Solving coupled nonlinear Schrödinger equations via a direct discontinuous Galerkin method*

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In this work, we present the direct discontinuous Galerkin (DDG) method for the one-dimensional coupled nonlinear Schrödinger (CNLS) equation. We prove that the new discontinuous Galerkin method preserves the discrete mass conservations corresponding to the properties of the CNLS system. The ordinary differential equations obtained by the DDG space discretization is solved via a third-order stabilized Runge–Kutta method. Numerical experiments show that the new DDG scheme gives stable and less diffusive results and has excellent long-time numerical behaviors for the CNLS equations.

Keywords: direct discontinuous Galerkin method, coupled nonlinear Schrödinger equation, mass conservation

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1. Introduction

In this paper, we consider the coupled nonlinear Schrödinger (CNLS) system

$$\begin{aligned} &\mathrm{i}U_t + U_{xx} + f(|U|^2, |V|^2)U = 0, \\ &\mathrm{i}V_t + V_{xx} + g(|U|^2, |V|^2)V = 0, \end{aligned} \tag{1}$$

where U and V are the wave amplitudes in two polarizations, β is the cross-phase modulation coefficient, $f(|U|^2, |V|^2) = |U|^2 + \beta|V|^2$, and $g(|U|^2, |V|^2) = \beta|U|^2 + |V|^2$. The initial conditions are

$$U(x, 0) = u_0(x), \quad V(x, 0) = v_0(x). \tag{2}$$

We assume the periodic boundary conditions for simplicity.

The CNLS equations play an important role in physical science. In hydrodynamics, the CNLS equations describe the propagation of two wave packets along a direction in which the group velocity projections overlap.[1] In optics, the propagation of short pulses in birefringent fibers also leads to a form of CNLS equation.[2]

The analytical solution for the CNLS equations normally can be found only for the special case, where the coefficients of the self phase and the cross phase modulation terms are equal, the so called Manakov model.[3–5] Thus, the numerical method has practical significance, and has drawn the attention of many researchers. In recent years, a great deal of literature gives various numerical schemes for solving the CNLS equations, most of them use the finite difference schemes.[6–8] There are also other numerical methods applied to the CNLS equations, for example, Ismail derived a finite element scheme to solve the equation,[9] Cheng proposed a direct perturbation method,[10] Mokhtari studied the generalized differential quadrature method,[11] Utsumi studied the CIP-basis set method.[12] The multi-symplectic methods can also be applied to the CNLS system.[13–15]

Xu and Shu[16] solved the CNLS equations using the local discontinuous Galerkin (LDG) method. In this work, we are going to solve the CNLS equations using the direct discontinuous Galerkin (DDG) method. The DDG method was first proposed by Liu and Yan for diffusion problems,[17] they also added interface corrections to get a $(k+1)$-th order of accuracy.[18] The DDG method not only inherits the original properties of the LDG method, such as the flexibility for arbitrarily unstructured meshes, a compact stencil, high parallelizability, and the ability to easily accommodate the h-p adaptivity, but also is more efficient than the LDG method, because rewrit-

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ing the equation into a 1-st order system is not needed. The ordinary differential equation (ODE) system obtained by the DDG space discretization technique can be solved by total variation diminishing (TVD) high order Runge–Kutta methods.[19]

The rest of this paper is organized as follows. In Section 2, we construct the DDG method for the CNLS equations. The mass conservation of the CNLS system in the DDG scheme is proved. In Section 3, numerical experiments for the collision behaviors of two soliton waves with various initial conditions are reported. The conclusion is given in Section 4.

2. DDG method

In this section, we will illustrate the algorithm of the DDG method for the CNLS equations. The computational domain of CNLS equation (1) is taken as $[x_{\rm l}, x_{\rm r}]$, which is then divided into N cells as $[x_{\rm l}, x_{\rm r}] = \bigcup_{j=1}^{N} I_j$, $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$. We denote the size of a cell as Δx. We define the finite element space consisting of complex piecewise polynomials as

$$V_h^k = \{v : v|_{I_j} \in \mathrm{P}^k(I_j), \quad j = 1, \ldots, N\}, \tag{3}$$

where $\mathrm{P}^k(I_j)$ denotes the set of complex polynomials with a degree of up to k on I_j. The $u^+_{j+1/2}$ and $u^-_{j+1/2}$ are values of u at $x_{j+1/2}$ from the right cell I_{j+1} and from the left cell I_j, respectively,

$$\begin{aligned} u^+_{j+1/2} &= \lim_{\epsilon \to 0^+} u(x_{j+1/2} + \epsilon), \\ u^-_{j+1/2} &= \lim_{\epsilon \to 0^+} u(x_{j+1/2} - \epsilon). \end{aligned}$$

We use the usual notations $\{u\} = (u^+ + u^-)/2$ and $[u] = u^+ - u^-$ to denote the mean and the jump of function u at the element boundary point, respectively.

The DDG scheme is applied to system (1). We need to find solutions $u, v \in V_h^k$ such that for all test functions $\phi, \psi \in V_h^k$,

$$\begin{aligned} &\mathrm{i}\int_{I_j} u_t\phi \mathrm{d}x - \int_{I_j} u_x\phi_x \mathrm{d}x + (\widehat{u_x}\phi)_{x_{j-1/2}}^{x_{j+1/2}} \\ &\quad - \frac{1}{2}[u](\phi_x)^-_{j+1/2} - \frac{1}{2}[u](\phi_x)^+_{j-1/2} \\ &\quad + \int_{I_j} f(|u|^2, |v|^2)u\phi \mathrm{d}x = 0, \\ &\mathrm{i}\int_{I_j} v_t\psi \mathrm{d}x - \int_{I_j} v_x\psi_x \mathrm{d}x + (\widehat{v_x}\psi)_{x_{j-1/2}}^{x_{j+1/2}} \\ &\quad - \frac{1}{2}[v](\psi_x)^-_{j+1/2} - \frac{1}{2}[v](\psi_x)^+_{j-1/2} \\ &\quad + \int_{I_j} g(|u|^2, |v|^2)v\psi \mathrm{d}x = 0, \end{aligned} \tag{4}$$

where the hat terms above are the numerical fluxes, which are defined as[17]

$$\widehat{u_x} = \frac{\beta_0}{\Delta x}[u] + \{u_x\}, \quad \widehat{v_x} = \frac{\beta_0}{\Delta x}[v] + \{v_x\}. \tag{5}$$

Next we will give the proof of the mass conservation of the CNLS system in the DDG scheme.

Theorem 1 System (1) has standard mass conserved quantities

$$\begin{aligned} \int_{x_l}^{x_{\rm r}} |U(x,t)|^2 \mathrm{d}x &= \int_{x_l}^{x_{\rm r}} |u_0(x)|^2 \mathrm{d}x, \\ \int_{x_l}^{x_{\rm r}} |V(x,t)|^2 \mathrm{d}x &= \int_{x_l}^{x_{\rm r}} |v_0(x)|^2 \mathrm{d}x. \end{aligned} \tag{6}$$

For the proof of Theorem 1, see Ref. [14]. It is desirable that the numerical schemes have the corresponding discrete mass conservation. Next we are going to prove that the DDG scheme preserves the conservation.

Theorem 2 The DDG scheme (4) with numerical flux (5) satisfies the discrete mass conservation

$$\begin{aligned} \sum_{j=1}^{N}\int_{I_j} |u|^2 \mathrm{d}x &= \sum_{j=1}^{N}\int_{I_j} |u_0(x)|^2 \mathrm{d}x, \\ \sum_{j=1}^{N}\int_{I_j} |v|^2 \mathrm{d}x &= \sum_{j=1}^{N}\int_{I_j} |v_0(x)|^2 \mathrm{d}x. \end{aligned} \tag{7}$$

Proof Sum up equalities (4) over $j = 1, \ldots, N$. The test functions ϕ and ψ are taken as u^* and v^*, which are the complex conjugates of u and v, respectively. Then we obtain

$$\begin{aligned} &\sum_{j=1}^{N}\int_{I_j} (\mathrm{i}u_t u^* - u_x u_x^*)\mathrm{d}x \\ &\quad - \sum_{j=1}^{N}(\widehat{u_x}[u^*] + [u]\{u_x^*\})_{j+1/2} \\ &\quad + \sum_{j=1}^{N}\int_{I_j} f(|u|^2, |v|^2)uu^* \mathrm{d}x = 0. \end{aligned} \tag{8}$$

Putting flux (5) into Eq. (8), we get

$$\begin{aligned} &\sum_{j=1}^{N}\int_{I_j} (\mathrm{i}u_t u^* - |u_x|^2)\mathrm{d}x \\ &\quad - \sum_{j=1}^{N}\left(\frac{\beta_0}{\Delta x}|[u]|^2 + \{u_x\}[u^*] + [u]\{u_x^*\}\right)_{j+1/2} \\ &\quad + \sum_{j=1}^{N}\int_{I_j} f(|u|^2, |v|^2)|u|^2 \mathrm{d}x = 0. \end{aligned} \tag{9}$$

Taking the imaginary part of Eq. (9), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{N}\int_{I_j}|u|^2\mathrm{d}x=0. \tag{10}$$

For the discrete conservation of variable v, the proof is same as the above. Then we get the discrete mass conservation (7).

For the actual numerical implementation, we decompose complex functions $u(x,t)$ and $v(x,t)$ into their real and imaginary parts by writing $u(x,t) = u_1(x,t) + \mathrm{i}u_2(x,t)$ and $v(x,t) = v_1(x,t) + iv_2(x,t)$. Then we apply the DDG scheme to the decomposed system.

3. Numerical results

In this section, we apply the DDG scheme to the CNLS equation (1) with different initial values and β.

First we give an accuracy test example to demonstrate the effectiveness of our DDG scheme. Let $\beta = 1$ and $u_0(x) = v_0(x)$, then system (1) reduces to the single NLS equation

$$\mathrm{i}U_t + U_{xx} + 2|U|^2U = 0, \tag{11}$$

with initial condition $u_0(x) = \mathrm{sech}(x)\,\mathrm{e}^{2\mathrm{i}x}$ and the periodic boundary condition. The computational domain is taken as $[-10, 20]$. We compute the errors in L_2 norm with different numbers of cells and obtain the orders of convergence shown in Table 1. We can compare these results with the numerical results in Ref. [7]. It shows the sufficient advantage of the DDG scheme, which has a high constructing accuracy.

Table 1. L_2 norm error of the difference solutions at $t = 1$.

N	P^2		P^3	
	L_2 error	Order	L_2 error	Order
20	1.1864×10^{0}	–	4.7130×10^{-1}	–
40	1.6260×10^{-1}	2.8672	1.7400×10^{-2}	4.7595
80	8.4000×10^{-3}	4.2748	9.5542×10^{-4}	4.1868
160	6.9214×10^{-4}	3.6013	5.8735×10^{-5}	4.0238

Then we compute the CNLS equation (1) with the following initial value:

$$\begin{aligned} u_0(x) &= \sqrt{2}r_1\,\mathrm{sech}\left(r_1x+\frac{d_0}{2}\right)\exp\left(\frac{\mathrm{i}v_0x}{4}\right),\\ v_0(x) &= \sqrt{2}r_2\,\mathrm{sech}\left(r_2x-\frac{d_0}{2}\right)\exp\left(\frac{-\mathrm{i}v_0x}{4}\right), \end{aligned} \tag{12}$$

where $r_1 = 1.2$, $r_2 = 1$, and β is a real-valued cross-phase modulation coefficient. When $\beta = 1$, the CNLS equations reduce to the Manakov equation, and the solitary waves collide elastically. Otherwise, the system is non-integrable, the collisions may be non-trivial, and interesting nonlinear phenomena can be expected. In this paper, we take various values of β to investigate the role of this parameter. All computations are done over region $-40 \le x \le 40$ and $0 \le t \le 50$ except the last one, in which $0 \le t \le 25$ is used. We take a space interval of $\Delta x = 0.5$ and a time step of $\Delta t = 0.002$. Here all problems are solved by the P^2 element, namely, $k = 2$ in Eq. (3). To solve the ordinary differential equation system obtained by the DDG scheme, we take the third order explicit Rung–Kutta method in time discretization.[19]

Along with the mass conservation (6), there is the energy conservation for the CNLS system (1)

$$\begin{aligned} I_3 = \int_{x_l}^{x_r}&\frac{1}{2}(|U|^2+|V|^2)+\beta|U|^2|V|^2\\ &-|U_x|^2-|V_x|^2\mathrm{d}x. \end{aligned} \tag{13}$$

To test the mass and the energy conservation properties of the present scheme (4) at time $t = n\Delta t$, the discretized expressions for Eqs. (6) and (13) are written as

$$\begin{aligned} I_1^n &= \Delta x\sum_{j=1}^{N}|u_j^n|^2, \quad I_2^n = \Delta x\sum_{j=1}^{N}|v_j^n|^2,\\ I_3^n &= \Delta x\sum_{j=1}^{N}\left(\frac{1}{2}(|u_j^n|^2+|v_j^n|^2)+\beta|u_j^n|^2|v_j^n|^2\right.\\ &\quad\left.-|(u_x)_j^n|^2-|(v_x)_j^n|^2\right), \end{aligned} \tag{14}$$

where $u_j^n = u(x_j, nt)$ and $v_j^n = v(x_j, nt)$ are taken at the cell center.

In the first case, we choose $\beta = 1$, $v_0 = 1$, and $d_0 = 25$, then the elastic interaction of two solitons is expected. From Fig. 1(a), we can see that after the colliding of the two solitons, the two waves move forward without changing their directions and velocities. Figure 1(b) shows the computation results for I_1^n, I_2^n, and I_3^n, which are constants in all time. We can see that the new DDG scheme preserves the mass and the energy conservations of the CNLS system.

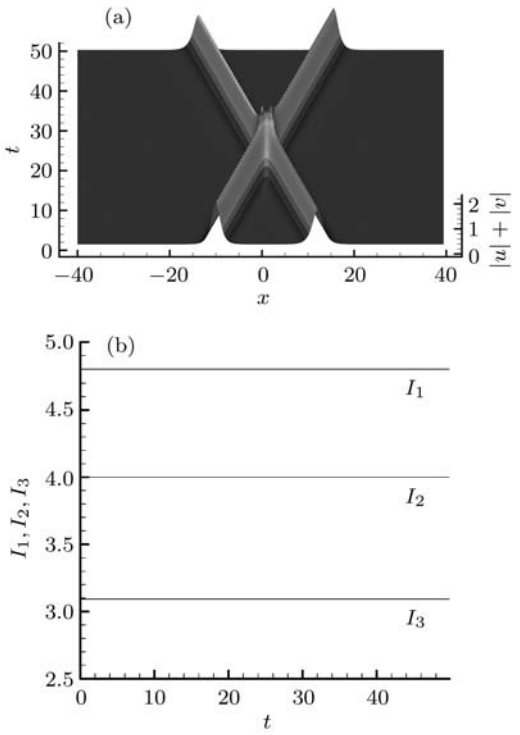


Fig. 1. (a) $|u| + |v|$, and (b) discrete mass and energy conservations for the numerical solution obtained with $\beta = 1,\ v_0 = 1$, and $d_0 = 25$.

In the second case, we choose $\beta = 2/3,\ v_0 = 1.3$, and $d_0 = 25$, which correspond to the real single-mode birefringent fibers. The transmission scenario is displayed in Fig. 2. From Fig. 2, we observe that after the colliding of the two solitons, the two waves are reshaped, and daughter waves are generated.

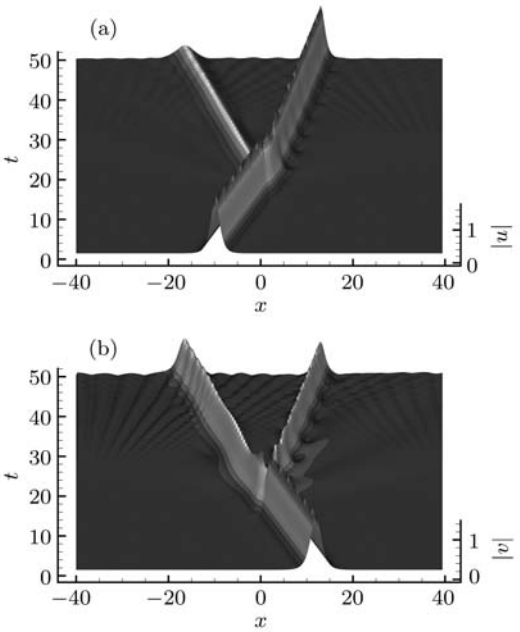


Fig. 2. (a) $|u|$ and (b) $|v|$ for the example with $\beta = \frac{2}{3},\ v_0 = 1.3$, and $d_0 = 25$.

Then we give two examples with β far away from 1. In Fig. 3, we take $\beta = 0.3,\ v_0 = 0.4$, and $d_0 = 12.5$. From Fig. 3, we can see that the two solitons fuse into one soliton after colliding. In Fig. 4, we take $\beta = 3,\ v_0 = 1.6$, and $d_0 = 25$. From Fig. 4, we notice that a new soliton is created after the collision of the two solitons.

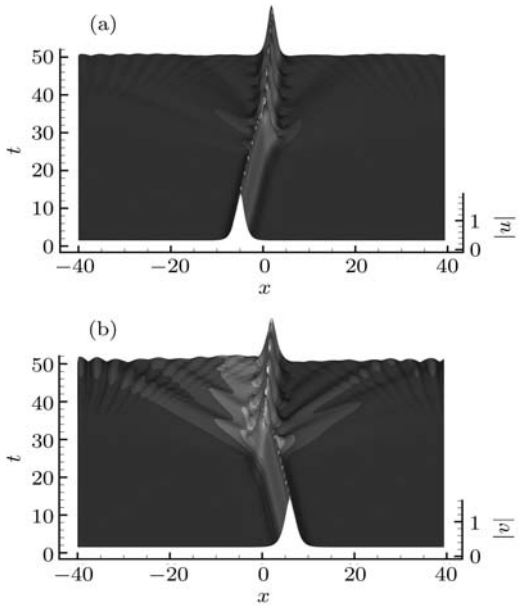


Fig. 3. (a) $|u|$ and (b) $|v|$ for the example with $\beta = 0.3,\ v_0 = 0.4$, and $d_0 = 12.5$.

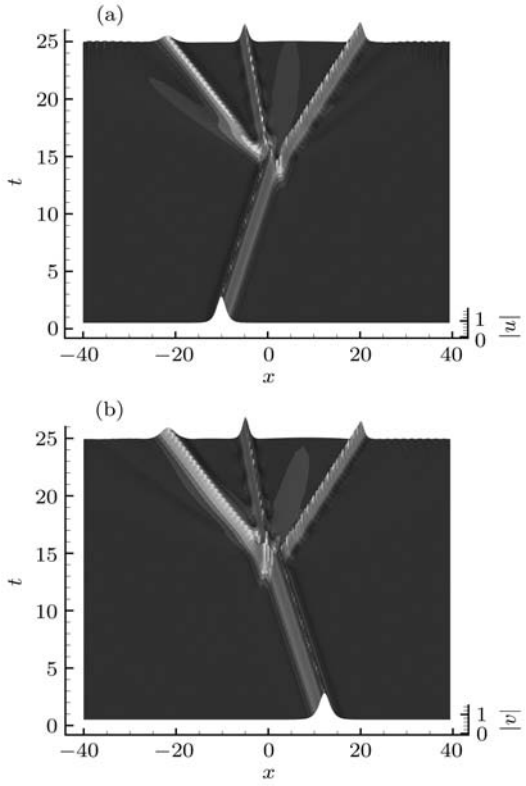


Fig. 4. (a) $|u|$ and (b) $|v|$ for the example with $\beta = 3,\ v_0 = 1.6$, and $d_0 = 25$.

4. Conclusion

In this paper, the DDG scheme for the coupled 1D nonlinear Schröinger system is proposed. The computational results show that the conservations of mass and energy are maintained accurately by the adopted numerical scheme, and the accuracy is improved considerably. We also prove that the method gives stable, low diffusion, and accurate results for long time computing. We can extend the DDG scheme to other nonlinear systems and expect it to work properly.

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