Asymptotic solving method for sea–air coupled oscillator ENSO model*

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The ENSO is an interannual phenomenon involved in the tropical Pacific ocean–atmosphere interaction. In this article, we create an asymptotic solving method for the nonlinear system of the ENSO model. The asymptotic solution is obtained. And then we can furnish weather forecasts theoretically and other behaviors and rules for the atmosphere–ocean oscillator of the ENSO.

Keywords: nonlinear system, asymptotic solution, sea–air coupled oscillator

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1. Introduction

The EI Niño/La Niña southern oscillation (ENSO) is an abnormal event that happens in the atmosphere and ocean of the tropical Pacific. Its physical cause had been a mystery for more than half a century until Bjerknes[1] visualized a close relation between the atmospheric Walker circulation and the sea surface temperature (SST) contrast fluctuation in the equatorial Pacific Ocean. The southern oscillation and the EI Niño/La Niño cycle describe a unified interannual climate variation simply referred to as ENSO in recent years. The complexity and the nonlinearity of the ocean–atmosphere interaction pose great difficulties for theoreticians to explain the ENSO using simple models. Consequently, numerical modeling becomes a popular approach. Some successes have been achieved during the last two decades in the numerical simulation of the ENSO with coupled intermediate models.[2,3]

The understanding obtained from numerical experiments has considerably advanced our knowledge of ENSO physics. At the same time, theoretical studies of the properties of the coupled ocean and atmosphere system were extensively pursued following the Bjerknes[4] hypothesis. Early theoretical analyses adopted prototype coupled models.[5] Philander *et al.*[6] presented the first rigorous stability analysis of a coupled shallow water system on an equatorial beta plane. Additional mechanisms and extensions were proposed in subsequent studies.[7−10] The development of a warm episode is now understood as resulting from a coupled instability of the atmosphere and ocean. The theoretical explanation for the temporal structure of the southern oscillation remains an outstanding challenge. A central question raised in Bjerknes'[1] pioneering work is how the turnabout from a cold state to a warm state takes place. Zebiak and Cane[11] first attempted to address Bjerknes' puzzle by examining the cause of the oscillation in their numerical model. They found an increase of the equatorial heat content prior to the warm events and a sharp decrease during the events. The delay that played a central role in the above argument resulted from the ocean wave dynamics as shown by Cane *et al.*[12] For studying nonlinear characteristics, on the basis of some simple coupled ocean–atmosphere models for the ENSO, many scientists studied the local

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and the whole properties of the ENSO with different methods.[13−28]

It is clear from the above discussion that great difficulties exist in theory to explain the physical mechanisms of the ENSO. Because the dynamics of the atmosphere and ocean in nature is nonlinear, it is very important to study the difficulties that result from the nonlinearity of the prediction model of the ENSO.

Recently, many scholars have investigated the nonlinear problem. Approximation methods have been developed and refined, including the method of averaging, the boundary layer method, the matched asymptotic expansion method, and the multiple-scale method.[29−32] Using the perturbation method, the authors[33] also have considered a class of nonlinear perturbation problems. In this article, we consider a class of ENSO models by using a simple and special method.[34]

2. Sea–air coupled oscillator model

The conceptual oscillator model should consider the variations of both the eastern and the western Pacific anomaly patterns. The oscillator model can be reduced to a simple one by using further simplification and assumption.

The western Pacific oscillator model of Wang *et al.*[13,14] emphasized the role of the western Pacific anomaly patterns in the ENSO. In particular, the off-equatorial SST variation of the date time initiates the equatorial wind anomalies in the western Pacific. Those wind anomalies lead to ocean responses that proceed eastward to affect the anomalies in the equatorial eastern Pacific. It does not necessarily need the explicit role of wave propagation.

Now we consider a sea–air coupled oscillator of the ENSO model

$$\frac{\mathrm{d}T}{\mathrm{d}t} = CT + Dh - \varepsilon\, f(T), \tag{1}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -ET - R_h h - \varepsilon\, g(h), \tag{2}$$

where T is the SST anomaly in the equatorial eastern Pacific, h is the thermocline depth anomaly, $\varepsilon\, f(T)$, $\varepsilon\, g(h)$ are the disturbed terms, with ε being a small positive parameter, and C, D, E, R_h are positive model parameters, for their physical definitions, the readers might be referred to Refs. [14]–[17].

When $\varepsilon = 0$, equations (1) and (2) reduce to

$$\frac{\mathrm{d}T}{\mathrm{d}t} = CT + Dh, \tag{3}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -ET - R_h h. \tag{4}$$

It is easy to see that the characteristic roots λ_i $(i = 1, 2)$ of the linear system (3) and (4) are

$$\lambda_1 = \frac{1}{2}\left[(C - R_h) - \sqrt{(C + R_h)^2 - 4DE}\right], \tag{5}$$

$$\lambda_2 = \frac{1}{2}\left[(C - R_h) + \sqrt{(C + R_h)^2 - 4DE}\right]. \tag{6}$$

As $C > R_h$, $(C+R_h)^2 < 4DE$, based on the theory of singular point, the zero solution of the linear system (3) and (4) is an unstable focal point. Because the nonlinear terms in Eqs. (1) and (2) are $-\varepsilon\, f(T)$ and $-\,\varepsilon\, g(h)$, we assume

$$\lim_{(T^2+h^2)\to 0} \varepsilon \sqrt{\frac{f^2 + g^2}{T^2 + h^2}} = 0.$$

Then the singular solution of the nonlinear coupling system (1) and (2) is an unstable focal point too.

It is easy to see that the solution $(\bar{T}(t), \bar{h}(t))$ of system (3) and (4) is

$$\begin{aligned}\bar{T}(t) = {} & \frac{A_1}{\beta\, E} \exp(\alpha\, t)\Big[\cos(-\beta\, t + B_1) \\ & + \frac{C - \alpha}{\beta} \sin(-\beta\, t + B_1)\Big],\end{aligned} \tag{7}$$

$$\bar{h}(t) = \frac{A_1}{\beta} \exp(\alpha\, t) \sin(-\beta\, t + B_1), \tag{8}$$

where $\alpha = (C - R_h)/2$, $\beta = \sqrt{4DE - (C + R_h)^2}/2$, and A_1, B_1 are arbitrary constants.

3. Asymptotic solution for disturbed sea–air oscillator model

Introducing a set of mapping

$$\begin{aligned}H_1(T, h, p) = {} & L_1(T, h) - L_1(u_0, v_0) \\ & + p(L_1(u_0, v_0) + \varepsilon\, f(T)),\end{aligned} \tag{9}$$

$$\begin{aligned}H_2(T, h, p) = {} & L_2(T, h) - L_2(u_0, v_0) \\ & + p(L_2(u_0, v_0) + \varepsilon\, g(h)),\end{aligned} \tag{10}$$

where p is a parameter, (u_0, v_0) is the zeroth order approximation of the solution of the disturbed ENSO system (1) and (2), and the linear operators L_i $(i = 1, 2)$ are

$$L_1(T, h) = \frac{\mathrm{d}T}{\mathrm{d}t} - CT - Dh,$$

$$L_2(T, h) = \frac{\mathrm{d}h}{\mathrm{d}t} + ET + R_h h. \tag{11}$$

Let $(u_0,\ v_0)$ be the solution $(\bar{T}(t), \bar{h}(t))$ of system (3) and (4). From Eqs. (7) and (8), we have

$$u_0(t) = \frac{A_1}{\beta E}\exp(\alpha t)\Big[\cos(-\beta t + B_1) + \frac{C-\alpha}{\beta}\sin(-\beta t + B_1)\Big], \tag{12}$$

$$v_0(t) = \frac{A_1}{\beta}\exp(\alpha t)\sin(-\beta t + B_1). \tag{13}$$

Obviously, $H_j(T, h,\ 1) = \lim_{p\to 1} H_j(T, h, p) = 0 \quad (j = 1, 2)$ are the same as system (1) and (2). Then the solution $(T(t), h(t))$ of system (1) and (2) is the same as the solution to $H_j(T, h, p) = 0 \ \ (j = 1, 2)$ with $p \to 1$.

Let

$$T = \sum_{k=0}^{\infty} T_k(t)p^k, \quad h = \sum_{k=0}^{\infty} h_k(t)p^k. \tag{14}$$

Considering Eqs. (9) and (10) developing to the power series of p and Eq. (14), from the coefficients of the zero-order power of p, we have

$$T_0(t) = \frac{A_1}{\beta E}\exp(\alpha t)\Big[\cos(-\beta t + B_1) + \frac{C-\alpha}{\beta}\sin(-\beta t + B_1)\Big], \tag{15}$$

$$h_0(t) = \frac{A_1}{\beta}\exp(\alpha t)\sin(-\beta t + B_1). \tag{16}$$

From the coefficients of the first-order power of p, we have

$$L_1(T_1, h_1) = -\varepsilon\, f(T_0), \tag{17}$$

$$L_2(T_1, h_1) = -\varepsilon\, g(h_0). \tag{18}$$

We can obtain the solution of Eqs. (17) and (18) with the zero initial conditions as

$$\begin{pmatrix} T_1(t) \\ h_1(t) \end{pmatrix} = -\varepsilon \int_0^t \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(s) \begin{pmatrix} f(T_0(s)) \\ g(h_0(s)) \end{pmatrix} \mathrm{d}s, \tag{19}$$

where

$$\boldsymbol{\Phi}(t) = \begin{pmatrix} D\exp(\lambda_1 t) & D\exp(\lambda_2 t) \\ (\lambda_1 - C)\exp(\lambda_1 t) & (\lambda_2 - C)\exp(\lambda_2 t) \end{pmatrix} \times \begin{pmatrix} \lambda_2 - C & -D \\ -(\lambda_2 - C) & D \end{pmatrix}.$$

It is easy to see that $\boldsymbol{\Phi}(t)$ is a real matrix.

From the coefficients of the second-order power of p, we have

$$L_1(T_2, h_2) = -\varepsilon \left[\frac{\mathrm{d}f}{\mathrm{d}T}(T_0)\right] T_1, \tag{20}$$

$$L_2(T_2, h_2) = -\varepsilon \left[\frac{\mathrm{d}g}{\mathrm{d}h}(h_0)\right] h_1. \tag{21}$$

We can obtain the solution of Eqs. (20) and (21) with the zero initial conditions as

$$\begin{pmatrix} T_2(t) \\ h_2(t) \end{pmatrix} = -\varepsilon \int_0^t \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(s) \times \begin{pmatrix} \dfrac{\mathrm{d}f}{\mathrm{d}T}(T_0(s))]T_1(s) \\ \dfrac{\mathrm{d}g}{\mathrm{d}h}(h_0(s))]h_1(s) \end{pmatrix} \mathrm{d}s. \tag{22}$$

From the coefficients of the k-th $(k = 3, 4, \ldots)$ order power of p, we have

$$L_1(T_k, h_k) = -\frac{\varepsilon}{(k-1)\,!}\left[\frac{\partial^{k-1} f}{\partial p^{k-1}}\left(\sum_{i=0}^{\infty} T_i p^i\right)\right]_{p=0}, \tag{23}$$

$$L_2(T_k, h_k) = -\frac{\varepsilon}{(k-1)\,!}\left[\frac{\partial^{k-1} g}{\partial p^{k-1}}\left(\sum_{i=0}^{\infty} h_i p^i\right)\right]_{p=0}. \tag{24}$$

Similarly, we can obtain the solution for Eqs. (23) and (24) with the zero initial conditions as

$$\begin{pmatrix} T_k(t) \\ h_k(t) \end{pmatrix} = -\varepsilon \int_0^t \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(s) \times \begin{pmatrix} \dfrac{1}{(k-1)\,!}\left[\dfrac{\partial^{k-1} f}{\partial p^{k-1}}\left(\sum\limits_{i=0}^{\infty} T_i p^i\right)\right]_{p=0} \\ \dfrac{1}{(k-1)\,!}\left[\dfrac{\partial^{k-1} g}{\partial p^{k-1}}\left(\sum\limits_{i=0}^{\infty} h_i p^i\right)\right]_{p=0} \end{pmatrix} \mathrm{d}s. \tag{25}$$

Thus from Eqs. (15), (16), (19), (22), and (25), we can obtain the n-th order asymptotic solution $(\bar{T}_n,\ \bar{h}_n)$ of the disturbed atmosphere–ocean oscillator of the ENSO model

$$\begin{pmatrix} \bar{T}_n(t) \\ \bar{h}_n(t) \end{pmatrix} = \begin{pmatrix} \dfrac{A_1}{\beta E}\exp(\alpha t)\left[\cos(-\beta t + B_1) + \dfrac{C-\alpha}{\beta}\sin(-\beta t + B_1)\right] \\ \dfrac{A_1}{\beta}\exp(\alpha t)\sin(-\beta t + B_1) \end{pmatrix} - \varepsilon \int_0^t \boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(s) \left[\begin{pmatrix} f(T_0(s)) \\ \\ g(h_0(s)) \end{pmatrix}\right.$$

$$+\left(\begin{array}{c}\dfrac{\mathrm{d}f}{\mathrm{d}T}(T_0(s))T_1(s)\\ \dfrac{\mathrm{d}g}{\mathrm{d}h}(h_0(s))h_1(s)\end{array}\right)+\left(\begin{array}{c}\sum\limits_{k=3}^{n}\dfrac{1}{(k-1)\,!}\left[\dfrac{\partial^{k-1}f}{\partial p^{k-1}}\left(\sum\limits_{i=0}^{\infty}T_ip^i\right)\right]_{p=0}\\ \sum\limits_{k=3}^{n}\dfrac{1}{(k-1)\,!}\left[\dfrac{\partial^{k-1}g}{\partial p^{k-1}}\left(\sum\limits_{i=0}^{\infty}h_ip^i\right)\right]_{p=0}\end{array}\right)\mathrm{d}s$$
$$+\,\mathrm{O}(\varepsilon^{n+1}),\quad 0<\varepsilon<<1. \tag{26}$$

4. Discussion

From the fixed point theorem,[29] we can also prove that

$$\left(\begin{array}{c}T(t)\\ h(t)\end{array}\right)=\lim_{n\to\infty}\left(\begin{array}{c}\bar{T}_n(t)\\ \bar{h}_n(t)\end{array}\right) \tag{27}$$

uniformly holds. Thus, equation (27) is a set of exact solutions for the atmosphere–ocean oscillator of the ENSO model (1) and (2).

It is clear that great difficulties exist in theory to explain the physical mechanisms of the ENSO. Because the dynamics of the atmosphere and ocean in nature is nonlinear, it is very important to study the difficulties that result from the nonlinearity of the prediction model of the ENSO. From the above result, we obtain the asymptotic solution ($\bar{T}_n$, $\bar{h}_n$) (Eq. (26)) of the disturbed atmosphere–ocean oscillator of the ENSO model (1) and (2), which can be used to analyze sequentially the change in nature of the disturbed oscillator of the ENSO model using the method of analytic operation.

From the discussion, we can furnish weather forecasts theoretically and other behaviors of the SST anomaly and the thermocline depth anomaly of the atmosphere–ocean oscillator.

5. Example

Consider a special sea–air coupled oscillator of the ENSO model. In the oscillator model, we assume that the winds in the western Pacific do not affect the sea surface temperature anomalies in the eastern Pacific, and the atmosphere is in a steady state or the Niño-4 (over 5°S–5°N, 160°E–150°W) zonal wind stress anomalies are linearly proportional to the Niño-3 (over 5°S–5°N, 150°–90°W) SST anomalies. We can select some peculiar parameters for the sea–air coupled oscillator of the disturbed ENSO model (1) and (2),[14,37] $C=0.97\times10^{-2}\ ^\circ\mathrm{C}\cdot\mathrm{m}^2\cdot\mathrm{N}^{-1}\cdot\mathrm{a}^{-1}$, $D=1.5\times10^{-2}\ ^\circ\mathrm{C}\cdot\mathrm{m}^2\cdot\mathrm{N}^{-1}\cdot\mathrm{a}^{-1}$, $E=-1.5\times10^{-2}\ ^\circ\mathrm{C}\cdot\mathrm{m}^2\mathrm{N}^{-1}\cdot\mathrm{a}^{-1}$, $R_h=0\times10^{-2}\ ^\circ\mathrm{C}\cdot\mathrm{m}^2\cdot\mathrm{N}^{-1}\cdot\mathrm{a}^{-1}$, and the disturbed terms are $f(T)=\varepsilon\sin T$, $g(h)=\varepsilon\cos\ h$. Then system (1) and (2) is given by

$$\frac{\mathrm{d}T}{\mathrm{d}t}=0.97T+1.5h-\varepsilon\ \sin\ T, \tag{28}$$
$$\frac{\mathrm{d}h}{\mathrm{d}t}=1.5T-0.5h-\varepsilon\ \cos\ h. \tag{29}$$

From the above ENSO model and Eqs. (7), (8), (15), and (16) with the initial conditions $T(0)=1$, $h(0)=0$, we have

$$\lambda_1=-1.42,\ \ \lambda_2=1.91,\ \ \alpha=0.24,$$
$$\beta=0.21,\ \ A_1=-0.32,\quad B_1=0.$$

From Eqs. (15), (16), (19), and (26), we obtain the first-order approximate solution ($T_{1\mathrm{app}}(t)$, $h_{1\mathrm{app}}(t)$) of the sea–air coupled oscillator of the ENSO model (28) and (29)

$$\left(\begin{array}{c}T_{1\mathrm{app}}(t)\\ h_{1\mathrm{app}}(t)\end{array}\right)=\left(\begin{array}{c}0.31\exp(0.24\,t)[\cos(-0.21\,t)-3.48\sin(0.21\,t)]\\ 1.52\exp(0.24\,t)\sin(0.21\,t)\end{array}\right)$$
$$-\varepsilon\int_0^t\boldsymbol{\Phi}(t)\boldsymbol{\Phi}^{-1}(s)\left(\begin{array}{c}\sin\ (0.31\exp(0.24\,t)[\cos(-0.21\,t)-3.48\sin(0.21\,t)])\\ \cos\ [1.52\exp(0.24\,t)\sin(0.21\,t)]\end{array}\right)\mathrm{d}s,$$
$$+\,\mathrm{O}(\varepsilon^2),\quad 0<\varepsilon<<1, \tag{30}$$

where

$$\boldsymbol{\Phi}(t)=\left(\begin{array}{cc}1.42[\exp(-1.42t)-\exp(1.97t)] & 2.25[-\exp(-1.42t)+\exp(1.97t)]\\ 0.94[0.45\exp(-1.42t)-0.94\exp(1.91t)] & 1.5[-0.45\exp(-1.42t)+0.94\exp(1.91t)]\end{array}\right).$$

Table 1. Numerical comparison of the simulated exact solution ($T_{\rm exa}$, $h_{\rm exa}$) and the approximate solution ($T_{\rm 1app}$, $h_{\rm 1app}$) ($\varepsilon = 0.1$).

t	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
$T_{\rm exa}$	1.00	1.00	1.00	0.93	0.64	0.31	0.05	– 0.11	– 0.18	– 0.18	– 0.15
$T_{\rm 1app}$	1.00	1.00	1.00	0.94	0.68	0.35	0.07	– 0.10	– 0.19	– 0.19	– 0.17
$h_{\rm exa}$	0.00	0.00	0.01	0.09	0.33	0.45	0.43	0.33	0.19	0.06	– 0.01
$h_{\rm 1app}$	0.00	0.00	0.00	0.07	0.31	0.46	0.45	0.35	0.21	0.07	– 0.01

From Eqs. (28)–(30), we can obtain the following simulation data of the exact solution ($T_{\rm exa}(t)$, $h_{\rm exa}(t)$) and the first-order approximate solution ($T_{\rm 1app}(t)$, $h_{\rm 1app}(t)$) for the sea–air coupled oscillator of the ENSO model (28) and (29) using the simulated numerical method (Table 1).

Thus, we know that the obtained approximate solutions using the approximate method possess a better degree of accuracy.

6. Summary

The ENSO is a complicated natural phenomenon. Hence we need to use reduced basic models. We solve them using approximate methods. In this article, a new special asymptotic method is shown to be valid. The method is an approximate analytical method, which differs from the general numerical method.

From our study, we know that the sea–air oscillator of the ENSO models (1) and (2), the sea surface temperature anomaly, and the zonal wind stress anomaly have an unstable focal point at the $O-T\ h$ phase plane, and the path curves are far away from the focal point.

Furthermore, in this article, on the basis of a class of oscillator of the ENSO models, using the asymptotic method, the higher-order approximate solutions of the corresponding system are gained. And we can furnish weather forecast more accurately.

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