Constrained consensuses in disconnected topologies*

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This paper is concerned with the constrained consensuses problem for a group of agents in disconnected topologies. By dividing the communication topology into a combination of directed trees, some necessary and sufficient conditions are derived for all the agents to asymptotically reach a single consensus and multiple consensuses, respectively. The obtained results indicate that arbitrary anticipant consensuses can be achieved, if additional constrained controllers are added to those agents with specific indexes. Some illustrative examples are provided to demonstrate the effectiveness of the proposed theoretical results.

Keywords: disconnected topologies, single consensus, multiple consensuses

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1. Introduction

In recent years, the consensus problem for multi-agent systems has attracted a lot of interest due to its broad applications including flocking, rendezvous, formation control, swarm aggregation, sensor networks, etc.[1−8] The formal study of consensus problems can be traced back to the 1970's in management science and statistics.[9] In networks of agents, consensus means to reach an agreement regarding a certain quantity of interest that depends on the update protocol and the initial states of all agents. Vicsek *et al.*[10] investigated the emergence of self-ordered motion in systems of particles with biologically motivated interaction. In their model, the particles were driven by a constant absolute velocity and each agent's heading is updated according to a local rule that based on the average of its own heading plus the headings of its neighbors. The theoretical framework for posing and solving consensus problems was introduced by Olfati-Saber and Murray in Refs. [11] and [12]. They studied the consensus problems for networks of dynamic agents with fixed and switching topologies and introduced two consensus protocols for networks with and without time-delays. A direct relation between the algebraic connectivity of the network and the performance of a linear consensus protocol was established, which required the generalization of algebraic connectivity in undirected graphs to digraphs. Tanner[13] studied the controllability of nearest neighbor interconnections and pointed out that the connectivity had an adverse effect on the controllability.

Note that the aforementioned results only considered that all the agents in a system could reach a single consensus. Moreover, the hidden condition, which should not be neglected, that the communication topology graph is supposed to contain a spanning tree when consensus is achieved. Yu and Wang[14] investigated a couple-group consensuses problem in networks of dynamic agents, where the agents could reach more than one consistent value asymptotically, but the communication topology graph was still connected. Moore[15] studied the coordination of multiple agents via nearest-neighbor negotiations over a consensus variable and extended the existing results for single consensus variable to the case of forced consensus. However, the topology graph related to single consensus problem still contains a spanning tree and the communication links between groups only consist in the root vertexes.

Inspired by the analyses mentioned above, some further extensions are considered in this paper. First, owing to the existence of packet drops, limited communication channels, external disturbances and other influencing factors, the hidden condition in a consensus problem that the communication topology graph contains a spanning tree is too strict in the real world.

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However, to the best of our knowledge, few results have broken this restriction, which motivates us to investigating the consensus problem in disconnected topologies. Secondly, in most existing results, the considered consensus is often fixed once the initial states of the agents are given. In some practical applications, however, the anticipant consensus may be expected to turn to a new one during the process, which may be impossible to be achieved through the original update protocol. Moreover, it should be pointed out that it is wasteful and impracticable to add a controller on every agent. To deal with this problem, we divide the communication topology graph into a combination of directed trees, based on which the indexes and the amount of the controlled agents are determined. Furthermore, the multiple consensuses problem (i.e., the agents are required to reach more than one homogeneous stationary states) in a disconnected topology is also investigated in this paper.

The organization of this paper is arranged as follows. In Section 2, some preliminary knowledge of graph theory and matrix theory is introduced. Section 3 is devoted to the analyses and syntheses of single consensus. The main results for multiple consensuses are given in Section 4. Illustrative examples are provided in Section 5. Finally, the conclusion is given in Section 6.

2. Preliminaries

The node indexes belong to a finite index set $\mathscr{I} = \{1, 2, \ldots, N\}$. Let $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ be a weighted directed graph with N nodes, where $\mathscr{V} = \{v_1, v_2, \ldots, v_N\}$ is the set of nodes, $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ is the set of edges, and $\mathscr{A} = [a_{i,j}]$ is a weighted adjacency matrix. An edge of $\mathscr{G}$ is denoted by (v_i, v_j), which means that the agent v_i can obtain information from the agent v_j and $(v_i, v_j) \in \mathscr{E}$ if and only if $a_{i,j} \neq 0$. Here, it is assumed that there are no self-loops, i.e., $a_{i,i} = 0$ for all $i \in \mathscr{I}$. The set of neighbors of sensor i is denoted by $\mathscr{N}_i = \{v_j \in \mathscr{V} \,|(v_i, v_j) \in \mathscr{E}\}$. For the edge (v_i, v_j), v_j is the parent node and v_i is the child node. A directed path is a sequence of ordered edges of the form $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_{m-1}}, v_{i_m})$. A directed tree is a directed graph, where every vertex, except the root vertex, has exactly one parent and the root vertex can transmit information to any other vertices through paths. Denote $T(v_i)$ as a directed tree with a root vertex v_i. A spanning tree of the directed graph $\mathscr{G}$ is a directed tree formed by $\mathscr{E}' \subset \mathscr{E}$ that includes all the vertices of the graph.

3. Single consensus

The consensus problem studied in this section consists of N agents, with the labels 1 through N. All the agents update their information through the following linear protocol:

$$\dot{x}_i(t) = \sum_{j \in \mathscr{N}_i} \alpha_{i,j}(x_j(t) - x_i(t)), \quad i = 1, 2, \ldots, N, \tag{1}$$

where x_i is the state of agent v_i and $\alpha_{i,j} > 0$ is a weighted value of the transmission channel from v_j to v_i.

Rewrite Eq. (1) in a matrix form as

$$\dot{x}(t) = -Lx(t), \tag{2}$$

where $L = [l_{ij}]$ is a Laplacian matrix[1)] with the elements

$$l_{ij} = \begin{cases} -\alpha_{i,j}, & j \neq i; \\ \sum\limits_{j \in \mathscr{N}_i} \alpha_{i,j}, & j = i. \end{cases} \tag{3}$$

The single consensus is defined that the states of the N agents asymptotically converge to a homogeneous stationary value x^*. In this paper, the link between any two agents in the communication topology graph is allowed to be both unidirectional and bidirectional. Thus, the related Laplacian matrix is not necessarily symmetric.

First, we introduce the following definition of spanning communication topology, which is different from the actual communication topology.

Definition 1 (spanning communication topology $g(n)$): Suppose that the actual communication topology is $\mathscr{G}$, then the spanning topology $g(n)$ is defined as $g(n) = \bigcup_{i=1}^{n} T_i$ with $\mathscr{V}(g(n)) = \mathscr{V}(\mathscr{G})$, $\mathscr{E}(g(n)) \subset \mathscr{E}(\mathscr{G})$, and $\sum_{i \neq j} |T_i \cap T_j| = 0$, where $|T_i|$ represents the number of vertexes contained in the directed tree T_i. Furthermore, denote $\langle g(n) \rangle$ by the amount of different directed trees T_i in $g(n)$.

According to Definition 1, the spanning communication topology is a combination of disjoint directed trees that cover all the nodes in $\mathscr{G}$. Obviously, there is a spanning tree that covers every vertex in $\mathscr{G}$ when $\min_n \langle g(n) \rangle = 1$. Then, the following lemma holds.

[1)] Note that the Laplacian matrix L is diagonally dominant with zero row sums, that is to say, 0 is an eigenvalue of L with the associated eigenvector $\mathbf{1} = [1, \ldots, 1]^{\mathrm{T}}$ and all of the nonzero eigenvalues of L have positive real parts.[16] Furthermore, 0 is a simple eigenvalue of L if the associated directed graph is strongly connected.[17]

Lemma 1 The following statements are equivalent:

(i) $\min_n \langle g(n)\rangle = 1$.

(ii) The consensus strategy (1) achieves global consensus asymptotically for $\mathscr{V}(\mathscr{G})$.

(iii) The Laplacian matrix L of the graph associated with $\mathscr{G}$ has a simple zero eigenvalue.

Proof The proof is followed by the Theorem 3.8 in Ref. [18] and thus omitted here.

Consider the case that $\min_n \langle g(n)\rangle = 1$, i.e., there exists a spanning tree $g(1) = T_1(v_{i_1})$ in the topology graph $\mathscr{G}$. If there is an additional link (v_{i_1}, v_{i_0}) with weighted value α_{i_1,i_0} added to $\mathscr{G}$ and denoted $\tilde{\mathscr{G}}$ by the new communication topology, then the following two lemmas hold.

Lemma 2[19] Let L and $\tilde{L}$ be the Laplacian matrices of $\mathscr{G}$ and $\tilde{\mathscr{G}}$, respectively. If $\mathscr{V}(\mathscr{G}) = \mathscr{V}(\tilde{\mathscr{G}})$ and L has a simple eigenvalue 0, then so does for $\tilde{L}$.

Lemma 3 If the consensus strategy (1) achieves global consensus asymptotically for $\mathscr{V}(\mathscr{G})$, then the consensus can still be reached for $\mathscr{V}(\tilde{\mathscr{G}})$.

Proof

Case 1 $v_{i_0} \in \mathscr{V}(\mathscr{G}) - \{v_{i_1}\}$ and $(v_{i_1}, v_{i_0}) \notin \mathscr{E}(\mathscr{G})$. In this case, according to Lemma 1, if the consensus can be asymptotically achieved under the communication topology graph $\mathscr{G}$, then the Laplacian matrix L has a simple zero eigenvalue. This leads to the fact that $\tilde{L}$ also has a simple zero eigenvalue according to Lemma 2, which follows that $\min_n \langle \tilde{g}(n)\rangle = 1$. Thus, with the statements in Lemma 1, the consensus can also be asymptotically achieved under the communication topology graph $\tilde{\mathscr{G}}$.

Case 2 $v_{i_0} \notin \mathscr{V}(\mathscr{G})$. In this case, $(v_{i_1}, v_{i_0}) \cup T_1(v_{i_1}) = \tilde{T}_1(v_{i_0}) = \tilde{g}(1)$, which implies that $\tilde{g}(1)$ is a directed tree with root vertex v_{i_0}. In this way, one has $\min_n \langle \tilde{g}(n)\rangle = 1$, and then the consensus can be achieved asymptotically for $\mathscr{V}(\tilde{\mathscr{G}})$. Thus, the proof is completed.

According to Lemma 3, if there is a spanning tree $T(v_{i_1})$ in $\mathscr{G}$, then the consensus strategy (1) can still achieve global consensus asymptotically if the weighted values $\alpha_{i,j}$ are adjusted (positive). Furthermore, if a new vertex links to the root vertex, then the $N+1$ agents still can reach a consensus. Thus, the consensus analysis on $\mathscr{G}$ can be simplified to the analysis on one of the spanning trees in $\mathscr{G}$.

Before proceeding further, the consensus invariability of a spanning tree with different label sequences is given as follows.

Lemma 4 Suppose $T(v_r)$ is a spanning tree of communication topology graph $\mathscr{G}$, with the labels 1 through N. If we relabel the vertexes with $1'$ to N', such that the labels of vertexes in depth i are greater than those of vertexes in depth $i-1$, then the new Laplacian matrix L' is a lower triangular matrix, and $\mathrm{rank}(L') = N-1$.

The following example visually shows the meaning of Lemma 4.

Example 1 Considering the topology graph that given in Fig. 1(a). It is easy to obtain that $\min_n \langle g(n)\rangle = 1$ and one of the spanning trees is shown in Fig. 1(b), with the associate Laplacian matrix

$$L = \begin{bmatrix} \alpha_{1,5} & 0 & 0 & 0 & -\alpha_{1,5} \\ 0 & \alpha_{2,5} & 0 & 0 & -\alpha_{2,5} \\ 0 & -\alpha_{3,2} & \alpha_{3,2} & 0 & 0 \\ -\alpha_{4,1} & 0 & 0 & \alpha_{4,1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Relabel the vertexes with $1'$ to $5'$, such that the labels of vertexes in depth i are greater than those of vertexes in depth $i-1$ (Fig. 1(c)), then the new Laplacian matrix L' is a lower triangular matrix

$$L' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -\alpha_{2',1'} & \alpha_{2',1'} & 0 & 0 & 0 \\ -\alpha_{3',1'} & 0 & \alpha_{3',1'} & 0 & 0 \\ 0 & -\alpha_{4',2'} & 0 & \alpha_{4',2'} & 0 \\ 0 & 0 & -\alpha_{5',3'} & 0 & \alpha_{5',3'} \end{bmatrix}.$$

Obviously, $0 = \lambda_1(L') < \lambda_2(L') \leq \lambda_3(L') \leq \lambda_4(L') \leq \lambda_5(L')$ and the consensus still can be achieved since the topology has not been changed.

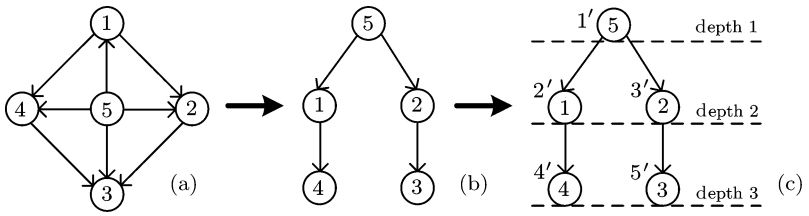


Fig. 1. The relabeling process: (a) topology graph with five vertexes, (b) one of the spanning trees, (c) the spanning tree after relabeling.

Note that the consensus depends on the initial states of all the agents. Once the initial states are given, the consensus under the strategy (1) is fixed. In some practical applications, however, the consensus may be expected to turn to a new one in some cases, which is tough to achieve under the consensus strategy (1). Inspired by these, we add an additional controller on some of the vertexes and the consensus strategy (1) is translated into

$$\dot{x}_i(t) = \sum_{j\in\mathscr{N}_i} \alpha_{i,j}(x_j(t) - x_i(t)) - d_i u_i(t), \qquad (4)$$

where $d_i \geq 0$, $u_i(t) = x_i(t) - x^*$, $i = 1, 2, \ldots, N$.

First, we consider the case that $\min_n \langle g(n)\rangle = 1$. Assume that the spanning topology $g(1) = T_1$ of $\mathscr{G}$ is non-unique. Denote $\hat{\mathscr{V}}(g) = \{v_{1_1}, v_{2_1}, \ldots, v_{k_1}\}$ by the set of root vertexes of $T_1(v_{i_1})$, $i = 1, 2, \ldots, k$, where k is the number of different $g(1)$. Then the following theorem holds.

Theorem 1 If $\min_n \langle g(n)\rangle = 1$, and there are k different spanning topology $g(1)$ in $\mathscr{G}$, then the consensus strategy (4) achieves global consensus x^* asymptotically for $\mathscr{V}(\mathscr{G})$, if and only if there is at least one $d_{i_1} > 0$, where d_{i_1} relates to $v_{i_1} \in \hat{\mathscr{V}}(g)$, $i = 1, 2, \ldots, k$.

Proof Sufficiency: Denote $e_i(t) = x_i(t) - x^*$, then

$$\dot{e}_i(t) = \sum_{j\in\mathscr{N}_i} \alpha_{i,j}(e_j(t) - e_i(t)) - d_i e_i(t). \qquad (5)$$

Denote $e = (e_1, e_2, \ldots, e_N)^{\mathrm{T}}$, then $\dot{e}(t) = -(L + D)e(t)$, where $D = \mathrm{diag}\{d_1, d_2, \ldots, d_N\}$.

For a specific spanning topology $g(1)$, since $\min_n \langle g(n)\rangle = 1$, then $g(1)$ is a spanning tree of $\mathscr{G}$. Relabel the vertexes with $1'$ to N', such that the new Laplacian matrix L' is a lower triangular matrix, then the eigenvalues of L' satisfy $0 = \lambda_1(L') < \lambda_2(L') \leq \cdots \leq \lambda_N(L')$. Thus, $\dot{e}'(t) = -(L'+D')e'(t)$, where $e' = \mathrm{diag}\{e_{1'}, e_{2'}, \ldots, e_{N'}\}$ and $D' = \mathrm{diag}\{d_{1'}, d_{2'}, \ldots, d_{N'}\}$. Without loss of generality, suppose v_{s_1} is the root vertex, then $v_{s_1} = v_{1'}$, $d_{s_1} = d_{1'}$ and all the elements of the 1st row in L' equal 0. If $d_{s_1} > 0$, then

$$\begin{aligned} &|\lambda I_{N\times N} - (L' + D')| \\ &= (\lambda - d_{s_1})\left|\lambda I_{(N-1)\times(N-1)} - (L'[1,1] + D'[1,1])\right| \\ &= (\lambda - d_{s_1})\prod_{j=2}^{N}(\lambda - \lambda_j(L') - d_{j'}), \end{aligned}$$

where $L'[1,1]$ and $D'[1,1]$ are sub-matrices that formed by deleting the 1th row and 1th column of L' and D', respectively.

Since $\lambda_1(L' + D') = d_{s_1}$, $\lambda_j(L' + D') = \lambda_j(L') + d_{j'} \geq \lambda_j(L') > 0$, $j = 2, \ldots, N$, then $\lambda_i(L' + D') > 0$, $i = 1, 2, \ldots, N$, i.e., all the eigenvalues of $L' + D'$ are positive, so the error system (5) is asymptotically stable. Thus, the consensus strategy (4) achieves global consensus x^* asymptotically for $\mathscr{V}(\mathscr{G})$.

Necessity: The consensus strategy (4) achieves global consensus x^* asymptotically for $\mathscr{V}(\mathscr{G})$ implies that the error system (5) is asymptotical stable, which requires that all the eigenvalues of $L+D$ have positive real part. If $d_{i_1} = 0$, $i = 1, 2, \ldots, k$, then the row sums in $L+D$ equal 0, i.e., $L+D$ has an eigenvalue 0, which is a contrary with $\mathrm{Re}(\lambda(L + D)) > 0$. Then, there is at least one $d_{i_1} > 0$. Thus, the proof is completed. □

Next, we will consider the case that $\min_n \langle g(n)\rangle > 1$. In this scenario, the spanning communication topology is $g(n) = \bigcup_{i=1}^{n} T_i$ with $n \geq 2$, which implies that all the vertexes of $\mathscr{V}(\mathscr{G})$ cannot be connected by one directed tree. Suppose $\min_n \langle g(n)\rangle = p \geq 2$, then each T_i can be augmented to $\mathscr{G}_i$, with $\mathscr{V}(T_i) = \mathscr{V}(\mathscr{G}_i)$, $\mathscr{V}(\mathscr{G}) = \mathscr{V}(\mathscr{G}_1)\cup\mathscr{V}(\mathscr{G}_2)\cup\ldots\cup\mathscr{V}(\mathscr{G}_p)$, $\mathscr{E}(\mathscr{G}_i) \subset \mathscr{E}(\mathscr{G})$, and $\mathscr{V}(\mathscr{G}_i)\cap\mathscr{V}(\mathscr{G}_j) = \varnothing$, $i \neq j$. Obviously, $\min_n \langle g_i(n)\rangle = 1$, $i = 1, 2, \ldots, p$. Then, according to Theorem 1, if there is no communication link between $\mathscr{G}_i$ and $\mathscr{G}_j$, $i \neq j$, the consensus strategy (4) achieves global consensus x^* asymptotically for $\mathscr{V}(\mathscr{G})$ if and only if there are at least one $d^i_{s_1} > 0$ in each $\mathscr{G}_i$, where $d^i_{s_1}$ relates to the root vertex $v^i_{s_1}$ of $T_i(v^i_{s_1})$. Of course, it is an ideal status that $\mathscr{G}_i$ and $\mathscr{G}_j$ are independent. When information exchanges exist between $\mathscr{G}_i$ and $\mathscr{G}_j$, the consensus strategy (4) is translated into

$$\begin{aligned} &\dot{x}_i(t) \\ &= \sum_{j\in\mathscr{N}_{i,i}} \alpha_{i,j}(x_j(t) - x_i(t)) + \sum_{k\neq i}\sum_{j\in\mathscr{N}_{i,k}} \alpha_{i,j}(x_j(t) \\ &\quad - x_i(t)) - d_i u_i(t), \end{aligned} \qquad (6)$$

where $d_i \geq 0$, $u_i(t) = x_i(t) - x^*$, $\mathscr{N}_{i,i}$ is the set of neighbors of node $v_i \in \mathscr{V}(\mathscr{G}_i)$ in $\mathscr{G}_i$ and $\mathscr{N}_{i,k}$ is the set of neighbors of node $v_i \in \mathscr{V}(\mathscr{G}_i)$ in $\mathscr{G}_k$, $k \neq i$, $i = 1, 2, \ldots, p$.

According to Lemma 3, the consensus analysis on $\mathscr{G}_i$ can be simplified to the analysis on one of the spanning topology of $\mathscr{G}_i$. Denote $e_i(t) = x_i(t) - x^*$, then

$$\dot{e}_i(t) = \sum_{j\in\mathscr{N}_{i,i}} \alpha_{i,j}(e_j(t) - e_i(t)) + \sum_{k\neq i}\sum_{j\in\mathscr{N}_{i,k}} \alpha_{i,j}(e_j(t)$$

$$- e_i(t)) - d_i e_i(t). \tag{7}$$

Denote $L_1, L_2, \ldots, L_p$ by the Laplacian matrix related to $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_p$, respectively, then equation (7) can be rewritten in a matrix form as

$$\dot{e}(t) = -(L + A + D)e(t), \tag{8}$$

where $L = \mathrm{diag}\,(L_1, L_2, \ldots, L_p)$, $D = \mathrm{diag}\,(D_1, D_2, \ldots, D_p)$ and

$$A = \begin{bmatrix} A_{11} & -A_{12} & \ldots & -A_{1p} \\ -A_{21} & A_{22} & \ldots & -A_{2p} \\ \vdots & \vdots & & \vdots \\ -A_{p1} & -A_{p2} & \ldots & A_{pp} \end{bmatrix},$$

where $A_{ii} = \mathrm{diag}\big\{ \sum_{k\neq i} \sum_{j\in\mathscr{N}^i_{1,k}} \alpha_{i,j}, \sum_{k\neq i} \sum_{j\in\mathscr{N}^i_{2,k}} \cdot \alpha_{i,j}, \ldots, \sum_{k\neq i} \sum_{j\in\mathscr{N}^i_{p,k}} \alpha_{i,j} \big\}$ and A_{ij} is the adjacency matrix of $\mathscr{G}_i$ and $\mathscr{G}_j$, $i \neq j$, $i = 1, 2, \ldots, p$.

Denote $\hat{\mathscr{V}}(g_i) = \left\{ v^i_{1_1}, v^i_{2_1}, \ldots, v^i_{k_{i1}} \right\}$ by the set of root vertexes of $T_i(v^i_{s_1})$ related to L_i, $s = 1, 2, \ldots, k_i$, where k_i is the number of different $g_i(1)$, $i = 1, 2, \ldots, p$. Then, the following theorem holds.

Theorem 2 The consensus strategy (6) achieves global consensus x^* asymptotically for $\mathscr{V}(\mathscr{G})$, if and only if for each $i \in \{1, \ldots, p\}$, there is at least one $d^i_{s_1} > 0$, where $d^i_{s_1}$ relates to $v^i_{s_1} \in \hat{\mathscr{V}}(g_i)$, $s = 1, 2, \ldots, k_i$.

Proof Sufficiency: Obviously, A is diagonally dominant, has zero row sum and nonnegative diagonal elements. Then, $\mathrm{Re}(\lambda_i(A)) \geq 0$, $i = 1, 2, \ldots, N$.

Relabel the vertexes of $\mathscr{G}$ with $1'$ to N', such that the new Laplacian matrix L' is a lower triangular matrix. Since $v^i_{s_1}$ is the root vertex of $T_i(v^i_{s_1})$, according to the definition of root vertex, all the elements of the 1st row in L'_i equal 0. If there exists a $d^i_{s_1} > 0$ for any $i \in \{1, \ldots, p\}$, similar to the proof in Theorem 1, we have $\lambda_i(L' + D') > 0$, $i = 1, 2, \ldots, N$. Thus, $\mathrm{Re}(\lambda_i(L' + A' + D')) > 0$, $i = 1, 2, \ldots, N$, i.e., all the eigenvalues of $L' + A' + D'$ have positive real part. In this way, the error system (7) is asymptotical stable, which implies that the global consensus x^* is asymptotically achieved.

Necessity: The consensus strategy (6) achieves global consensus x^* for $\mathscr{V}(\mathscr{G})$ implies that the error system (7) is asymptotically stable, which follows that all the eigenvalues of $L + A + D$ have positive real part. According to the definition of root vertex, all the elements of the row related to $v^i_{s_1}$ in L and A equal 0, so $L + A + D$ has a eigenvalue 0 if all the $d^i_{s_1} = 0$, $s = 1, 2, \ldots, k_i$, which is a contrary to $\mathrm{Re}(\lambda_i(L + A + D)) > 0$. Thus, there is at least one $d^i_{s_1} > 0$. This completes the proof.

Note that owing to the choice of different $T_i(v^i_{s_i})$, the spanning communication topology $g(p) = \bigcup_{i=1}^p T_i$ is non-unique for a specific p. Suppose there are m different $g(p)$, the related augmented topology graphs are

$$\begin{gathered} S_1 : \left(\mathscr{G}^1_1, \mathscr{G}^1_2, \ldots, \mathscr{G}^1_p\right), \\ S_2 : \left(\mathscr{G}^2_1, \mathscr{G}^2_2, \ldots, \mathscr{G}^2_p\right), \\ \vdots \\ S_m : \left(\mathscr{G}^m_1, \mathscr{G}^m_2, \ldots, \mathscr{G}^m_p\right), \end{gathered}$$

where $\min_n \left\langle g^l_i(n) \right\rangle = 1$, $i = 1, 2, \ldots, p$, $l = 1, 2, \ldots, m$. Then Theorem 2 is developed into the following one.

Theorem 3 The consensus strategy (6) achieves global consensus x^* asymptotically for $\mathscr{V}(\mathscr{G})$, if and only if there exists a $l \in \{1, \ldots, m\}$, such that for each $i \in \{1, \ldots, p\}$, there is at least one $d^l_{i,s_1} > 0$, where d^l_{i,s_1} relates to the root vertexes v^l_{i,s_1} of $T^l_i\left(v^l_{i,s_1}\right)$.

Proof The proof is similar to the one in Theorem 2 and omitted here. □

4. Multiple consensuses

In this section, we consider the multiple consensuses problem. Suppose that the system is composed of p groups of agents, with the related topology graph $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_p$, respectively. Denote the index set

$$\mathscr{I}_k = \left\{ 1 + \sum_{i=1}^{k-1} q_i,\ 2 + \sum_{i=1}^{k-1} q_i,\ \ldots,\ q_k + \sum_{i=1}^{k-1} q_i \right\},$$

and the set of nodes $\mathscr{V}_k = \{v_{k,1}, v_{k,2}, \ldots, v_{k,q_k}\}$, where q_k are the amount of agents in group k, $k = 1, 2, \ldots, p$. The set of the neighbors of agent $v_{k,i}$ in $\mathscr{G}_l$ is defined by $\mathscr{N}^l_{k,i} = \{v_{k,j} \in \mathscr{V}_l | (v_{k,i}, v_{k,j}) \in \mathscr{E}\}$, $l = 1, 2, \ldots, p$. Then the multiple consensuses can be defined that each group of agents asymptotically converge to its own homogeneous stationary state, i.e.,

$$x^i_1 = x^i_2 = \cdots = x^i_{q_i} \to x^*_i, \text{ as } t \to \infty,\ i = 1, 2, \ldots, p.$$

Combining the in-group information transmission and out-group transmission, the consensus strategy can be written as

$$\dot{x}_i(t)=\begin{cases}\sum\limits_{j\in\mathscr{N}_{1,i}^1}\alpha_{i,j}(x_j(t)-x_i(t))+\sum\limits_{l\neq 1}\sum\limits_{j\in\mathscr{N}_{1,i}^l}\alpha_{i,j}(x_j(t)-x_i(t))-d_iu_i^1(t)-u_i^2(t), & i\in\mathscr{I}_1;\\ \quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots & \\ \sum\limits_{j\in\mathscr{N}_{p,i}^p}\alpha_{i,j}(x_j(t)-x_i(t))+\sum\limits_{l\neq p}\sum\limits_{j\in\mathscr{N}_{p,i}^l}\alpha_{i,j}(x_j(t)-x_i(t))-d_iu_i^1(t)-u_i^2(t), & i\in\mathscr{I}_p,\end{cases}\tag{9}$$

where $u_i^1(t)$ and $u_i^2(t)$ are additional consensus compensation laws that will be designed later. Obviously, the consensus strategy (9) cannot achieve arbitrary consensuses when $u_i^1(t)=u_i^2(t)=0$.

Denote the smallest set of root vertex in $\mathscr{G}$ and $\mathscr{G}_i$, respectively, as

$$\hat{\mathscr{V}}=\left\{v_{s,k}\in\mathscr{V}(\mathscr{G})\,\middle|\,r=\min\langle g(n)\rangle,\quad g(r)=\bigcup_{k=1}^{r}T(v_{s,k})\right\},$$
$$\hat{\mathscr{V}}_i=\left\{v_{i,k}\in\mathscr{V}(\mathscr{G}_i)\,\middle|\,m=\min\langle g_i(n)\rangle,\quad g_i(m)=\bigcup_{k=1}^{m}T_i(v_{i,k})\right\},$$

and the set of the vertex in $\mathscr{G}_i$ that links with another one in $\mathscr{G}_j$, $i\neq j$, is defined as

$$\breve{\mathscr{V}}_i=\{v_{i,k}\in\mathscr{V}(\mathscr{G}_i)\,|v_{i,k}\ \text{ can recieve the information from }\mathscr{G}_j,i\neq j\}.$$

Then, the consensus compensation laws are designed as

$$u_i^1(t)=\begin{cases}x_i(t)-x_1^*,\ i\in\mathscr{I}_1;\\ \quad\vdots\\ x_i(t)-x_p^*,\ i\in\mathscr{I}_p.\end{cases}\tag{10}$$

$$u_i^2(t)=\begin{cases}\sum\limits_{j\in\mathscr{N}_{1,i}^k,k\neq 1}\alpha_{i,j}(x_k^*-x_1^*), & v_{1,i}\in\breve{\mathscr{V}}_1,i\in\mathscr{I}_1;\\ \quad\vdots & \\ \sum\limits_{j\in\mathscr{N}_{p,i}^k,k\neq p}\alpha_{i,j}(x_k^*-x_p^*), & v_{p,i}\in\breve{\mathscr{V}}_p,i\in\mathscr{I}_p.\end{cases}\tag{11}$$

Similarly to the analysis in Section 3, the consensus analysis on $\mathscr{G}$ can be simplified to the analysis on the spanning topology of $\mathscr{G}$. Denote $e_i(t)=x_i(t)-x_i^*$, then

$$\dot{e}_i(t)=\begin{cases}\sum\limits_{j\in\mathscr{N}_{1,i}^1}\alpha_{i,j}(e_j(t)-e_i(t))+\sum\limits_{l\neq i}\sum\limits_{j\in\mathscr{N}_{1,i}^l}\alpha_{i,j}(e_j(t)-e_i(t))-d_ie_i(t), & i\in\mathscr{I}_1;\\ \quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots & \\ \sum\limits_{j\in\mathscr{N}_{p,i}^p}\alpha_{i,j}(e_j(t)-e_i(t))+\sum\limits_{l\neq i}\sum\limits_{j\in\mathscr{N}_{p,i}^l}\alpha_{i,j}(e_j(t)-e_i(t))-d_ie_i(t), & i\in\mathscr{I}_p.\end{cases}\tag{12}$$

Rewrite Eq. (12) in a matrix form as

$$\dot{e}(t)=-(L+D)e(t),\tag{13}$$

where $D=\mathrm{diag}(D_1,D_2,\ldots,D_p)$, $D_i=\mathrm{diag}(d_{i,1},d_{i,2},\ldots,d_{i,q_i})$,

$$L=\begin{bmatrix}L_1 & L_{12} & \ldots & L_{1p}\\ L_{21} & L_2 & \ldots & L_{2p}\\ \vdots & \vdots & & \vdots\\ L_{p1} & L_{p2} & \ldots & L_p\end{bmatrix}$$

with

$$L_i = \begin{bmatrix} \sum\limits_{j\in\mathscr{N}^i_{i,1}} \alpha_{1,j} + \sum\limits_{l\neq i}\sum\limits_{j\in\mathscr{N}^l_{i,1}} \alpha_{1,j} & -\alpha_{1,2} & \cdots & -\alpha_{1,q_i} \\ -\alpha_{2,1} & \sum\limits_{j\in\mathscr{N}^i_{i,2}} \alpha_{2,j} + \sum\limits_{l\neq i}\sum\limits_{j\in\mathscr{N}^l_{i,2}} \alpha_{2,j} & & -\alpha_{2,q_i} \\ & & & \\ -\alpha_{q_i,1} & -\alpha_{q_i,2} & \cdots & \sum\limits_{j\in\mathscr{N}^i_{i,q_i}} \alpha_{q_i,j} + \sum\limits_{l\neq i}\sum\limits_{j\in\mathscr{N}^l_{i,q_i}} \alpha_{q_i,j} \end{bmatrix},$$

and

$$L_{ij} = \begin{bmatrix} -\alpha_{\hat{i}+1,\hat{j}+1} & \cdots & -\alpha_{\hat{i}+1,\hat{j}+q_j} \\ \vdots & & \vdots \\ -\alpha_{\hat{i}+q_i,\hat{j}+1} & \cdots & -\alpha_{\hat{i}+q_i,\hat{j}+q_j} \end{bmatrix}, \quad \hat{i} = \sum_{k=1}^{i-1} q_k, \quad \hat{j} = \sum_{k=1}^{j-1} q_k, \quad i \neq j, \quad i,j = 1,2,\ldots,p.$$

If $\hat{\mathscr{V}}$ and $\hat{\mathscr{V}}_i$, $i = 1,2,\ldots,p$, are unique, then the following theorem holds.

Theorem 4 The consensus strategy (9) achieves multiple consensuses asymptotically for $\mathscr{V}(\mathscr{G})$, if and only if all the $d_{s,k} > 0$, where $d_{s,k}$ relates to $v_{s,k} \in \hat{\mathscr{V}}$. Especially, if $\bigcup_{i=1}^{p}\left(\hat{\mathscr{V}}_i \cap \breve{\mathscr{V}}_i\right) = \varnothing$, then $\hat{\mathscr{V}} = \bigcup_{i=1}^{p} \hat{\mathscr{V}}_i$.

Proof Note that the consensus strategy (9) achieves multiple consensuses

$$(\underbrace{x_1^*,\ldots,x_1^*}_{q_1},\underbrace{x_2^*,\ldots,x_2^*}_{q_2},\ldots,\underbrace{x_p^*,\ldots,x_p^*}_{q_p})^{\mathrm{T}}$$

asymptotically for $\mathscr{V}(\mathscr{G}) = \bigcup_{k=1}^{p} \mathscr{V}_k$, is equivalent that the error update strategy (12) achieves a consensus $e^* = 0$ asymptotically for $\mathscr{V}(\mathscr{G}) = \bigcup_{k=1}^{p} \mathscr{V}_k$, i.e., $e_i(t) \to e^* = 0$, as $t \to \infty$, $i = 1,2,\ldots,\sum_{i=1}^{p} q_i$. Then the multiple consensuses is translated to a single one and the state of agent v_i is e_i, $i = 1,2,\ldots,\sum_{i=1}^{p} q_i$. Similar to the analysis in Theorem 2, the error update strategy (12) achieves a global consensus $e^* = 0$ asymptotically, if and only if all the $d_{s,k} > 0$, where $d_{s,k}$ relates to $v_{s,k} \in \hat{\mathscr{V}}$.

If $\bigcup_{i=1}^{p}\left(\hat{\mathscr{V}}_i \cap \breve{\mathscr{V}}_i\right) = \mathrm{O}$, then $\min_n \langle g(n)\rangle = \sum_{i=1}^{p} \min_n \langle g_i(n)\rangle$, which implies that $\hat{\mathscr{V}} = \bigcup_{i=1}^{p} \hat{\mathscr{V}}_i$. Thus, the proof is completed.

Not that for given r and m, the $g(r) = \bigcup_{k=1}^{r} T(v_{s,k})$ and $g_i(m) = \bigcup_{k=1}^{m} T_i(v_{i,k})$ can be multifarious, which leads to the fact that both $\hat{\mathscr{V}}$ and $\hat{\mathscr{V}}_i$ may be non-unique. Considering these, denote $\Omega(\hat{\mathscr{V}})$ as the set of different $\hat{\mathscr{V}}$. Suppose $\min_n \langle g_i(n)\rangle = p_i$ and there are m_i different $g_i(p_i)$, then the related augmented topology graphs are

$$S : \left(\widehat{\mathscr{G}}_1, \widehat{\mathscr{G}}_2, \ldots, \widehat{\mathscr{G}}_p\right),$$

where

$$\widehat{\mathscr{G}}_i \in \left\{\left(\mathscr{G}^1_{i,1},\mathscr{G}^2_{i,1},\ldots,\mathscr{G}^{p_i}_{i,1}\right),\left(\mathscr{G}^1_{i,2},\mathscr{G}^2_{i,2},\ldots,\mathscr{G}^{p_i}_{i,2}\right),\ldots,\left(\mathscr{G}^1_{i,m_i},\mathscr{G}^2_{i,m_i},\ldots,\mathscr{G}^{p_i}_{i,m_i}\right)\right\},\ i = 1,2,\ldots,p.$$

Then, Theorem 4 is developed into the following one.

Theorem 5 The consensus strategy (9) achieves multiple consensuses asymptotically for $\mathscr{V}(\mathscr{G})$, if and only if there exists a set $\hat{\mathscr{V}} \in \Omega(\hat{\mathscr{V}})$, such that all the $d_{s,k} > 0$, where $d_{s,k}$ relates to $v_{s,k} \in \hat{\mathscr{V}}$. Especially, for the case that

$$\bigcup_{i=1}^{p}\left(\bigcup_{j=1}^{m_i}\left(\hat{\mathscr{V}}_i(\mathscr{G}^r_{i,j}) - \breve{\mathscr{V}}_i(\mathscr{G}^r_{i,j})\right)\right) = \varnothing,$$

then the multiple consensuses can be achieved if and only if for each $i \in \{1,2,\ldots,p\}$ and $r \in \{1,2,\ldots,p_i\}$, there exists a $j \in \{1,2,\ldots,m_i\}$, such that there is at least one $d^r_{i,j} > 0$, where $d^r_{i,j}$ relates to $v^r_{i,j} \in \hat{\mathscr{V}}_i\left(\mathscr{G}^r_{i,j}\right)$.

5. Simulation results

Example 2 Consider a multi-agent system with the communication topology graph given in Fig. 2. From Fig. 2, it is easy to obtain that $\min_n \langle g(n)\rangle = 1$, and the set of root vertexes of different $T(v_{i_1})$ is $\hat{\mathscr{V}}(g) = \{v_1, v_2\}$. According to Theorem 2, we know that under the consensus strategy (4), all the agents achieve global consensus x^* asymptotically if and only if $d_1 > 0$ or $d_2 > 0$. Choose the anticipant consensus $x^* = 6$, with the initial state $x(0) =$

$[1,2,3,4,5,6,7,8,9]^{\mathrm{T}}$, then the state trajectories of agents are shown in Fig. 3.

From Fig. 4, we can know that when $d_i = 0$, all the agents can achieve a global consensus that consists of the linear combination of $x_i(0)$, $i = 1, 2, \ldots, 9$. And an arbitrary anticipant (static) consensus can be reached if $d_1 = 1$, or $d_2 = 1$, or $d_1 = 1$ and $d_2 = 1$.

Example 3 Consider a multi-agent system with the communication topology graph given in Fig. 4. From Fig. 4, we can get that the $\min_n |g(n)| = 2$ and the set of root vertexes of different $T(v_{i_1})$ is $\hat{\mathscr{V}}(g) = \{v_1, v_9\}$ or $\hat{\mathscr{V}}(g) = \{v_2, v_9\}$. According to Theorem 3, we know that under the consensus strategy (6), all the agents achieve global consensus x^* asymptotically if and only if $d_1 > 0$ and $d_9 > 0$, or $d_2 > 0$ and $d_9 > 0$. Choose the anticipant consensus $x^* = 6$, the state trajectories of agents with initial state $x(0) = [1,2,3,4,5,6,7,8,9]^{\mathrm{T}}$ are shown in Fig. 5.

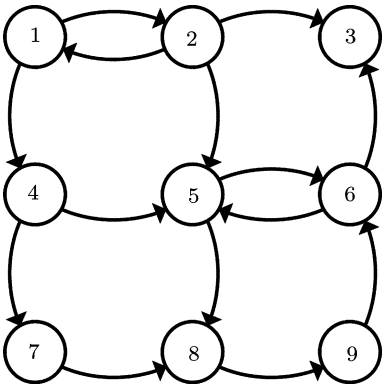


Fig. 2. Topology graph (Example 2).

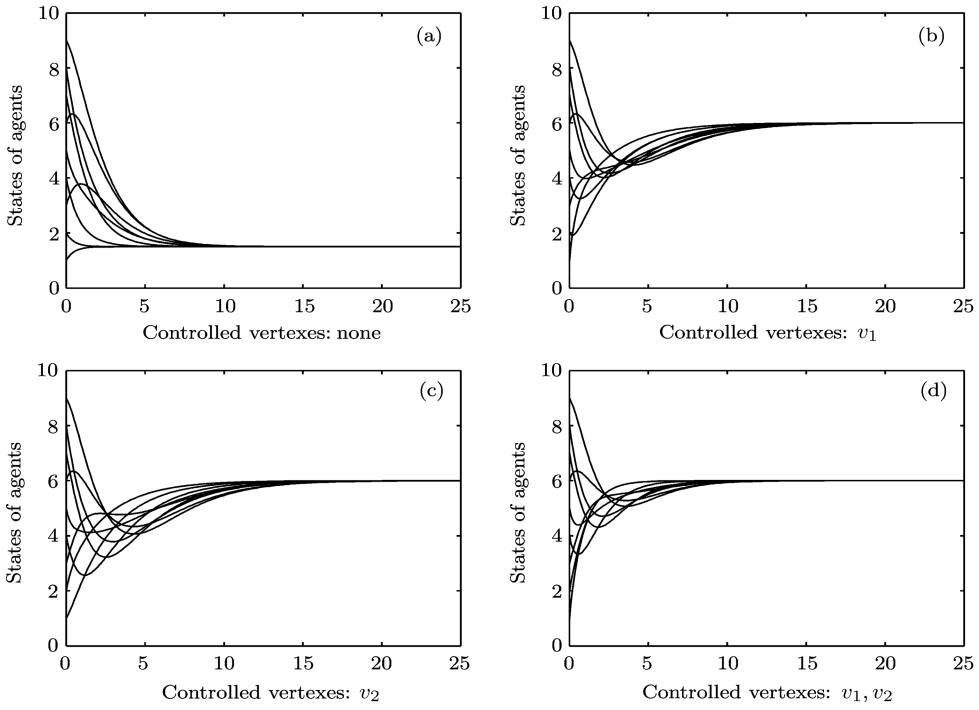


Fig. 3. The state trajectories of nine agents (Example 2).

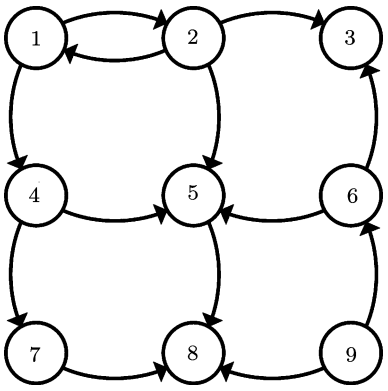


Fig. 4. Topology graph (Example 3).

From Fig. 5, it can be seen that under the consensus strategy with no constrained control in a disconnected topology, all the agents can not reach a global consensus. And an arbitrary anticipant (static) consensus can be reached if $d_1 = 1$ and $d_9 = 1$, or $d_2 = 1$ and $d_9 = 1$.

Example 4 For a system consists of twelve agents, the topology graph is given in Fig. 6. All the agents are divided into three groups, with the index set $\mathscr{I}_1 = \{1,2,3,4\}$, $\mathscr{I}_2 = \{5,6,7,8\}$ and $\mathscr{I}_3 = \{9,10,11,12\}$, respectively. Both in-group information communication and out-group information

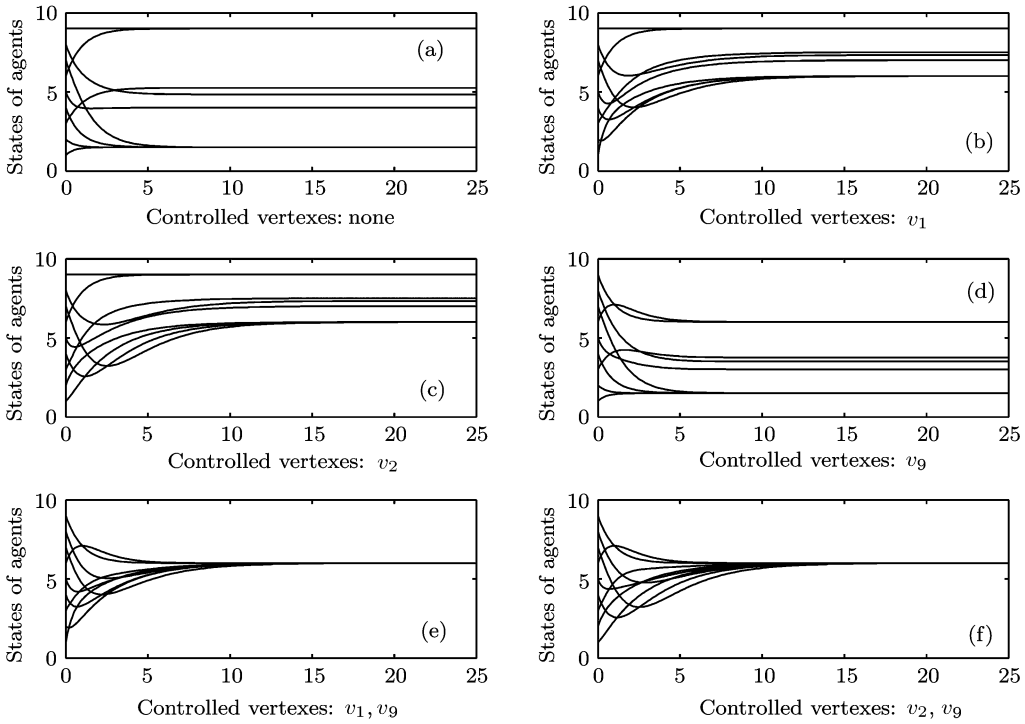


Fig. 5. The state trajectories of nine agents (Example 3).

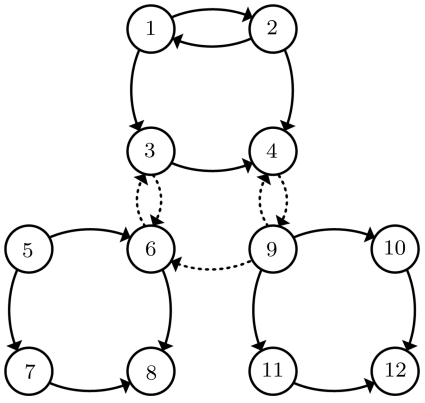


Fig. 6. Topology graph (Example 4).

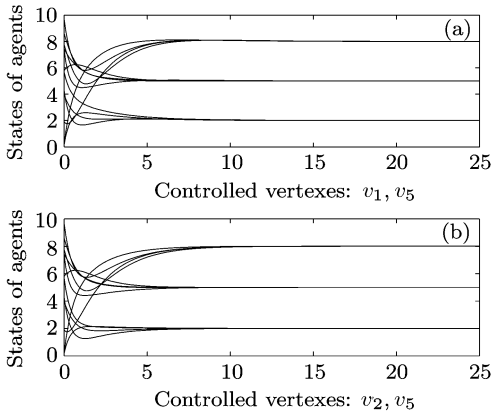


Fig. 7. The state trajectories of twelve agents (Example 4).

communication are existing, represented by real line and broken line in Fig. 6, respectively. It is easy to obtain that $\hat{\mathscr{V}}_1 = \{1\} \vee \{2\}$, $\hat{\mathscr{V}}_2 = \{5\}$, $\hat{\mathscr{V}}_3 = \{9\}$, $\breve{\mathscr{V}}_1 = \{3,4\}$, $\breve{\mathscr{V}}_2 = \{6\}$, $\breve{\mathscr{V}}_3 = \{9\}$, then $\hat{\mathscr{V}} = \{1,5\} \vee \{2,5\}$. Set the consensus values $x_1^* = 2$, $x_2^* = 5$, $x_3^* = 8$. According to Theorem 5, choose $d_1 = d_5 = 1$ and $d_2 = d_5 = 1$, respectively, the state trajectories of agents with stochastic initial state $x_i(0) \in [0, 10]$, $i = 1, 2, \ldots, 12$, can be obtained as shown in Fig. 7.

6. Conclusion

In this paper, the constrained consensuses problem is investigated for a group of agents in disconnected topologies. According to the fact that the consensus analysis on a communication topology can be simplified to the analysis on one of the spanning topology, the communication topology is divided into a combination of directed trees, based on which some necessary and sufficient conditions are derived for both single consensus and multiple consensuses and the indexes and the amount of the controlled agents are determined. The obtained results indicate that arbitrary anticipant consensuses can be achieved, if additional constrained controllers are added on those agents with specific indexes.

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