Modified adaptive controller for synchronization of incommensurate fractional-order chaotic systems*

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We investigate the synchronization of a class of incommensurate fractional-order chaotic systems, and propose a modified adaptive controller for fractional-order chaos synchronization based on the Lyapunov stability theory, the fractional order differential inequality, and the adaptive strategy. This synchronization approach is simple, universal, and theoretically rigorous. It enables the synchronization of0 fractional-order chaotic systems to be achieved in a systematic way. The simulation results for the fractional-order Qi chaotic system and the four-wing hyperchaotic system are provided to illustrate the effectiveness of the proposed scheme.

Keywords: synchronization, modified adaptive controller, incommensurate fractional-order chaotic system

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1. Introduction

Even though the fractional calculus is a mathematical topic with a more than 300-year history, its application to physics and engineering has attracted lots of attention only in recent years. It has been found that many systems in interdisciplinary fields can be described by fractional differential equations, such as viscoelastic systems,[1] dielectric polarization,[2] electrode–electrolyte polarization,[3], some finance systems, and electromagnetic waves.[4] Meanwhile, it has been found that some fractional-order differential systems can demonstrate chaotic behavior, such as the fractional-order Chua circuit,[5] the fractional order lorenz system,[6] the fractional order Liu system,[7] the fractional order Chen system,[8] the fractional order Lü system,[9] the fractional order conjugate Chen system,[10] and the fractional order Qi system.[11]

Recently, studying fractional-order chaotic systems has become an active research field. The synchronization of the fractional-order chaotic systems starts to attract increasing attention due to its potential applications in secure communication and control processing. Some approaches have been proposed to achieve chaos synchronization in fractional-order chaotic systems, such as PC control, the nonlinear state observer method, adaptive control, impulsive control, sliding mode control, and the scalar transmitted signal method.[12−25] However, in most of those studies, stability theory is based on the stability criteria of the linear fractional order systems or the Laplace transform theory.

In this paper, we propose a modified adaptive control approach to synchronize a class of incommensurate fractional-order systems based on the Lyapunov stability theory and the adaptive control technique. Our method is more general and can synchronize almost all well-known incommensurate fractional-order chaotic systems. In this method, the nonlinear terms are not removed, and the controller contains a time-varying feedback gain k. The estimate of k is unknown in advance, it will automatically converge to a suitable constant k^*. To show the wide applicability of our method, we give illustrations by using the fractional-order chaotic Qi system and the four-wing hyperchaotic system. The numerical simulations are used to verify the effectiveness of the proposed synchronization scheme. The rest of this paper is organized as follows. In Section 2, preliminaries and definitions are given. Section 3 gives the main results. In Section 4, two examples are used to verify the effectiveness of the proposed scheme. The paper

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is concluded in Section 5.

2. Preliminaries and definitions

2.1. Definitions of fractional derivatives

Fractional calculus plays an important role in modern science. In this paper, we use both the Reimann–Liouville and the Caputo fractional operators as our main tools. The Caputo fractional derivative is defined as[26]

$$\mathrm{D}_t^\alpha x(t) = \mathrm{J}^{m-\alpha} x^{(m)}(t), \quad \alpha \in (0,1), \tag{1}$$

where $m = [\alpha]$, i.e., m is the smallest integer that is not less than α, $x^{(m)}$ is the m-order derivative in the usual sense, $\mathrm{J}^\beta\,(\beta > 0)$ is the β-order Reimann–Liouville integral operator

$$\mathrm{J}^\beta y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} y(\tau)\,\mathrm{d}\tau, \tag{2}$$

and Γ is the Gamma function.

2.2. Properties of fractional derivatives

Two general properties of the fractional-order derivative will be used, which are recalled here.[27,28]

Property 1 The additive index law

$$\mathrm{D}_t^\alpha \mathrm{D}_t^\beta x(t) = \mathrm{D}_t^\beta \mathrm{D}_t^\alpha x(t) = \mathrm{D}_t^{\alpha+\beta} x(t) \tag{3}$$

holds some reasonable constraints on $x(t)$.

Property 2 The Caputo fractional derivative operator is a linear operator

$$\mathrm{D}_t^\alpha(\lambda x(t) + \mu y(t)) = \lambda \mathrm{D}_t^\alpha x(t) + \mu \mathrm{D}_t^\alpha y(t), \tag{4}$$

where λ, μ are real constants. In particular, $\mathrm{D}_t^\alpha x(t) = \mathrm{D}_t^\alpha(x(t)+0) = \mathrm{D}_t^\alpha x(t) + \mathrm{D}_t^\alpha 0$, then we have $\mathrm{D}_t^\alpha 0 = 0$.

3. Main results

Here, based on the Lyapunov stability theory, a modified adaptive controller with only one control parameter is proposed to synchronize two identical incommensurate fractional-order chaotic systems.

Let the fractional-order chaotic system be given as

$$\mathrm{D}_t^\alpha \boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}(\boldsymbol{x}), \tag{5}$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^{\mathrm{T}}$ with $0 < \alpha_i < 1$ $(i = 1, 2, \ldots, n)$, $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathrm{R}^n$ is the state vector of the system (if $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, system (5) is a commensurate fractional-order system, otherwise is an incommensurate fractional-order system[29]) $\boldsymbol{A} \in \mathrm{R}^{n\times n}$ is a constant matrix, and $\boldsymbol{f}: \mathrm{R}^n \to \mathrm{R}^n$ is a continuous vector function. We assume that

$$\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}}(\boldsymbol{y} - \boldsymbol{x}) \tag{6}$$

for a bounded matrix $\boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}}$ with its elements depending on $\boldsymbol{x}$ and $\boldsymbol{y}$. At this point, it is important to note that many known fractional-order chaotic systems can be described by Eqs. (5) and (6).

Let system (5) be the drive system, then the controlled response system is given by

$$\mathrm{D}_t^\alpha \boldsymbol{y} = \boldsymbol{A}\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{u}, \tag{7}$$

where $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in \mathrm{R}^n$ is the state vector, and $\boldsymbol{u} \in \mathrm{R}^n$ is a controller to be determined.

Define $\boldsymbol{e} = \boldsymbol{y} - \boldsymbol{x}$, the error system from Eqs. (5) and (7) is

$$\mathrm{D}_t^\alpha \boldsymbol{e} = (\boldsymbol{A} + \boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}})\boldsymbol{e} + \boldsymbol{u}. \tag{8}$$

It is obvious that the synchronization between systems (5) and (7) is achieved if and only if error system (8) is asymptoticlly stable at the zero point. So, the main objective of this paper is to design a suitable controller $\boldsymbol{u}(t)$ such that error system (8) is asymptotically stable at the zero point. To study the stabilization of error system (8), we define

$$\mathrm{D}_t^{1-\alpha} \boldsymbol{s} = \boldsymbol{e}, \tag{9}$$

where $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)^{\mathrm{T}}$. The adaptive controller is modified to

$$\boldsymbol{u} = -k_1 \boldsymbol{s}, \tag{10}$$

where $k_1 \in \mathrm{R}^1$ is a time-varying parameter, which will be duly adapted according to the following update law:

$$\dot{k}_1 = \gamma \boldsymbol{s}^{\mathrm{T}} \boldsymbol{s} = \gamma \sum_{i=1}^{n} s_i^2, \tag{11}$$

with γ being an arbitrary constant.

Remark 1 For Eq. (9), according to Property 1 of the fractional derivatives, we have

$$\mathrm{D}_t^\alpha \boldsymbol{e} = \mathrm{D}_t^\alpha \mathrm{D}_t^{1-\alpha} \boldsymbol{s} = \mathrm{D}_t^1 \boldsymbol{s} = \dot{\boldsymbol{s}}. \tag{12}$$

Remark 2 For Eq. (9), according to Property 2 of the fractional derivatives, if $\boldsymbol{s} = 0$, then $\boldsymbol{e} = 0$. Hence there exists a positive number l such that

$$||\boldsymbol{e}|| \le l||\boldsymbol{s}||. \tag{13}$$

Remark 3 If $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, the synchronization of the incommensurate fractional-order

chaotic systems will be reduced to the synchronization of the commensurate systems. In particular, if $\alpha_1 = \alpha_2 = \cdots = \alpha_n$=1, from Eq. (9), we have $\boldsymbol{s} = \boldsymbol{e}$, the synchronization of the incommensurate fractional-order chaotic systems will be reduced to the synchronization of the integer chaotic systems.

Theorem 1 Error system (8) is asymptotically stable at the zero point under controller (10) and update law (11), i.e., the two systems (5) and (7) are globally asymptotically synchronized.

Proof Since $\boldsymbol{A}$ is a constant matrix and $\boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}}$ is a bounded matrix with elements depending on $\boldsymbol{x}$ and $\boldsymbol{y}$, there exist two nonnegative constants δ and η such that $||\boldsymbol{A}||_2 \leq \delta$ and $||\boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}}||_2 \leq \eta$.

Define the following Lyapunov functional candidate:

$$V(t) = \frac{1}{2}\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} + \frac{1}{2\gamma}(k_1 - k_1^*)^2, \tag{14}$$

where $k_1^* > l(\delta + \eta)$ is a constant. Differentiating V with respect to time t along the solution of Eq. (14), we can obtain

$$\begin{aligned}
\dot{V} &= \boldsymbol{s}^{\mathrm{T}}\dot{\boldsymbol{s}} + (k_1 - k_1^*)\dot{k}/\gamma \\
&= \boldsymbol{s}^{\mathrm{T}}\mathrm{D}_t^{\alpha}\boldsymbol{e} + (k_1 - k_1^*)\boldsymbol{s}^2 \\
&= \boldsymbol{s}^{\mathrm{T}}[(A + \boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}})\boldsymbol{e} - k_1\boldsymbol{s}] + (k_1 - k_1^*)\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} \\
&\leq ||\boldsymbol{s}^{\mathrm{T}}||(\delta + \eta)||\boldsymbol{e}|| - k_1^*\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} \\
&\leq l(\delta + \eta)\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} - k_1^*\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} \\
&= [l(\delta + \eta) - k_1^*]\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s} \\
&\leq 0.
\end{aligned}$$

Using Barbalat's lemma,[30] we can conclude that $\boldsymbol{s}, \dot{\boldsymbol{s}} \in L_\infty$. As $t \to \infty$, $\boldsymbol{s}$ approaches zero, which shows that error $\boldsymbol{e}$ asymptotically converges to zero and the two systems (5) and (7) are globally asymptotically synchronized.

4. Applications

In this section, in order to show the effectiveness of the proposed scheme, numerical examples on two fractional-order chaotic (hyperchaotic) systems are provided. When numerically solving the fractional differential equations, we adopt the predictor–corrector method.[31]

4.1. Sychronization of fractional-order chaotic Qi system

The fractional-order Qi system[11,32] is given by

$$\begin{aligned}
\mathrm{D}_t^{\alpha_1} x_1 &= a(x_2 - x_1) + x_2 x_3, \\
\mathrm{D}_t^{\alpha_2} x_2 &= c x_1 - x_2 - x_1 x_3, \\
\mathrm{D}_t^{\alpha_3} x_3 &= x_1 x_2 - b x_3,
\end{aligned} \tag{15}$$

where x_1, x_2, x_3 are the state variables, and a, b, c are the parameters. For $a = 35$, $b = 8/3$, $c = 80$, we find that chaos exists in the fractional-order Qi system, the fractional order is $(\alpha_1, \alpha_2, \alpha_3) = (0.95, 0.96, 0.97)$, and the phase portraits are shown in Fig. 1.

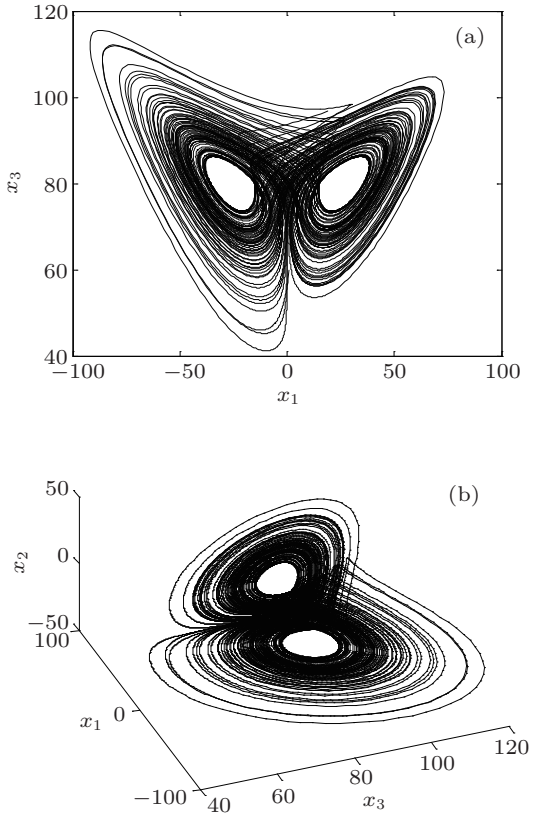


Fig. 1. Chaotic attractors in the fractional-order Qi chaotic system shown in (a) x_1–x_3 plane and (b) x_1–x_2–x_3 space.

We take system (15) as the drive system, and the response system is described by

$$\begin{aligned}
\mathrm{D}_t^{\alpha_1} y_1 &= a(y_2 - y_1) + y_2 y_3 + u_1, \\
\mathrm{D}_t^{\alpha_2} y_2 &= c y_1 - y_2 - y_1 y_3 + u_2, \\
\mathrm{D}_t^{\alpha_3} y_3 &= y_1 y_2 - b y_3 + u_3,
\end{aligned} \tag{16}$$

where $\boldsymbol{u} = (u_1,\ u_2,\ u_3)^{\mathrm{T}}$ is the controller vector.

We define the state errors as $e_i = y_i - x_i$ $(i = 1, 2, 3)$, and $\mathrm{D}_t^{1-\alpha_1} s_1 = e_1$, $\mathrm{D}_t^{1-\alpha_2} s_2 = e_2$, $\mathrm{D}_t^{1-\alpha_3} s_3 = e_3$. The controller is taken as

$$u_1 = -k_1 s_1, \quad u_2 = -k_1 s_2, \quad u_3 = -k_1 s_3,$$

with

$$\dot{k}_1 = \sigma(s_1^2 + s_2^2 + s_3^2), \quad (\sigma > 0).$$

According to Theorem 1, the synchronization of drive system (15) and response system (16) can be achieved asymptotically.

In the simulation, $u_1 = -k_1 s_1$, $u_2 = u_3 = 0$, $\dot{k}_1 = s_1^2$, $(\alpha_1, \alpha_2, \alpha_3) = (0.95, 0.96, 0.97)$, the parameter vector is $(a,\ b,\ c) = (35, 8/3, 80)$ and the initial values are set as $\boldsymbol{x}(0) = (1, -2, 60)^{\mathrm{T}}$, $\boldsymbol{y}(0) = (-1, -3, 6)^{\mathrm{T}}$, and $k_1(0) = 0$. The corresponding numerical results are shown in Figs. 2 and 3. Figure 2 shows the temporal evolution of the synchronization error between systems (15) and (16), which implies that the synchronization can be realized. Figure 3 displays that s_1 tends to zero and feedback gain k_1 tends to a positive constant.

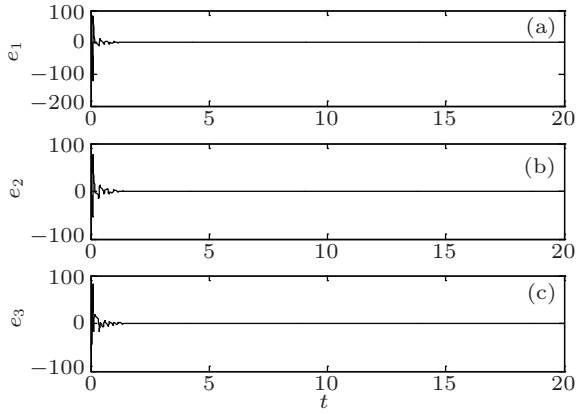


Fig. 2. Synchronization errors (a) e_1, (b) e_2, and (c) e_3 in the fractional-order Qi chaotic system.

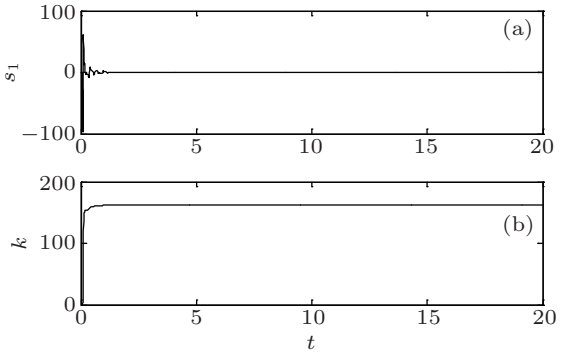


Fig. 3. (a) Variable s_1 and (b) feedback gain k_1.

4.2. Sychronization of a fractional-order four-wing hyperchaotic system

In order to further illustrate the effectiveness of the proposed scheme, in this subsection, we consider the fractional-order four-wing hyperchaotic system,[33] which is described by the following equations:

$$\begin{aligned}
\mathrm{D}_t^{\alpha_1} x_1 &= a x_1 - x_2 x_3 + x_4,\\
\mathrm{D}_t^{\alpha_2} x_2 &= -b x_2 + x_1 x_3,\\
\mathrm{D}_t^{\alpha_3} x_3 &= x_1 x_2 - c x_3 + x_1 x_4,\\
\mathrm{D}_t^{\alpha_4} x_4 &= -x_2,
\end{aligned} \tag{17}$$

where x_1, x_2, x_3, and x_4 are the state variables, a, b, and c are the parameters. For $a = 8$, $b = 40$, $c = 49$, we find that chaos exists in the fractional-order four-wing hyperchaotic system. The fractional order is $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1,\ 0.95,\ 0.90,\ 0.85)$, and the phase portraits are shown in Fig. 4. The response system corresponding to system (17) is

$$\begin{aligned}
\mathrm{D}_t^{\alpha_1} y_1 &= a y_1 - y_2 y_3 + y_4 + u_1,\\
\mathrm{D}_t^{\alpha_2} y_2 &= -b y_2 + y_1 y_3 + u_2,\\
\mathrm{D}_t^{\alpha_3} y_3 &= y_1 y_2 - c y_3 + y_1 y_4 + u_3,\\
\mathrm{D}_t^{\alpha_4} y_4 &= -y_2 + u_4,
\end{aligned} \tag{18}$$

where y_1, y_2, y_3, and y_4 are the state vectors, and $\boldsymbol{u} = (u_1, u_2, u_3, u_4)^{\mathrm{T}}$ is the controller vector.

We define the state errors as $e_i = y_i - x_i$ $(i = 1, 2, 3, 4)$, and $\mathrm{D}_t^{1-\alpha_1} s_1 = e_1$, $\mathrm{D}_t^{1-\alpha_2} s_2 = e_2$, $\mathrm{D}_t^{\alpha_3} s_4 = e_3$, $\mathrm{D}_t^{\alpha_4} s_4 = e_4$. The controller is taken as

$$u_1 = -k_1 s_1,\ \ u_2 = -k_1 s_2,\ \ u_3 = -k_1 s_3,\ \ u_4 = -k_1 s_4,$$

with

$$\dot{k}_1 = \sigma(s_1^2 + s_2^2 + s_3^2 + s_4^2), \quad (\sigma > 0).$$

According to Theorem 1**,** the synchronization of drive system (17) and response system (18) can be achieved asymptotically.

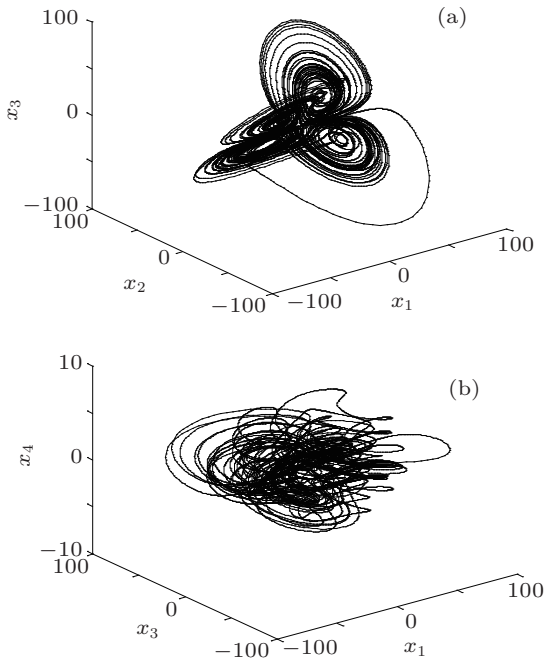


Fig. 4. Chaotic attractors in the fractional-order Four-wing hyperchaotic system shown in (a) x_1–x_2–x_3 and (b) x_1–x_3–x_4 spaces.

In the simulation, $u_1 = -k_1 s_1$, $u_3 = -k_1 s_3$, $u_2 = u_4 = 0$, $\dot{k}_1 = s_1^2 + s_3^2$, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1,\ 0.95,\ 0.90,\ 0.85)$. The corresponding numerical results are shown in Figs. 5 and 6, where the initial values are set as $\boldsymbol{x}(0) = (1,\ -2,\ 3,\ 1)^{\mathrm{T}}$, $\boldsymbol{y}(0) = (-1,\ -3,\ 6,\ -1)^{\mathrm{T}}$, and $k_1(0) = 0$. Figure 5 shows the temporal evolution of the synchronization error between systems (17) and (18), which implies that synchronization can be realized. Figure 6 displays that s_1 and s_3 tend to zero and feedback gain k_1 tends to a positive constant.

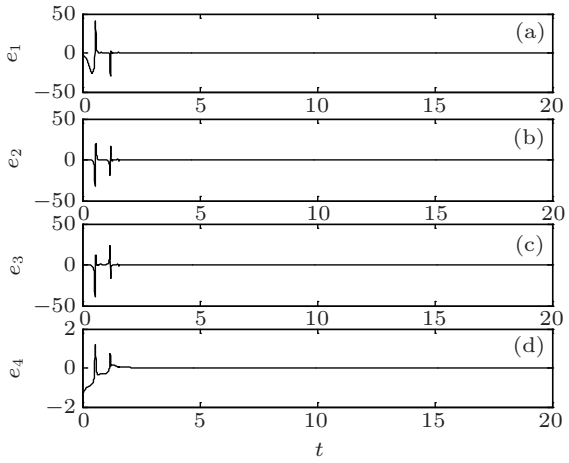


Fig. 5. Synchronization errors (a) e_1, (b) e_2, (c) e_3, and (d) e_4 of the fractional-order four-wing hyperchaotic systems.

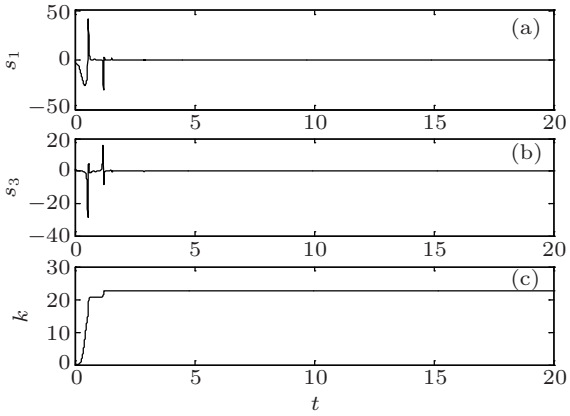


Fig. 6. Variables (a) s_1 and (b) s_3, and (c) feedback gain k_1.

5. Conclusion

In this paper, based on the Lyapunov stability theory and the fractional order differential inequality, we investigate the synchronization of the incommensurate fractional-order chaotic sysems. A modified adaptive controller is proposed to achieve the synchronization of the fractional-order chaotic systems. The proposed scheme is simple, universal, and theoretically rigorous. The synchronization of the fractional-order chaotic Qi system and the four-wing hyperchaotic system are used to illustrate the effectiveness of the proposed synchronization scheme. It should be noted that the proposed method is also applicable for the synchronization of commensurate fractional-order chaotic systems and integer-order chaotic systems.

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