Dispersion relation of excitation mode in spin-polarized Fermi gas*

Liu Ke(刘 可)† and Chen Ji-Sheng(陈继胜)

Department of Physics and Institute of Particle Physics, Central China Normal University, Wuhan 430079, *China*

(Received 4 August 2011; revised manuscript received 9 October 2011)

We study the dispersion relation of the excitation mode in a spin-polarized Fermi gas. In the frame of the imaginary-time finite temperature field theory, the polarization tensor is calculated by taking the random phase approximation. The population imbalance effects on the dispersion relation of the excitation mode and the spin–spin correlation susceptibility are investigated. The numerical results in terms of the imbalance ratio indicate the polarization effects on the dispersion relation and susceptibility χ.

Keywords: dispersion relation, polarized fermions, polarization tensor

PACS: 03.75.Ss, 05.30.Fk, 67.85.Lm **DOI:** 10.1088/1674-1056/21/3/030309

1. Introduction

The interacting atomic Fermi gas becomes the focus topic in contemporary physics with the realization of the artificial degeneracy of the Fermi atom gas.[1,2] It is expected that the Fermi atom system can be used to explore the physics hidden in the realistic complex system.

The relevant thermodynamics of strongly interacting fermions has attracted much attention in recent years. For cold fermions, one can control the interaction strength between the Fermi atoms through the Feshbach resonance by tuning the external magnetic field.[3] For a dilute system, the two-particle interaction can be characterized by the scattering length a. By changing the external magnetic field strength, the scattering length a can be $\pm\infty$ at the Feshbach resonance point. At resonance, scattering reaches the unitarity limit, and the scattering cross section will be saturated as $\sigma \sim 4\pi/k^2$, where k is the relative momentum of the Fermi atoms.[4−6] The relevant universal thermodynamics of the strongly interacting unitary Fermi gas is the focus of theoretical and experimental studies in the literature.[7−11] In our previous work, we have tried a quasilinear approximation method to investigate the finite temperature thermodynamics of the unitary Fermi gas.[12−16]

In addition to the general finite temperature thermodynamics and the zero-temperature ground state energy, the dispersion relation of the excitation mode characterizes the collective property of the interacting fermions. Most of the important physical quantities can be represented by the dispersion relation, such as the width of the energy spectrum and the particle lifetime. In principle, the kinetic transport and finite temperature field theory can be used to explore the dispersion relation excited in the interacting fermion system. In the language of the finite temperature field theory, the dispersion relation of the excitation mode is determined by the full fermion propagator. Technically, the full propagator can be obtained by using the Dyson–Schwinger equations through the proper self-energy or the polarization tensor.[17] In Ref. [18], we have studied the dispersion relation of the excitation mode for the spin-balanced case with equal spin-up and spin-down components.

In the meantime, the polarized fermion matter has attracted lots of attention from both theoretical and experimental sides.[19−24] It remains an intriguing task to study the population imbalance effects on the dispersion relation of the excitation mode. In this work, we will use the imaginary-time finite temperature field theory formalism to investigate the effects of the imbalance ratio and temperature/density as well as the interaction strength on the dispersion relation.

This paper is organized as follows. In Section 2, the polarization tensor is calculated with the Dyson–Schwinger equation formalism. In Section 3, we give the particle number density expressions in terms of

*Project supported by the National Natural Science Foundation of China (Grant No. 10875050 and 11178001).

†Corresponding author. E-mail: liuke@phy.ccnu.edu.cn

© 2012 Chinese Physical Society and IOP Publishing Ltd

the corresponding chemical potentials, and the dispersion relations of the excitation mode with different imbalance ratios are discussed with two particle interaction $U_{\rm r}$. The spin–spin correlation susceptibility is obtained in the static infrared limit of the polarization tensor in Section 4. The conclusion is made in the final section. The natural units $k_{\rm B} = \hbar = 1$ are employed throughout this paper.

2. Polarization tensor obtained with Dyson–Scwhinger equations

The Hamiltonian of a spin-polarized Fermi gas with a finite contact interaction is usually taken to be

$$\begin{aligned}\widehat{H} =& -\sum_{\sigma}\psi_{\sigma}^{\dagger}\frac{\boldsymbol{\nabla}^2}{2m}\psi_{\sigma} + U\psi_{\uparrow}^{\dagger}\psi_{\downarrow}^{\dagger}\psi_{\downarrow}\psi_{\uparrow} - \mu(\psi_{\uparrow}^{\dagger}\psi_{\uparrow} + \psi_{\downarrow}^{\dagger}\psi_{\downarrow}) \\ & -h(\psi_{\uparrow}^{\dagger}\psi_{\uparrow} - \psi_{\downarrow}^{\dagger}\psi_{\downarrow}),\end{aligned} \tag{1}$$

where $U = 4\pi\hbar^2 a/m$ is the coupling constant, a is the scattering length, m is the bare fermion mass, and $\sigma = \uparrow, \downarrow$ represents the spin projection. For the convenience of the discussion, we define $\mu = (\mu_{\uparrow} + \mu_{\downarrow})/2$ and $h = (\mu_{\uparrow} - \mu_{\downarrow})/2$, where μ_{σ} is the chemical potential of spin component σ. In this paper, we will set the spin-up fermions as the majority component, i.e., $h > 0$ with $\mu_{\uparrow} > \mu_{\downarrow}$.

In a many-body environment, the interaction between particles will be inflected by the medium, which can be taken into account by the polarization correction. With the proper polarization Π^*, which is a polarization that cannot be separated into two parts by cutting a single interaction line, the interaction between two particles can be written as an equation of the exact interaction, the lowest-order interaction, and the proper polarization using the Dyson–Schwinger equation formalism. For a homogeneous system, the corresponding algebraic equation takes the form

$$\begin{aligned}U(\omega,\boldsymbol{p}) =& \, U_0(\omega,\boldsymbol{p}) + U_0(\omega,\boldsymbol{p})\Pi^*(\omega,\boldsymbol{p})U(\omega,\boldsymbol{p}) \\ =& \, \frac{U_0(\omega,\boldsymbol{p})}{1 - \Pi^*(\omega,\boldsymbol{p})U_0(\omega,\boldsymbol{p})},\end{aligned} \tag{2}$$

where $U(\omega,\boldsymbol{p})$ and $U_0(\omega,\boldsymbol{p})$ denote the exact and the lowest-order interactions, respectively, and $\Pi^*(\omega,\boldsymbol{p})$ is the proper polarization.

There is no suitable perturbation method in the strongly interacting unitary regime. In quantum many-body theory, the random phase approximation (RPA) is usually taken to replace the true interaction $U(\omega,\boldsymbol{p})$, which is realized by retaining only the lowest-order proper polarization $\Pi^* \approx \Pi^0$ shown in Fig. 1. How to go beyond the RPA is usually considered to be a formidable task. Within the RPA, the higher-order self-energy and the other Feynman diagrams are neglected. In this work, we will explore the polarization effect in the RPA framework.

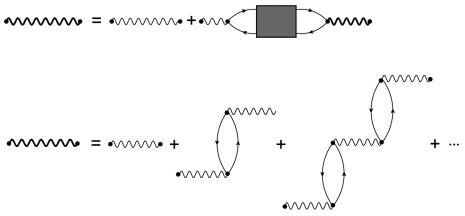


Fig. 1. In-medium interaction with Dyson–Schwinger equation, and random phase approximation.

In addressing the polarization effects in a many-body environment with the RPA, the fundamental ingredient is calculating the polarization tensor $\Pi^0(\omega_n,\boldsymbol{p})$. In the momentum space, it is given by

$$\begin{aligned}&\Pi^0(\omega_n,\boldsymbol{p}) \\ &= T\sum_{\omega_m}\int_k \mathcal{G}_{\uparrow}(\omega_m,\boldsymbol{k})\mathcal{G}_{\downarrow}(\omega_m+\omega_n,\boldsymbol{k}+\boldsymbol{p}) \\ &= T\sum_{\omega_m}\int_k \frac{1}{{\rm i}\omega_m - (\varepsilon_{\boldsymbol{k}_{\uparrow}} - \mu_{\uparrow})} \\ &\quad \times \frac{1}{{\rm i}(\omega_m+\omega_n) - (\varepsilon_{\boldsymbol{k}+\boldsymbol{p}_{\downarrow}} - \mu_{\uparrow})},\end{aligned} \tag{3}$$

where $\int_k = 1/(2\pi)^3 \int {\rm d}^3k$ represents the momentum integral.

In order to perform the above summation, one can insert a convergent factor ${\rm e}^{{\rm i}\omega_m\eta}, \eta \to 0_+$ to ensure its absolute convergence. With the Matsubara discrete frequency summing formulae for fermions,[17]

$$\lim_{\eta\to 0}\sum_n \frac{{\rm e}^{{\rm i}\omega_n\eta}}{{\rm i}\omega_n - x} = \frac{\beta\hbar}{{\rm e}^{\beta\hbar x}+1}, \tag{4}$$

the polarization tensor $\Pi^0(\omega_n,\boldsymbol{p})$ can be reduced to

$$\Pi^0(\omega_n,\boldsymbol{p}) = \int_k \frac{f_{\boldsymbol{k}_{\uparrow}} - f_{\boldsymbol{k}+\boldsymbol{p}_{\downarrow}}}{{\rm i}\omega_n + \varepsilon_{\boldsymbol{k}_{\uparrow}} - \varepsilon_{\boldsymbol{k}+\boldsymbol{p}_{\downarrow}} - 2h}. \tag{5}$$

In Eq. (5), we have employed the Fermi–Dirac distribution function

$$f_{\boldsymbol{k}_{\sigma}} = \frac{1}{{\rm e}^{\beta(k^2/2m - \mu_{\sigma})}+1}, \tag{6}$$

where $\beta = 1/T$, and $\sigma = \uparrow, \downarrow$ represents the spin projection.

Substituting $\int \mathrm{d}^3k = \int_0^\infty k^2\mathrm{d}k \int_0^\pi \sin\theta\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi$ into Eq. (5), we can get the final expression for the polarization tensor

$$\begin{aligned}\Pi^0(\omega_n,\boldsymbol{p}) &= \frac{m}{2\pi^2}\\ &\times \int_0^\infty \frac{k\mathrm{d}k}{2p}\left(f_{\boldsymbol{k}\uparrow}\ln\left|\frac{p^2-2pk-2mi\omega_n+4mh}{p^2+2pk-2mi\omega_n+4mh}\right|\right.\\ &\left.+f_{\boldsymbol{k}\downarrow}\ln\left|\frac{p^2-2pk+2mi\omega_n-4mh}{p^2+2pk+2mi\omega_n-4mh}\right|\right), \end{aligned} \tag{7}$$

where $p=|\boldsymbol{p}|$, and $k=|\boldsymbol{k}|$.

In addition to the many-body medium effect manifested by the temperature and the chemical potential, the polarization tensor is an explicit function of h. In the limit $h\to 0$, the polarization tensor will return to the result of the symmetric system. In the following sections, we will explore the polarization effects on the dispersion relation of the excitation mode and the susceptibility in detail.

3. Particle number density and dispersion relation with different spin polarization strengths

Before further discussion, let us give the expression for the particle number density. It can also be given by the fermion propagator as[17]

$$n_\sigma = T\int_k \sum \mathrm{e}^{\mathrm{i}\omega_n\eta}\mathcal{G}_\sigma(\omega_n,\boldsymbol{k}), \tag{8}$$

which can be further presented with the standard Fermi integral formalism as

$$n_\sigma = \frac{1}{\lambda^3} f_{3/2}(\mathrm{e}^{\mu_\sigma/T}), \tag{9}$$

where $\lambda=\sqrt{2\pi/mT}$ is the thermodynamical wavelength, and $f_\nu(z)=1/\Gamma(\nu)\int_0^\infty x^{\nu-1}\mathrm{d}x/(z^{-1}\mathrm{e}^x+1)$ is the Fermi–Dirac integral function. For the symmetric case with $\mu=\mu_\uparrow=\mu_\downarrow$, the total particle number density $n=n_\uparrow+n_\downarrow$ can be reduced to $n=(2/\lambda^3)f_{3/2}(\mathrm{e}^{\mu/T})$.

Now we discuss the dispersion relation of the collective excitation mode. After the analytical continuation of the discrete frequency ω_n in the polarization tensor in terms of $\mathrm{i}\omega_n\to\omega+\mathrm{i}\epsilon$ with $\epsilon\to 0^+$, we can obtain the in-medium two-particle effective interaction U_r depending on frequency ω and momentum $\boldsymbol{p}$

$$U_\mathrm{r}(\omega,\boldsymbol{p}) = \frac{U_0(\omega,\boldsymbol{p})}{1-\Pi^0(\omega,\boldsymbol{p})U_0(\omega,\boldsymbol{p})}. \tag{10}$$

In the linear response theory, the pole of Eq. (10) determines the dispersion relation of the collective excitation mode

$$1-\mathrm{Re}\Pi^0(\omega,\boldsymbol{p})U_0(\omega,\boldsymbol{p})=0, \tag{11}$$

where U_0 is the lowest-order effective contact interaction. According to the Born approximation, U_0 is related to the S-wave scattering length a in terms of $U_0(\omega,\boldsymbol{p})=4\pi a/m$.

The numerical result will be rescaled by the Fermi characterized temperature $T_\mathrm{F}=k_\mathrm{F}^2/(2m)$. Here, the Fermi wave number k_F is given by the total particle number density in terms of $n=k_\mathrm{F}^3/(3\pi^2)$. For the spin polarized system, the following notation is conventionally taken to measure the population imbalance

$$P=\frac{n_\uparrow-n_\downarrow}{n_\uparrow+n_\downarrow}, \tag{12}$$

where P is called spin polarization in the literature.

For given P and n, the corresponding chemical potentials $\mu_{\uparrow(\downarrow)}$ and hence h can be fixed numerically with Eqs. (9) and (12). The dispersion relation curves for different spin polarization P are presented in Fig. 2. As an example, we give the results for the case at $T/T_\mathrm{F}=10$ with the interaction parameter $1/(k_\mathrm{F}a)=0.5\times10^{-3}$.

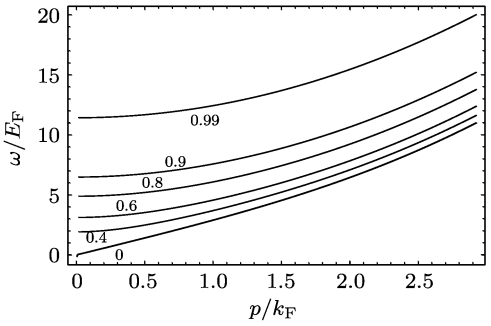


Fig. 2. Rescaled frequency ω/E_F versus rescaled momentum p/k_F of excitation mode for different spin polarization P ($T/T_\mathrm{F}=10$).

As shown in Fig. 2, the population imbalance increases, the energy–momentum curve departs from that of the symmetric case. For a small spin polarization with $P<0.5$, the dispersion relation curves are very close together. The curves depart from each other rapidly for large values of population imbalance.

Especially, we can see that the characteristic of the spin polarized case is quite different from that of the symmetric one. For the unpolarized symmetric case, the dispersion curve crosses through the point of origin ($\omega=0, p=0$). However, for the population imbalance system with $P\neq 0$, the dispersion curves will not pass through the origin.

4. Susceptibility for a spin polarized fermion system

Susceptibility is an important physical quantity in analyzing the polarized medium. To further analyze the population imbalance effect on the collective property, we discuss the susceptibility. In the frame of the linear response theory, it can be determined by the calculated proper polarization tensor at the static infrared limit, i.e., $\omega = 0, p \to 0$,

$$\chi \equiv -\lim_{p\to 0} \Pi^0(0, \boldsymbol{p}). \tag{13}$$

Theoretically, spin response susceptibility is different from that of density correlation. In the spin-balanced case, however, the former is identical to density response susceptibility. We have investigated density response susceptibility in Ref. [18]. When $P = 0$ with $n_\uparrow = n_\downarrow = n/2$, $\mu = \mu_\uparrow = \mu_\downarrow$, and $f_{\boldsymbol{k}\uparrow} = f_{\boldsymbol{k}\downarrow} = f_k$, the well known result $\chi_0 = f_{1/2}(\mathrm{e}^{\mu/T})/T\lambda^3$ can be recovered.[25] At zero temperature, the density response susceptibility can be reduced to $\chi_{00} = mk_\mathrm{F}/(2\pi^2)$.

Using a similar method in discussing the dispersion relation, we can give the polarization susceptibility for the spin population imbalance case. The numerical results in Fig. 3 for the susceptibility are rescaled by the zero-temperature density susceptibility χ_{00} for the ideal symmetric system.

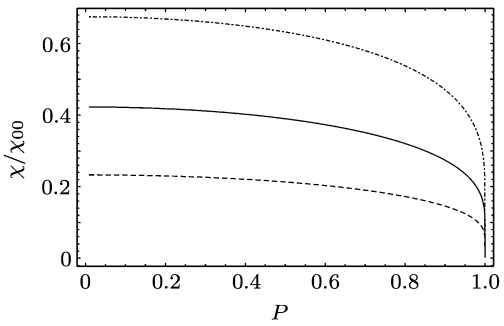


Fig. 3. Spin response susceptibility χ rescaled by χ_{00} versus spin polarization P. Dotted-dashed line is for $T/T_\mathrm{F} = 5$, solid line for $T/T_\mathrm{F} = 10$, and dashed line for $T/T_\mathrm{F} = 20$.

In principle, susceptibility should depend on temperature. With an increase in temperature, susceptibility will be changed to a smaller value. This general characteristic of the response susceptibility of the Fermi gas is shown in Fig. 3. In addition to the temperature effects, the numerical results also indicate the significant effect of the imbalance population on susceptibility. As spin polarization increases, the absolute value of susceptibility χ decreases. For a small polarization, susceptibility χ varies slowly; for a large polarization or the nearly full-polarized case, $P \longrightarrow 1$, susceptibility χ varies fast. In particular, as the imbalance population P approaches 1, there will a clear trend toward the zero limit value. This numerical result is consistent with the general expectation that in the single component spin-up system, there is no polarization response because there is no interaction between the spin-up and the spin-down components.

5. Conclusion

In summary, the dispersion relation of the excitation mode in a strongly interacting spin-polarized Fermi gas is studied in terms of the imaginary-time finite temperature field theory. With the calculation of the polarization tensor, the dispersion relations and the spin response susceptibility of the interacting fermions are discussed for different spin polarizations.

In addition to the general effects of temperature, density (chemical potential), and interaction parameter, we find that the imbalance population ratio P plays an important role in describing the dispersion relation. Different from the result for the ideal symmetric case, the dispersion curves of the unequal spin-up/down polarized system will not pass through the origin.

As expected, the temperature and the polarization effects will tend to reduce susceptibility. As the spin polarization P increases, susceptibility decreases faster. Especially, when P approaches 1, susceptibility χ will approach 0.

References

[1] Giorgini S, Pitaevskii L P and Stringari S 2008 *Rev. Mod. Phys.* **80** 1215
[2] Bloch I, Dalibard J and Zwerger W 2008 *Rev. Mod. Phys.* **80** 885
[3] Chin C, Grimm R, Julienne P and Tiesinga E 2010 *Rev. Mod. Phys.* **82** 1225
[4] Nozières P and Schmitt-Rink S 1985 *J. Low Temp. Phys.* **59** 195
[5] Chin C, Bartenstein M, Altmeyer A, Riedl S, Jochim S, Denschlag J H and Grimm R 2004 *Science* **305** 1128
[6] Kinnunen J, Rodríguez M and Törmä P 2004 *Science* **305** 1131
[7] Heiselberg H 2001 *Phys. Rev.* A **63** 043606
[8] Ho T L 2004 *Phys. Rev. Lett.* **92** 090402
[9] Cao C, Elliott E, Joseph J, Wu H, Petricka J, Schäfer T and Thomas J E 2011 *Science* **331** 58
[10] Chen J S, Li J R, Wang Y P and Xia X J 2008 *J. Stat. Mech.* P12008

[11] Partridge G B, Li W H, Kamar R I, Liao Y A and Hulet R G 2006 *Science* **311** 503

[12] Chen J S, Cheng C M, Li J R and Wang Y P 2007 *Phys. Rev.* A **76** 033617

[13] Chen J S 2007 *Chin. Phy. Lett.* **24** 1825

[14] Chen J S 2007 *Commun. Theor. Phys.* **48** 99

[15] Chen J S, Qin F and Wang Y P 2009 *Science in China* G **52** 1324

[16] Liu K and Chen J S 2011 *Chin. Phys.* B **20** 020501

[17] Fetter A L and Walecka J D 1971 *Quantum Theory of Many-Particle Systems* (New York: McGraw-Hill)

[18] Wang Y P and Chen J S 2008 *Chin. Phy.* B **17** 4401

[19] Yu Z Q and Yin L 2010 *Phys. Rev.* A **80** 013605

[20] Hao Y J 2011 *Chin. Phy.* B **20** 060307

[21] Chevy F and Mora C 2010 *Rep. Prog. Phys.* **73** 112401

[22] Pilati S and Giorgini S 2008 *Phys. Rev. Lett.* **100** 030401

[23] Parish M M, Marchetti F M, Lamacraft A and Simons B D 2007 *Nat. Phys.* **3** 124

[24] Zwierlein M W, Schirotzek A, Schunck C H and Ketterle W 2006 *Science* **311** 492

[25] Pathria R K *Statistical Mechanics* 1996 (2nd Edn.) (Oxford: Butterworth-Heinemann)