The Wigner distribution functions of coherent and partially coherent Bessel–Gaussian beams

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Based on the integral representation of the Bessel functions and the generating function of the Tricomi function, an analytical expression of the Wigner distribution function (WDF) for a coherent or partially coherent Bessel–Gaussian beam is presented. The reduced two-dimensional WDFs are also demonstrated graphically, which reveals the dependence of the reduced WDFs on the beam parameters.

Keywords: partially coherent Bessel–Gaussian beam, Wigner function, optical angular momentum

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1. Introduction

The Wigner distribution function (WDF) was originally aimed at understanding the quantum corrections to the classical behavior of certain statistical systems described by the Boltzmann formula.[1] Since then, it has become a powerful tool in many fields. For example, it is extensively applied in the descriptions of both coherent and partially coherent beams and their passage through first-order systems.[2−10] The WDF that corresponds to a Hermite–Gaussian mode has been known since the early days of quantum mechanics. Later, WDFs associated with other beam modes including the Laguerre–Gaussian modes were obtained,[11,12] and the light beams passing through a circular or rectangular aperture were also determined.[13,14]

However, to the best of our knowledge, the WDF that corresponds to a Bessel–Gaussian mode has not been studied. In fact, the Bessel–Gaussian beam is one of the typical examples of singularity beams carrying orbital angular momentums and has attracted a lot of attention because of its nondiffracting property in free propagation and the advantages over other hollow light beams for guiding or trapping atoms.[15−19] In this paper, we obtain an analytical form of the WDF for the Bessel–Gaussian beam by virtue of the integral representation of the Bessel function. Based on the integral representation of the Bessel function and the generating function of the Tricomi function, we obtain the analytic expression of the Wigner distribution function associated with a partially coherent Bessel–Gaussian beam in Section 2. And in Section 3, some demonstrations on the dependence of the Wigner distribution function on the beam parameters are given with numerical and graphical methods. Finally, in Section 4, some conclusions are presented.

2. The analytical Wigner distribution function of partially coherent Bessel–Gaussian beams

A partially coherent Bessel–Gaussian beam is characterized by the cross-spectral density[19]

$$\begin{aligned} \Gamma(\boldsymbol{\rho_1}, \boldsymbol{\rho_2}) = {} & \mathrm{J}_n(\beta\rho_1)\,\mathrm{J}_n(\beta\rho_2) \\ & \times \exp\left[-\frac{\boldsymbol{\rho}_1^2+\boldsymbol{\rho}_2^2}{w_0^2} - \frac{(\boldsymbol{\rho_1}-\boldsymbol{\rho_2})^2}{2\sigma_0^2} \right. \\ & \left. + \mathrm{i}\,n(\theta_1-\theta_2)\right], \end{aligned} \tag{1}$$

where $\boldsymbol{\rho}_i = (\rho_{xi}, \rho_{yi}) = (\rho_i\cos\theta_i, \rho_i\sin\theta_i)$ ($i = 1$ or 2) are two arbitrary points in the transverse plane, w_0 is the beam waist width of the fundamental Gaussian mode, β characterizes the oscillatory variation in the radial direction, n is the topological charge associated with the coherent Bessel–Gaussian beam, σ_0 is

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the transverse coherence length, and $\mathrm{J}_n(x)$ is the n-th order Bessel function of the first kind. Equation (1) reduces to a complete coherent Bessel–Gaussian beam under the condition of $\sigma_0 \to \infty$, and just represents a Gaussian–Schell model beam for $n = \beta = 0$, which has been widely studied in the free space and turbulent atmosphere propagation.[20−23]

Based on the Bessel integral representation

$$\int_0^{2\pi} \exp\left(\mathrm{i}\, z\cos\theta - \mathrm{i}\, n\theta\right)\mathrm{d}\theta = 2\pi \mathrm{i}^n \mathrm{J}_n\left(z\right),$$

the cross-spectral density (1) may be rewritten as

$$\begin{aligned}\Gamma\left(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2\right) = {} & \frac{1}{\left(2\pi\right)^2}\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\theta\,\mathrm{d}\theta' \Gamma_{\theta\theta'}\left(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2\right) \\ & \times \exp\left[\mathrm{i}\, n\left(\theta - \theta'\right)\right], \end{aligned} \tag{2}$$

with

$$\begin{aligned}\Gamma_{\theta\theta'}\left(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2\right) = {} & \exp\left[-\frac{\boldsymbol{\rho}_1^2 + \boldsymbol{\rho}_2^2}{w_0^2} - \frac{\left(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\right)^2}{2\sigma_0^2}\right] \\ & \times \exp[\mathrm{i}\,\beta(\rho_{x1}\cos\theta + \rho_{y1}\sin\theta \\ & - \rho_{x2}\cos\theta' - \rho_{y2}\sin\theta')]. \end{aligned} \tag{3}$$

The WDF is defined as[1,2]

$$\begin{aligned} W\left(\boldsymbol{r}, \boldsymbol{u}\right) = {} & \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty} \mathrm{d}^2\xi \exp\left(-\mathrm{i}\, k\boldsymbol{u}\cdot\boldsymbol{\xi}\right) \\ & \times \Gamma\left(\boldsymbol{r} + \frac{1}{2}\boldsymbol{\xi}, \boldsymbol{r} - \frac{1}{2}\boldsymbol{\xi}\right) \\ = {} & \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\theta\,\mathrm{d}\theta' W_{\theta\theta'}\left(\boldsymbol{r}, \boldsymbol{u}\right) \\ & \times \exp\left[\mathrm{i}\, n\left(\theta - \theta'\right)\right], \end{aligned} \tag{4}$$

where $\boldsymbol{r} = (x, y)$, and $k\boldsymbol{u} = (ku_x, ku_y)$ are the wave vector components along the x and the y axes, respectively, with $k = 2\pi/\lambda$ representing the wave number. Respectively replacing $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ with $\boldsymbol{r} + \frac{1}{2}\boldsymbol{\xi}$, $\boldsymbol{r} - \frac{1}{2}\boldsymbol{\xi}$ in Eq. (3), we turn $\Gamma_{\theta\theta'}\left(\boldsymbol{r} + \frac{1}{2}\boldsymbol{\xi}, \boldsymbol{r} - \frac{1}{2}\boldsymbol{\xi}\right)$ into

$$\begin{aligned} & \Gamma_{\theta\theta'}\left(\boldsymbol{r} + \frac{1}{2}\boldsymbol{\xi}, \boldsymbol{r} - \frac{1}{2}\boldsymbol{\xi}\right) \\ = {} & \exp\left[-\frac{2\boldsymbol{r}^2}{w_0^2} - \frac{A}{2}\boldsymbol{\xi}^2 + \mathrm{i}\left(\boldsymbol{Q}_-\cdot\boldsymbol{r} + \frac{\boldsymbol{Q}_+\cdot\boldsymbol{\xi}}{2}\right)\right], \end{aligned} \tag{5}$$

where $A = \frac{1}{w_0^2} + \frac{1}{\sigma_0^2}$, and $\boldsymbol{Q}_\pm$ formally denotes the two-dimensional vectors

$$\boldsymbol{Q}_\pm = \beta\left(\cos\theta \pm \cos\theta', \sin\theta \pm \sin\theta'\right).$$

We have

$$\begin{aligned} & W_{\theta\theta'}\left(\boldsymbol{r}, \boldsymbol{u}\right) \\ = {} & \left(\frac{k}{2\pi}\right)^2 \int_{-\infty}^{\infty} \mathrm{d}^2\boldsymbol{\xi} \exp\left(-\mathrm{i}\, k\boldsymbol{u}\cdot\boldsymbol{\xi}\right) \\ & \times \Gamma_{\theta\theta'}\left(\boldsymbol{r} + \frac{1}{2}\boldsymbol{\xi}, \boldsymbol{r} - \frac{1}{2}\boldsymbol{\xi}\right) \\ = {} & \Omega\left(\boldsymbol{r}, \boldsymbol{u}\right)\exp\left[\Theta_\theta + \Theta_{\theta'}^* - \frac{\beta^2}{4A}\cos 2\left(\theta - \theta'\right)\right], \end{aligned} \tag{6}$$

where

$$\Omega\left(\boldsymbol{r}, \boldsymbol{u}\right) = \Omega_0 \exp\left(-\frac{2\boldsymbol{r}^2}{w_0^2} - kN_\sigma \boldsymbol{u}^2\right), \tag{7}$$

$$\Theta_\theta = \beta\left(U_x\cos\theta + U_y\sin\theta\right), \tag{8}$$

$$\Omega_0 = \frac{k^2}{2\pi A}\exp\left(-\frac{\beta^2}{4A}\right), \quad U_x = N_\sigma u_x + \mathrm{i}\, x,$$
$$U_y = N_\sigma u_y + \mathrm{i}\, y, \quad N_\sigma = N_w/\left(1 + w_0^2/\sigma_0^2\right),$$

with $N_w = \frac{1}{2}kw_0^2$ being the Rayleigh range related to the Gaussian beam, and $*$ representing the complex conjugate. Inserting Eq. (6) into Eq. (4) and making use of $\exp\left(z\cos\theta\right) = \sum\limits_{n=-\infty}^{\infty} \mathrm{I}_n\left(z\right)\mathrm{e}^{\mathrm{i}\, n\theta}$, we get

$$W\left(\boldsymbol{r}, \boldsymbol{u}\right) = \frac{\Omega\left(\boldsymbol{r}, \boldsymbol{u}\right)}{\left(2\pi\right)^2}\sum_{m=-\infty}^{\infty} \mathrm{I}_m\left(-\frac{\beta^2}{4A}\right) G_m G_m^*, \tag{9}$$

$$G_m = \int_0^{2\pi} \mathrm{d}\theta \exp\left[\mathrm{i}\left(n + 2m\right)\theta + \Theta_\theta\right]. \tag{10}$$

Recalling the generating function $\exp\left(t - \frac{z}{t}\right) = \sum\limits_{n=-\infty}^{\infty} t^n \mathrm{C}_n\left(z\right)$ of the Tricomi function $\mathrm{C}_n\left(z\right)$,[24−26] formally, for arbitrary complex parameters α and η, we may have

$$\begin{aligned} & \exp\left(\alpha\cos\theta + \eta\sin\theta\right) \\ = {} & \exp\left(\frac{\alpha - \mathrm{i}\,\eta}{2}\,\mathrm{e}^{\mathrm{i}\,\theta} + \frac{\alpha + \mathrm{i}\,\eta}{2}e^{-\mathrm{i}\,\theta}\right) \\ = {} & \sum_{l=-\infty}^{\infty}\left(\frac{\alpha - \mathrm{i}\,\eta}{2}\right)^l \mathrm{C}_l\left(-\frac{\alpha^2 + \eta^2}{4}\right)\mathrm{e}^{\mathrm{i}\, l\theta}. \end{aligned} \tag{11}$$

Therefore, with the help of $\int_0^{2\pi} \mathrm{e}^{\mathrm{i}\, m\theta}\mathrm{d}\theta = 2\pi\delta_{m,0}$, we can perform the integral in Eq. (10),

$$\begin{aligned} G_m = {} & \mathrm{C}_{-n-2m}\left(-\beta^2\frac{U_x^2 + U_y^2}{4}\right) \\ & \times \left(\beta\frac{U_x - \mathrm{i}\, U_y}{2}\right)^{-n-2m}, \end{aligned} \tag{12}$$

$$\begin{aligned} G_m^* = {} & \mathrm{C}_{n+2m}\left(-\beta^2\frac{U_x^{*2} + U_y^{*2}}{4}\right) \\ & \times \left(\beta\frac{U_x^* - \mathrm{i}\, U_y^*}{2}\right)^{n+2m}. \end{aligned} \tag{13}$$

Due to $\mathrm{C}_{-n}\left(z\right) = \left(-z\right)^n \mathrm{C}_n\left(z\right)$,[24−26] we have

$$\mathrm{C}_{-n}\left(-\alpha^2 - \eta^2\right)\left(\alpha - \mathrm{i}\,\eta\right)^{-n}$$

$$= \mathrm{C}_n\left(-\alpha^2-\eta^2\right)(\alpha+\mathrm{i}\,\eta)^n, \tag{14}$$

and G_m becomes

$$G_m = \mathrm{C}_{n+2m}\left(-\beta^2\frac{U_x^2+U_y^2}{4}\right) \times\left(\beta\frac{U_x+\mathrm{i}\,U_y}{2}\right)^{n+2m}. \tag{15}$$

Inserting Eqs. (13) and (15) into Eq. (9), we have

$$W(\boldsymbol{r},\boldsymbol{u}) = \Omega(\boldsymbol{r},\boldsymbol{u})\sum_{m=-\infty}^{\infty}(-1)^m\,\mathrm{I}_m\left(\frac{\beta^2}{4A}\right) \times\left|\mathrm{C}_{n+2m}\left(-\beta^2\frac{U_x^2+U_y^2}{4}\right)\right|^2 \times\left(\frac{\beta}{2}\left|U_x+\mathrm{i}\,U_y\right|\right)^{2(n+2m)}. \tag{16}$$

Additionally, due to the relation between the Tricomi function and the Bessel function[24–26]

$$\mathrm{C}_n(x) = \begin{cases} x^{-n/2}\mathrm{J}_n\left(2\sqrt{x}\right), & \text{for } x>0,\\ (-x)^{-n/2}\,\mathrm{I}_n\left(2\sqrt{-x}\right), & \text{for } x\le 0, \end{cases} \tag{17}$$

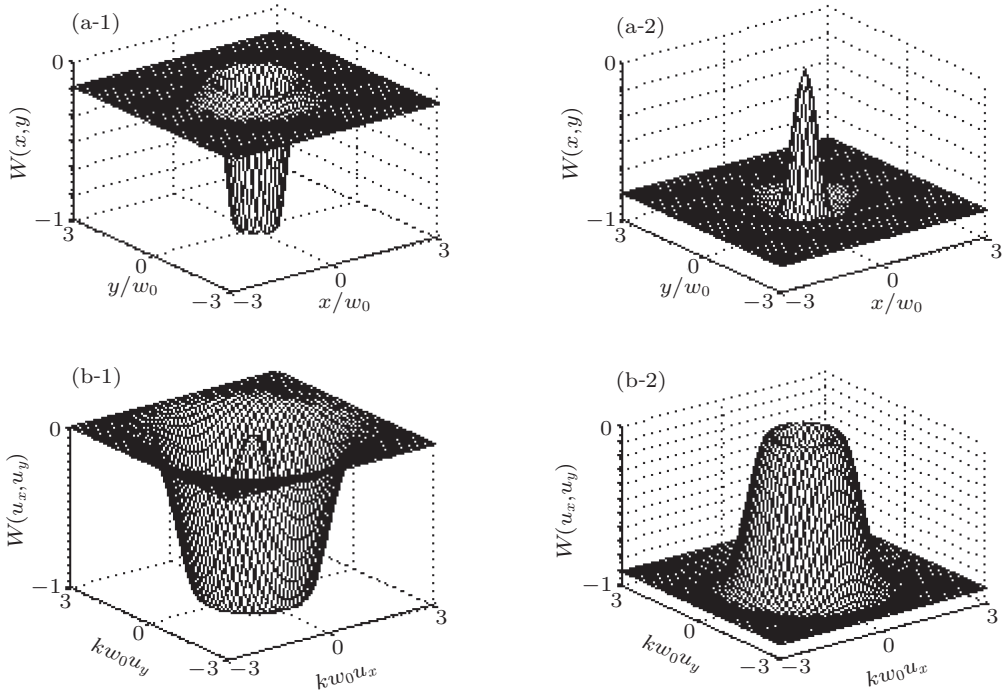
we finally obtain

$$W(\boldsymbol{r},\boldsymbol{u}) = \Omega(\boldsymbol{r},\boldsymbol{u})\sum_{m=-\infty}^{\infty}(-1)^m\,\mathrm{I}_m\left(\frac{\beta^2}{4A}\right) \times\left|\mathrm{I}_{n+2m}\left(\beta\sqrt{U_x^2+U_y^2}\right)\right|^2\left|\frac{U_x+\mathrm{i}\,U_y}{U_x-\mathrm{i}\,U_y}\right|^{n+2m}. \tag{18}$$

3. Numerical and graphical result analyses

It is a difficult task to directly study four-dimensional WDF (16) or (18) in detail. However, it is possible to reduce it into a function with two variables by choosing the other two to be specified values. For different choices, WDF can be written as six reduced functions, respectively, each involving only two variables (2D). The 2D reduced WDFs may be $W_2^{u_x=u_y=0}(x,y)$, $W_2^{x=y=0}(u_x,u_y)$, $W_2^{y=u_y=0}(x,u_x)$, $W_2^{x=u_x=0}(y,u_y)$, $W_2^{y=u_x=0}(x,u_y)$, and $W_2^{x=u_y=0}(y,u_x)$. In fact, $W_2^{y=u_y=0}(x,u_x)$ and $W_2^{x=u_x=0}(y,u_y)$ are the same in the functional form, so are $W_2^{y=u_x=0}(x,u_y)$ and $W_2^{x=u_y=0}(y,u_x)$, so in the following graphical demonstrations, we only draw four of the reduced WDFs.

Based on Eq. (15), the numerical calculations can be carried out, and some representative results are shown in Figs. 1 and 2. From these figures, we can see that the 2D reduced WDFs sensitively depend on the topological charge n and the beam parameter β. For example, from Fig. 1, it can be seen that for $\beta w_0 = 3$, the reduced WDFs $W_2^{x=y=0}(u_x,u_y)$ and $W_2^{y=u_y=0}(x,u_x)$ may be always negative for $n=3$ but may become positive for $n=4$. Moreover, as shown in Fig. 2, the reduced WDF $W_2^{u_x=u_y=0}(x,y)$ evidently depends on parameter β.



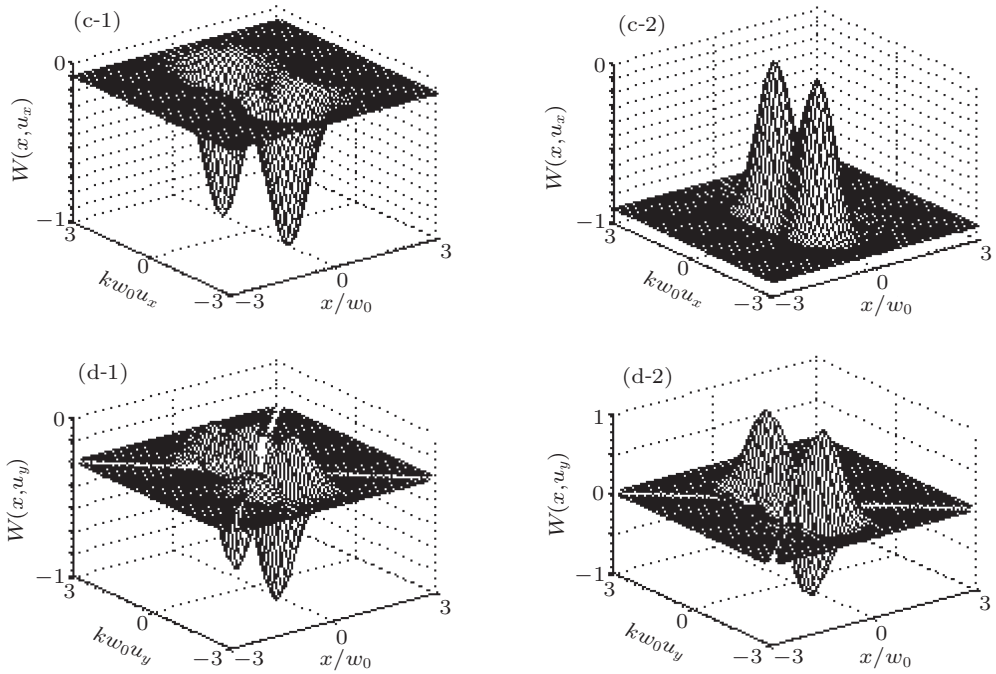


Fig. 1. Reduced WDFs (a) $W_2^{u_x=u_y=0}(x,y)$, (b) $W_2^{x=y=0}(u_x,u_y)$, (c) $W_2^{y=u_y=0}(x,u_x)$ and (d) $W_2^{y=u_x=0}(x,u_y)$ (in arb. units) for $\beta w_0=3$, $\sigma_0=0$ and $n=3$ (left), 4 (right).

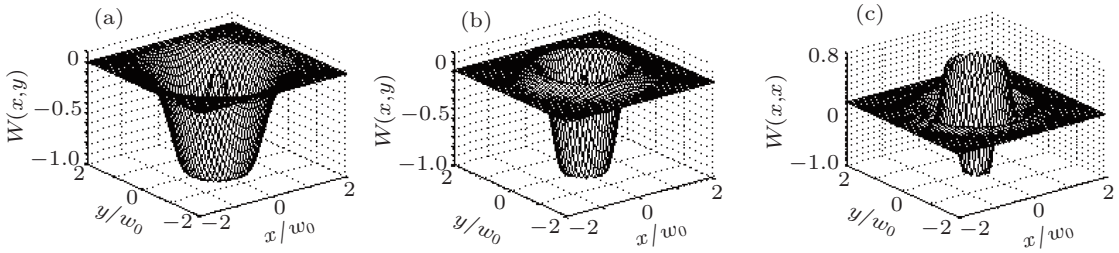


Fig. 2. Reduced WDF $W_2^{u_x=u_y=0}(x,y)$ (in arb. units) for βw_0 being (a) 1, (b) 3 and (c) 6 with $n=3$ and $\sigma_0=0$.

4. Conclusion

In conclusion, the analytical expression for the Wigner distribution function of a coherent or partially coherent Bessel–Gaussian beam has been derived for the first time, and some reduced two-dimensional WDFs are demonstrated graphically, which reveals the dependence of the reduced WDFs on the topological charge index n and the beam parameter β.

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