu^2}},\quad \sin\varphi=\frac{\nu}{\sqrt{\lambda^2+\nu^2}}. \quad (17)$$

On the other hand, using Eqs. (12), (13) and (16), we have

$$\mathrm{e}^{-\mathrm{i}g(\lambda X+\nu P)}=\int_{-\infty}^{\infty}\mathrm{d}y\left|y\right\rangle_{\lambda,\nu\ \lambda,\nu}\left\langle y\right|\mathrm{e}^{-\mathrm{i}gy}$$

$$= \int_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}p\Delta\left(p,x\right) \mathrm{e}^{-\mathrm{i}g'\left(x\cos\varphi+p\sin\varphi\right)}. \quad (18)$$

The right-hand side can be considered as a Fourier transformation, then its inverse transformation leads to

$$\begin{aligned}
&\Delta\left(p,x\right)\\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}g'|g'|\int_0^{\pi}\mathrm{d}\varphi\,\mathrm{e}^{\mathrm{i}g'\left(x\cos\varphi+p\sin\varphi\right)}\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}y\left|y\right\rangle_{\lambda,\nu\ \lambda,\nu}\left\langle y\right|\mathrm{e}^{-\mathrm{i}g'y/\sqrt{\lambda^2+\nu^2}}\\
&= \frac{1}{4\pi^2}\int\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}g'|g'|\int_0^{\pi}\mathrm{d}\varphi\left|y\right\rangle_{\lambda,\nu\ \lambda,\nu}\left\langle y\right|\\
&\quad\times\mathrm{e}^{-\mathrm{i}g'\left[y/\sqrt{\lambda^2+\nu^2}-\left(x\cos\varphi+p\sin\varphi\right)\right]}. \quad (19)
\end{aligned}$$

Multiplying $h\left(p,x\right)$ to both sides of Eq. (19) and referencing to Eq. (15), we have

$$\begin{aligned}
\rho &= \int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}p\Delta\left(p,x\right)h\left(p,x\right)\\
&= \frac{1}{4\pi^2}\int\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}g'|g'|\\
&\quad\times\int_0^{\pi}\mathrm{d}\varphi\left|y\right\rangle_{\lambda,\nu\ \lambda,\nu}\left\langle y\right|\mathrm{e}^{-\mathrm{i}g'\left(y/\sqrt{\lambda^2+\nu^2}\right)}\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}ph\left(p,x\right)\mathrm{e}^{\mathrm{i}g'\left(x\cos\varphi+p\sin\varphi\right)}\\
&= \frac{1}{2\pi}\int\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}g'|g'|\int_0^{\pi}\mathrm{d}\varphi\left|y\right\rangle_{\lambda,\nu\ \lambda,\nu}\left\langle y\right|\\
&\quad\times\mathrm{e}^{-\mathrm{i}g'\left(y/\sqrt{\lambda^2+\nu^2}\right)}S\left(g',\varphi\right). \quad (20)
\end{aligned}$$

Then using

$$\begin{aligned}
&\int_{-\infty}^{\infty}\mathrm{d}y\left|y\right\rangle_{\lambda,\nu\ \lambda,\nu}\left\langle y\right|\mathrm{e}^{-\mathrm{i}g'\left(y/\sqrt{\lambda^2+\nu^2}\right)}\\
&= \mathrm{e}^{-\mathrm{i}g'\left(\lambda X+\nu P\right)/\sqrt{\lambda^2+\nu^2}}, \quad (21)
\end{aligned}$$

we have

$$\begin{aligned}
\rho &= \int_{-\infty}^{\infty}\frac{\mathrm{d}g'}{2\pi}|g'|\int_0^{\pi}\mathrm{d}\varphi\\
&\quad\times\mathrm{e}^{-\mathrm{i}g'\left(\lambda X+\nu P\right)/\sqrt{\lambda^2+\nu^2}}S\left(g',\varphi\right), \quad (22)
\end{aligned}$$

which is the new expansion formula of the density operator in terms of $S\left(g',\varphi\right)$. When $\sqrt{\lambda^2+\nu^2}=1$, equation (22) becomes

$$\begin{aligned}
\rho &= \int_{-\infty}^{\infty}\frac{\mathrm{d}g}{2\pi}|g|\int_0^{\pi}\mathrm{d}\varphi\\
&\quad\times\mathrm{e}^{-\mathrm{i}g\left(X\cos\varphi+P\sin\varphi\right)}S\left(g,\varphi\right). \quad (23)
\end{aligned}$$

As indicated by Eq. (15), $S\left(g,\varphi\right)$ is the Fourier slice transformation of ρ's classical Weyl correspondence, and correspondingly, $\mathrm{e}^{-\mathrm{i}g\left(X\cos\varphi+P\sin\varphi\right)}$ is the Fourier slice transform of Wigner operator $\Delta\left(p,x\right)$ (see Eq. (4)). This is the physical meaning of the expansion of ρ in Eq. (23).

3. $S\left(g,\varphi\right)$ as the Fourier transformation of the quantum tomogram of ρ

We now analyze $S\left(g,\varphi\right)$ in detail. For a pure state $\rho=\left|\psi\right\rangle\left\langle\psi\right|$, $h\left(p,x\right)=2\pi\mathrm{Tr}\left[\rho\Delta\left(p,x\right)\right]=2\pi\left\langle\psi\right|\Delta\left(p,x\right)\left|\psi\right\rangle$. Using Eq. (15) and the Weyl correspondence, we have

$$\begin{aligned}
&S\left(g,\varphi\right)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}ph\left(p,x\right)\mathrm{e}^{\mathrm{i}g\left(x\cos\varphi+p\sin\varphi\right)}\\
&= \left\langle\psi\left|\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}p\Delta\left(p,x\right)\mathrm{e}^{\mathrm{i}g\left(x\cos\varphi+p\sin\varphi\right)}\right|\psi\right\rangle\\
&= \left\langle\psi\left|\mathrm{e}^{\mathrm{i}g\left(X\cos\varphi+P\sin\varphi\right)}\right|\psi\right\rangle. \quad (24)
\end{aligned}$$

For a mixed state ρ, from the Fourier slice theorem, we know $S\left(g,\varphi\right)$ can be expressed as a projector's Fourier transformation

$$S\left(g,\varphi\right)=\int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{e}^{\mathrm{i}gy}P_{\varphi}\left(y\right). \quad (25)$$

Referencing Eq. (7), we have

$$\begin{aligned}
&P_{\varphi}\left(y\right)\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}g\,\mathrm{e}^{-\mathrm{i}gy}S\left(g,\varphi\right)\\
&= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}ph\left(p,x\right)\\
&\quad\times\int_{-\infty}^{\infty}\mathrm{d}g\,\mathrm{e}^{-\mathrm{i}g\left(y-x\cos\varphi-p\sin\varphi\right)}\\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}ph\left(p,x\right)\delta\left(y-x\cos\varphi-p\sin\varphi\right)\\
&= \mathrm{Tr}\left[\rho\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}p\Delta\left(p,x\right)\delta\left(y-x\cos\varphi-p\sin\varphi\right)\right]\\
&= \mathrm{Tr}\left(\rho\left|y\right\rangle_{\varphi\ \varphi}\left\langle y\right|\right), \quad (26)
\end{aligned}$$

thus

$$S\left(g,\varphi\right)=\int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{e}^{\mathrm{i}gy}\mathrm{Tr}\left[\rho\left|y\right\rangle_{\varphi\ \varphi}\left\langle y\right|\right]. \quad (27)$$

Physically, $\mathrm{Tr}(\rho\left|y\right\rangle_{\varphi\ \varphi}\left\langle y\right|)$ is just the quantum tomogram of ρ,[2–4] so $S\left(g,\varphi\right)$ is its Fourier transformation. Finally, we mention that in Ref. [17], we have found the relationship between the optical Fresnel transformation and the quantum tomography. For the quantum tomography in the multi-dimensional phase space, the readers can refer to Ref. [18].

4. Conclusion

In summary, using the Weyl quantization scheme and based on the Fourier slice transformation of the Wigner operator, we have constructed a new expansion formula for the density operator in terms of its quantum tomogram, and the physical meaning of the expansion function is explained. The IWOP technique and the coordinate–momentum intermediate representation as the Radon transformation of the Wigner operator have been used.

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