Comparison of nonclassicality between photon-added and photon-subtracted squeezed vacuum states*

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In this paper, we introduce photon-added and photon-subtracted squeezed vacuum state (PASV and PSSV) and obtain their normalized factors, which have the similar forms involved in Lengendre polynomials. Moreover, we give the compact expressions of Wigner function, which are related to single-variable Hermite polynomials. Especially, we compare their nonclassicality in terms of Mandel Q-factor and the negativity of Wigner function.

Keywords: photon-added and photon-subtracted squeezed vacuum state, nonclassicality, Wigner function

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1. Introduction

Nonclassical states play an important role in quantum computation and quantum information procession.[1] The nonclassicality of quantum states can be well-described by some nonclassical properties, such as sub-Poissonian photon statistics,[2] squeezing in one of the quadratures of the field,[3] and negativity of Wigner function (WF).[4] Especially, the partial negativity of WF implies the highly nonclassical properties of quantum states. Many experimental schemes have been proposed to generate nonclassical states of optical fields. In recent years, subtracting photons from and/or adding photons to quantum states have been paid much attention because these fields exhibit abundant nonclassical properties and may give access to a complete engineering of quantum states and to fundamental quantum phenomena.[5−10] For example, a single-photon addition was experimentally performed by Zavatta *et al.*,[11] which unveils the nonclassical features associated with the excitation of a classical coherent field by a single light quantum. For the single photon-subtracted squeezed vacuum state (PSSV), its nonclassical properties and decoherence was investigated theoretically in two different decoherent channels (amplitude decay and phase damping) by Biswas and Agarwal.[12] They indicated that the WF losses its non-Gaussian nature and becomes Gaussian at long times in the amplitude decay case.

On the other hand, several universal methods to quantify the nonclassicality have been proposed. First the so called Mandel Q-factor to describe the photon number distribution of the state from Poissonian statistics has been introduced by Mandel.[13] Second 'nonclassical distance' in terms of the trace norm of the difference between the density operator of the quantum state and that of the nearest classical state to measure the nonclassicality has been defined by Hillery *et al.*[14] Next the nonclassical depth of the radiation has been given by Lee.[15] Later a measure named the entanglement potential for quantifying the nonclassicality of the single mode optical field has been proposed.[16] Recently the question of how robust the nonclassicalities are against the losses has been investigated by Li *et al.*[17] However, all these criteria cannot reveal all the various quantum effects of the quantum states and it is difficult to quantify precisely how nonclassical a quantum state is.

In this paper, we introduce photon-added and photon-subtracted squeezed vacuum states (PASV and PSSV) and compare their nonclassicality. This paper is arranged as follows: In Section 2, we introduce two kinds of quantum states (PASV and PSSV), give their normalization factors (Legendre polyno-

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mial in the form, a remarkable result) and calculate their explicit analytical expressions of WF (related to single-variable Hermite polynomials). Moreover, we plot the WF in the phase space and find the negative region of WF in phase space. In Section 3, we compare the nonclassicality between PASV and PSSV in terms of Mandel Q-parameter and the negativity of Wigner function. Conclusions are involved in the last section.

2. Photon-added and phaton-subtracted squeezed vacuum state

It is well known that the single-mode squeezed vacuum state[3] can be given as

$$|\psi\rangle = S(r)|0\rangle, \tag{1}$$

where $|0\rangle$ is single mode vacuum state, a ($a^\dagger$) is the Bose annihilation (creation) operator and $S(r) = \exp[\frac{r}{2}(a^{\dagger 2} - a^2)]$ is the single-mode squeezing operator with real squeezing parameter r, satisfying the following unitary transformations

$$\begin{aligned} S^\dagger(r) a S(r) &= a\cosh r + a^\dagger \sinh r, \\ S^\dagger(r) a^\dagger S(r) &= a^\dagger \cosh r + a \sinh r. \end{aligned} \tag{2}$$

Since $S(r)$ can be expanded in the following explicit form

$$\begin{aligned} S(r) \;=\; & \sqrt{\mathrm{sech}r}\exp\left(\frac{\tanh r}{2}a^{\dagger 2}\right) \\ & \times \exp(a^\dagger a \ln \mathrm{sech}r)\exp\left(-\frac{\tanh r}{2}a^2\right), \end{aligned} \tag{3}$$

it leads to $S(r)|0\rangle$ in the number state basis as follows:

$$|\psi\rangle = \sqrt{\mathrm{sech}r}\sum_{k=0}^{\infty}\frac{\sqrt{(2k)!}(\tanh r)^k}{2^k k!}|2k\rangle. \tag{4}$$

Wigner function now becomes a very popular tool to study the nonclassical properties of quantum states.[18] For a single-mode system ρ, the Wigner function is given by[19–21]

$$W(\alpha,\alpha^*) = \mathrm{Tr}[\rho\Delta(\alpha,\alpha^*)], \tag{5}$$

where $\alpha = (q + \mathrm{i}\,p)\sqrt{2}$ and the Wigner operator $\Delta(\alpha,\alpha^*)$ in the coherent representation $|z\rangle$ can be expressed as

$$\Delta(\alpha,\alpha^*) = \frac{\mathrm{e}^{2|\alpha|^2}}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}^2 z}{\pi}|z\rangle\langle -z|\,\mathrm{e}^{-2(z\alpha^* - z^*\alpha)}. \tag{6}$$

It is easy to obtain the Wigner function for $S(r)|0\rangle$ as follows:

$$W_{|\psi\rangle}(\alpha,\alpha^*) = \frac{\mathrm{e}^{-2|\bar{\alpha}|^2}}{\pi} = \frac{\mathrm{e}^{-q^2\,\mathrm{e}^{-2r} - p^2\,\mathrm{e}^{2r}}}{\pi}, \tag{7}$$

where $\bar{\alpha} = \alpha\cosh r - \alpha^*\sinh r$. The Wigner function in Eq. (7) is Gaussian and positive in phase space, which indicates that $S(r)|0\rangle$ is a Gaussian state.

Employing non-Gaussian operation on the squeezed vacuum states, we may obtain the photon-added squeezed vacuum state (PASV) and photon-subtracted squeezed vacuum state (PSSV), two kinds of non-Gaussian states, which exhibit an abundance of nonclassical properties and may give access to a complete engineering of quantum states and to fundamental quantum phenomena. From another point of view, the non-Gaussianity of quantum state can also be characterized by the presence of negativity of Wigner function in phase space. Since non-Gaussian state is always nonclassical, but nonclassical state cannot be non-Gaussian for certain cases. For example, a squeezed state is a typical nonclassical state, but it is not a Gaussian state and its Wigner function is nonnegative.

2.1. PASV

Theoretically, the single-mode PASV can be obtained by repeatedly operating the photon creation operator $a^\dagger$ on a squeezed vacuum state $S(r)|0\rangle$,

$$|r\rangle_{am} \equiv N_{am}a^{\dagger m}S(r)|0\rangle, \tag{8}$$

where N_{am} is the normalization constant determined by (See Appendix A, case (i))

$$N_{am}^{-2} = m!\cosh^m r \mathrm{P}_m(\cosh r), \tag{9}$$

where $\mathrm{P}_m(x)$ is Legendre polynomial. From Eq. (4), we find that the PASV in the number state basis can be represented as (the relations $a^\dagger|l\rangle = \sqrt{l+1}|l+1\rangle$)

$$\begin{aligned} |r\rangle_{am} \;\equiv\; & N_{am}\sqrt{\mathrm{sech}r}\sum_{k=0}^{\infty}\frac{\sqrt{(2k)!}(\tanh r)^k}{2^k k!}a^{\dagger m}|2k\rangle \\ =\; & N_{am}\sqrt{\mathrm{sech}r}\sum_{k=0}^{\infty}\frac{\sqrt{(2k+m)!}(\tanh r)^k}{2^k k!} \\ & \times |2k+m\rangle. \end{aligned} \tag{10}$$

The Wigner function of PASV can be calculated as follows (See Appendix B, case (I))

$$W_{am}(\alpha,\alpha^*) \;=\; \frac{N_{am}^2\sinh^m 2r}{4^m\pi}\exp(-2|\alpha|^2\cosh 2r$$

$$+\alpha^2\sinh 2r\ +\alpha^{*2}\sinh 2r)\times\sum_{k=0}^{m}\frac{(m!)^2(-2\coth r)^k}{k![(m-k)!]^2}\times|\mathrm{H}_{m-k}(-\mathrm{i}\sqrt{2\coth r}\bar{\alpha})|^2, \quad (11)$$

where H_m is Hermite polynomial.

2.2. PSSV

Similarly, by repeatedly operating the photon annihilation operator a on a squeezed vacuum state $S(r)|0\rangle$, one can generate another non-Gaussian state, i.e. photon-subtracted squeezed vacuum state (PSSV)

$$|r\rangle_{sn}\equiv N_{sn}a^nS(r)|0\rangle, \quad (12)$$

where the normalization constant N_{sn} determined by (See Appendix A, case (ii))

$$N_{sn}^{-2}=n!(-\mathrm{i}\sinh r)^n\mathrm{P}_n(\mathrm{i}\sinh r). \quad (13)$$

From Eq. (4), we find that the PSSV in the number state basis can be represented as (the relations $a|l\rangle=\sqrt{l}|l-1\rangle$)

$$\begin{aligned}|r\rangle_{sn}&\equiv N_{sn}\sqrt{\mathrm{sech}r}\sum_{k=0}^{\infty}\frac{\sqrt{(2k)!}(\tanh r)^k}{2^kk!}a^n|2k\rangle\\&=N_{sn}\sqrt{\mathrm{sech}r}\sum_{k=0}^{\infty}\frac{(2k)!(\tanh r)^k}{2^kk!\sqrt{(2k-n)!}}|2k-n\rangle\\&=N_{sn}\sqrt{\mathrm{sech}r}\sum_{k>n/2}^{\infty}\frac{(2k)!(\tanh r)^k}{2^kk!\sqrt{(2k-n)!}}\times|2k-n\rangle. \quad (14)\end{aligned}$$

The Wigner function of PSSV can be calculated as follows (See Appendix B, case (II))

$$W_{sn}(\alpha,\alpha^*)=\frac{N_{sn}^2\sinh^n 2r}{4^n\pi}\exp(-2|\alpha|^2\cosh 2r+\alpha^2\sinh 2r+\alpha^{*2}\sinh 2r)\times\sum_{l=0}^{n}\frac{(n!)^2(-2\tanh r)^l}{l![(n-l)!]^2}\times|\mathrm{H}_{n-l}(\mathrm{i}\,\bar{\alpha}\sqrt{2\tanh r})|^2. \quad (15)$$

In fact, for either PASV or PSSV, if only acting single-photon operation, i.e., $m=n=1$, they all reduce to the same quantum state, i.e.

$$|r\rangle_{am=1}=|r\rangle_{sn=1}=S(r)|1\rangle, \quad (16)$$

that is the squeezed single photon Fock state whose Wigner function is

$$W_{S(r)|1\rangle}(\alpha,\alpha^*)=\frac{\mathrm{e}^{-2|\bar{\alpha}|^2}}{\pi}(4|\bar{\alpha}|^2-1), \quad (17)$$

which is non-Gaussian form because of the non-Gaussian item.

As it is well known, when $m=n=0$, the squeezed vacuum state $S(r)|0\rangle$ is a Gaussian, which can be judged from the Gaussian form of its Wigner function in Eq. (7), namely, a Gaussian form in phase space (See Fig. 1). While equations (11) and (15) are a non-Gaussian Wigner functions in phase space due to the presence of non-Gaussian item, which further show that the PASV and PSSV are non-Gaussian states. It is just the addition (subtraction) of photon to (from) the squeezed vacuum state that leads to loss the character of Gaussian. In order to see clearly the non-Gaussianity of PASV and PSSV, we plot its Wigner function in Fig. 2, where there is some negative region of the Wigner function in phase space which implies the nonclassicality of PASV and PSSV. Moreover, the Wigner function of PASV and PSSV is non-Gaussian in phase space and presents one peak in the middle and two negative valleys in both sides (See Fig. 2).

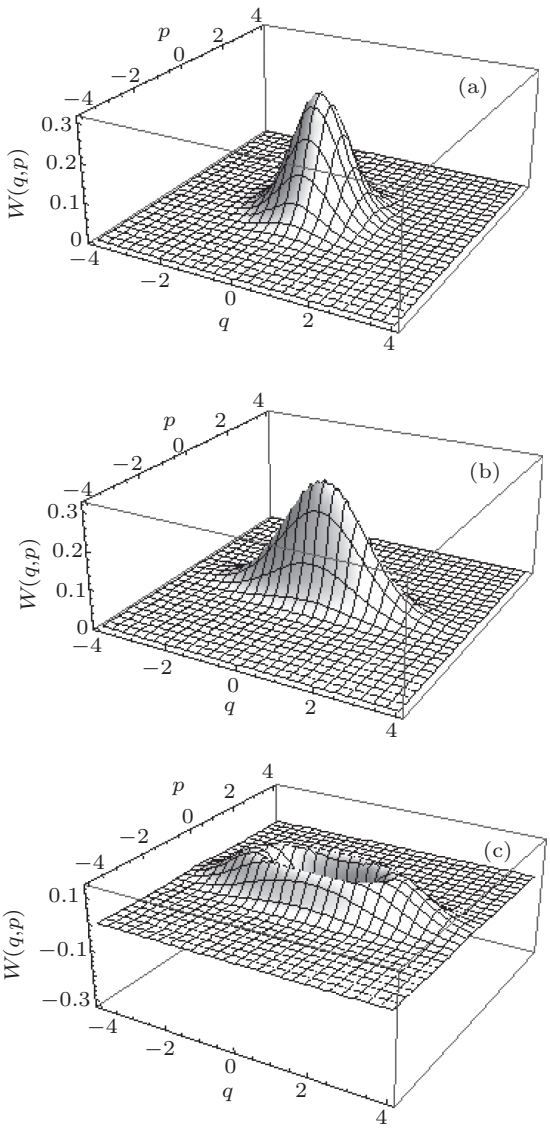


Fig. 1. Wigner function of the states for (a) $|0\rangle$, (b) $S(r)|0\rangle$, (c) $S(r)|1\rangle$ with $r=0.5$.

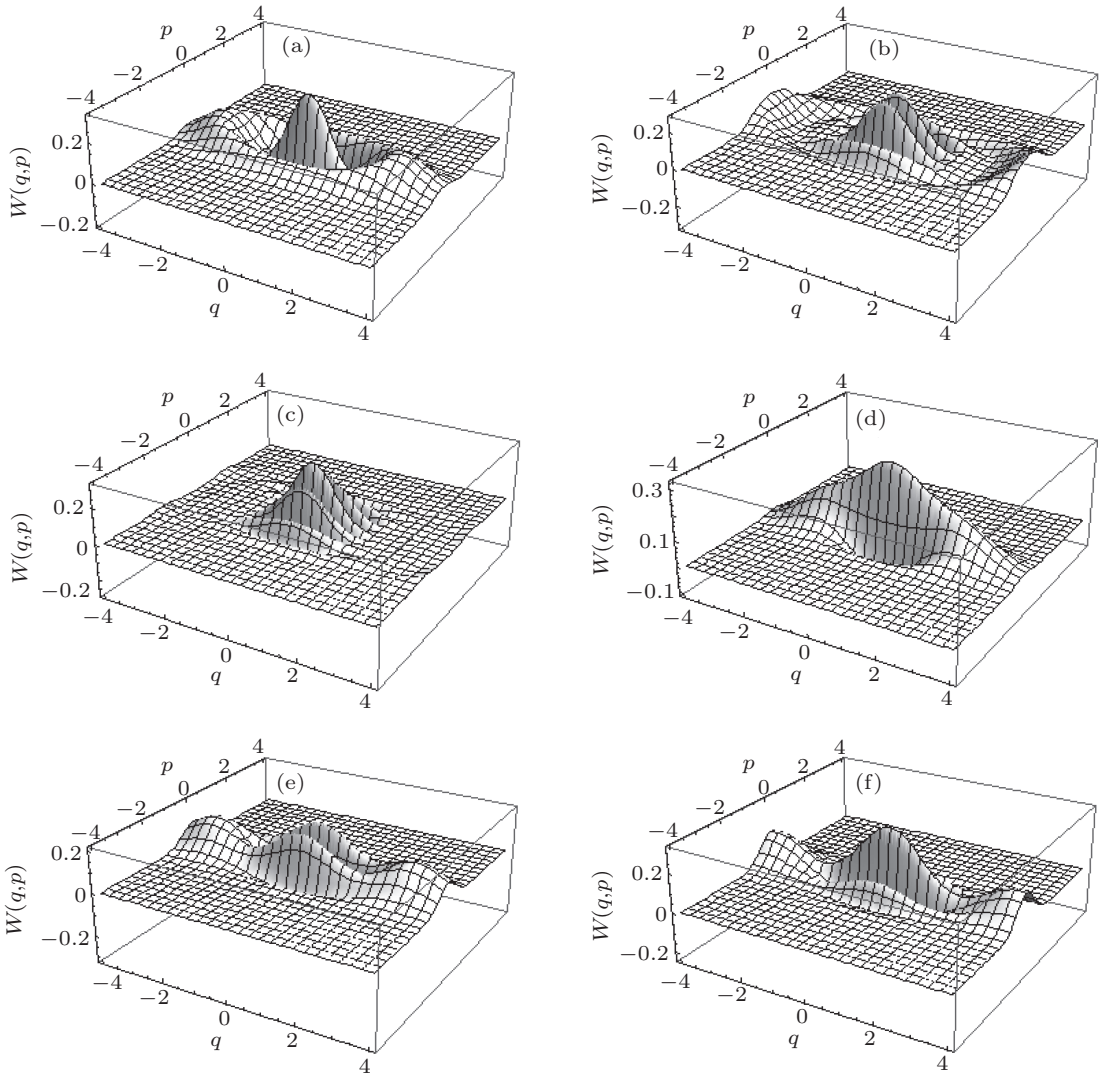


Fig. 2. Wigner function of the states for PASVs (a) $m = 2$, (b) $m = 5$, (c) $m = 8$, and PSSVs (d) $n = 2$, (e) $n = 5$, (f) $n = 8$ with $r = 0.5$.

3. Nonclassicality of PASV and PSSV

In this section, we shall compare some nonclassicality of PASV and PSSV by virtue of Mandel Q-factor and negativity of Wigner function.

3.1. Mandel Q-factor

In early days, Mandel introduced the so-called Mandel Q-factor[13] to described the violation from the photon number distribution of the state of Poissonian statistics. To see clearly the photon statistical properties of quantum optical field, we examine the Mandel's Q-parameter defined by

$$Q = \frac{\langle a^{\dagger 2} a^2 \rangle}{\langle a^\dagger a \rangle} - \langle a^\dagger a \rangle, \tag{18}$$

which measures the deviation of the photon number statistics from the Poissonian distribution, characteristic of coherent states. If $Q = 0$, the radiation field has Poissonian photon statistics, while if $Q > 0$ ($Q < 0$) we say that the radiation field has super- (sub-) Poissonian photon statistics.

According to Eq. (2), for the quantum state $S(r)|0\rangle$, we can easily obtain

$$\begin{aligned} \langle a^\dagger a \rangle &= \sinh^2 r, \\ \langle a^{\dagger 2} a^2 \rangle &= \sinh^2 r \cosh^2 r + 2 \sinh^4 r, \end{aligned} \tag{19}$$

which leads to the Mandel Q-factor of $S(r)|0\rangle$ as follows:

$$Q_{|\psi\rangle} = \cosh 2r. \tag{20}$$

Because $\cosh 2r$ is always positive, we say that $S(r)|0\rangle$ has super-Poissonian distribution.

3.1.1. Photon-added case

Setting $N_{am} \equiv N_{a,m}$, we know

$$\begin{aligned}\langle a^\dagger a\rangle_{am} &= \langle aa^\dagger\rangle_{am} - 1 = \frac{N_{a,m}^2}{N_{a,m+1}^2} - 1,\\ \langle a^{\dagger 2}a^2\rangle_{am} &= \langle a^2a^{\dagger 2}\rangle_{am} - 4\langle aa^\dagger\rangle_{am} + 2\\ &= \frac{N_{a,m}^2}{N_{a,m+2}^2} - 4\frac{N_{a,m}^2}{N_{a,m+1}^2} + 2. \end{aligned} \tag{21}$$

Thus the Q-parameter of PASV can be given by

$$\begin{aligned}Q_{am} &= \frac{N_{a,m}^2N_{a,m+1}^2/N_{a,m+2}^2 - 4N_{a,m}^2 + 2N_{a,m+1}^2}{N_{a,m}^2 - N_{a,m+1}^2}\\ &\quad - \left(\frac{N_{a,m}^2}{N_{a,m+1}^2} - 1\right). \end{aligned} \tag{22}$$

In particular, the Q-parameter for single-photon-added case $m = 1$ is given by

$$Q = \frac{3\sinh^2 2r}{3\cosh 2r - 1} - 1. \tag{23}$$

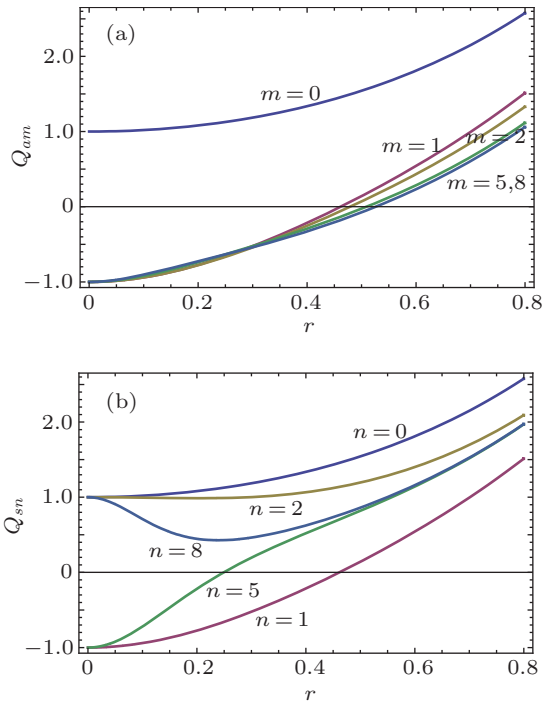


Fig. 3. (colour online) Mandel Q parameter for (a) PASVs $m = 0$, 1, 2, 5, 8; and (b) PSSVs $n = 0$, 1, 2, 5, 8.

From Eq. (23) we find that Q becomes negative for $m = 1$ which is satisfied for the squeezing parameter $r \lesssim 0.46$ similar to the result of Q for the single-photon subtracted case.[22] In order to see clearly the variation of Q-parameter with r, we show the plots of Q-parameter in Fig. 3, from which one can clearly see that Q-parameter becomes negative ($m \neq 0$) when r is less than a certain threshold value which increases as m increases; while for $m = 0$, Q is always positive. This implies that the nonclassicality is enhanced by adding photons to the squeezed state. We should emphasize that the WF has negative region for all r, and thus the PASV is nonclassical.

3.1.2. Photon-subtracted case

In the similar way, setting $N_{sn} \equiv N_{s,n}$, we know

$$\langle a^\dagger a\rangle_{sn} = \frac{N_{s,n}^2}{N_{s,n+1}^2}, \quad \langle a^{\dagger 2}a^2\rangle_{sn} = \frac{N_{s,n}^2}{N_{s,n+2}^2}. \tag{24}$$

Thus the Q-parameter of PSSV can be given by

$$\begin{aligned}Q_{sn} &= \frac{N_{s,n}^2/N_{s,n+2}^2}{N_{s,n}^2/N_{s,n+1}^2} - \frac{N_{s,n}^2}{N_{s,n+1}^2}\\ &= \frac{N_{s,n+1}^2}{N_{s,n+2}^2} - \frac{N_{s,n}^2}{N_{s,n+1}^2}. \end{aligned} \tag{25}$$

It is clearly seen from Fig. 3 that Q-parameter is generally proportional to squeezing parameter r. Moreover, the negativity of Q-parameter concerns added or subtracted photon number m, n. In the case of PASVs, when m increases, the negativity of Q-parameter also increases. In the case of PSSVs, the special property of Q-parameter is that when n equals even, the negativity exists and proportional to squeezing parameter r, when n equals odd, the Q-parameter has no negativity. It is well known that the negativity of the Q-parameter refers to sub-Poissonian statistics of the state. However, a state may be nonclassical even though Q is positive. This case is true for the present state. From the discussion below, the Wigner function of PASV and PSSV has negative region in phase space which shows that the state is nonclassical and non-Gaussian.

3.2. Negativity of Wigner function

In Ref. [22], a measure of non-classicality based on the volume of the negative part of the WF is proposed. Its definition for a state $|\psi\rangle$ may be written as

$$\begin{aligned}\delta(\psi) &= \frac{1}{2}\int[|W_\psi(q,p)| - W_\psi(q,p]\mathrm{d}q\mathrm{d}p\\ &= \frac{1}{2}\left(\int|W_\psi(q,p)|\mathrm{d}q\mathrm{d}p - 1\right). \end{aligned} \tag{26}$$

Hence we shall treat δ as a parameter characterizing the properties of the PASV and PSSV. By definition, the quantity δ is equal to zero for squeezed vacuum state, for which the WF is non-negative.

In Fig. 4, δ of the PASV and PSSV with different (m, n) values is plotted as the function of r. From Fig. 4, one can also see that the quantity δ of PASVs and PSSVs have different properties. For PASVs, δ decreases with the increase of squeezing parameter r, and the rate of decrease becomes fast with the increase of added photon number m. For PSSVs, the quantity δ increases with the increase of squeezing parameter r and the increase of subtracted photon number n.

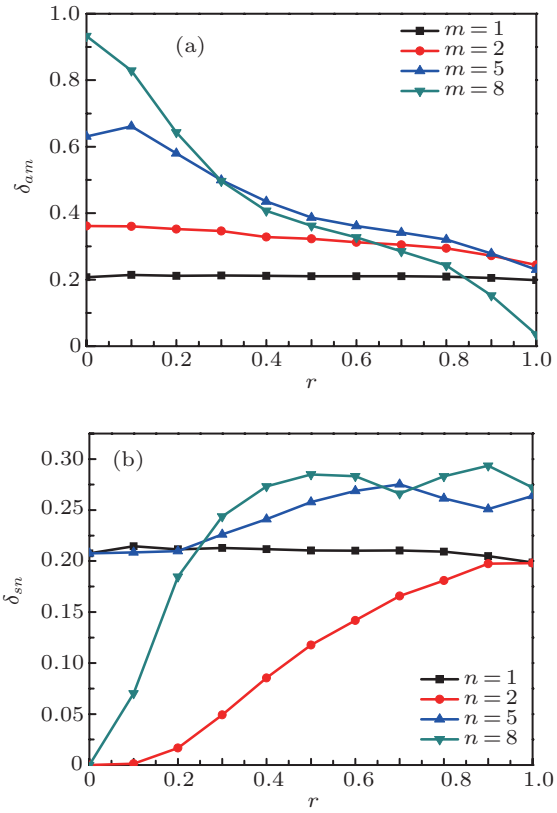


Fig. 4. (colour online) Negativity of Wigner function as a function of squeezing parameter r for (a) PASVs $m = 1$, 2, 5, 8; and (b) PSSVs $n = 1$, 2, 5, 8.

4. Conclusion

In summary, we introduced photon-added and photon-subtracted squeezed vacuum state (PASV and PSSV) and obtained their normalized factors, which have the similar forms involved in Lengendre polynomials. Moreover, we gave their compact expressions of Wigner function, which are all related to single-variable Hermite polynomials. Especially, we mainly compared the nonclassicality between PASV and PSSV, by virtue of Mandel Q-factor and negativity of Wigner function. From the comparison we find that PASV and PSSV have different nonclassicalities.

Appendix A: Normalizations of PASV and PSSV

(i) The photon-added case

For

$$|r\rangle_{am} \equiv N_{am} a^{\dagger m} S(r)|0\rangle,$$

due to ${}_{am}\langle r, m|r, m\rangle_{am} = 1$, we have

$$N_{am}^{-2} = \mathrm{sech}r\langle 0| \exp\left(\frac{\tanh r}{2} a^2\right) a^m a^{\dagger m} \times \exp\left(\frac{\tanh r}{2} a^{\dagger 2}\right)|0\rangle. \tag{A1}$$

Inserting the completeness relation

$$\int \frac{\mathrm{d}^2 z}{\pi} |z\rangle\langle z| = 1$$

into Eq. (A1), we have

$$N_{am}^{-2} = \mathrm{sech}r \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 z}{\pi} z^m z^{*m} \exp\left(-|z|^2 + \frac{\tanh r}{2} z^2 + \frac{\tanh r}{2} z^{*2}\right)$$
$$= \mathrm{sech}r \frac{\partial^{2m}}{\partial t^m \partial \tau^m} \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 z}{\pi} \exp\left(-|z|^2 + tz + \tau z^* + \frac{\tanh r}{2} z^2 + \frac{\tanh r}{2} z^{*2}\right)\Big|_{t=\tau=0}. \tag{A2}$$

After employing the integration, we obtain

$$N_{am}^{-2} = \frac{\partial^{2m}}{\partial t^m \partial \tau^m} \exp\left(\frac{\sinh 2r}{4} t^2 + \frac{\sinh 2r}{4} \tau^2 + t\tau \cosh^2 r\right)\Big|_{t=\tau=0} = m! \cosh^m r \mathrm{P}_m(\cosh r), \tag{A3}$$

where we used the newly found generating function of Legendre polynomial,[23]

$$\frac{\partial^{2m}}{\partial t^m \partial \tau^m} \exp\left(-t^2 - \tau^2 + \frac{2x\tau t}{\sqrt{x^2-1}}\right)\Big|_{t,\tau=0} = \frac{2^m m!}{(x^2-1)^{m/2}} \mathrm{P}_m(x). \tag{A4}$$

(ii) The photon-subtracted case

For $|r\rangle_{sn} \equiv N_{sn} a^n S(r)|0\rangle$, due to ${}_{sn}\langle r|r\rangle_{sn} = 1$, we have

$$N_{sn}^{-2} = \mathrm{sech}r \langle 0| \exp\left(\frac{\tanh r}{2} a^2\right) a^{\dagger n} a^n \exp\left(\frac{\tanh r}{2} a^{\dagger 2}\right)|0\rangle. \tag{A5}$$

Inserting the completeness relation for two times into Eq. (A5), we have

$$\begin{aligned} N_{sn}^{-2} &= \mathrm{sech}r \frac{\partial^{2n}}{\partial t^n \partial \tau^n} \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 z_1}{\pi} \exp\left(-|z_1|^2 + t z_1^* + \frac{\tanh r}{2} z_1^2\right) \\ &\quad \times \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 z_2}{\pi} \exp\left(-|z_2|^2 + \tau z_2 + z_1^* z_2 + \frac{\tanh r}{2} z_2^{*2}\right)\Big|_{t=\tau=0} \\ &= \frac{\partial^{2n}}{\partial t^n \partial \tau^n} \exp\left(\frac{\sinh 2r}{4} t^2 + \frac{\sinh 2r}{4} \tau^2 + t\tau \sinh^2 r\right)\Big|_{t=\tau=0} = n!(-\mathrm{i}\sinh r)^n \mathrm{P}_n(\mathrm{i}\sinh r). \end{aligned} \tag{A6}$$

Appendix B: Wigner functions of PASV and PSSV

(I) The photon-added case

Substituting $\rho_{am} = |r\rangle_{am\,am}\langle r|$ into Eq. (5), we have

$$\begin{aligned} W_{am}(\alpha, \alpha^*) &= N_{am}^2 \mathrm{sech}r \frac{\mathrm{e}^{2|\alpha|^2}}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 z}{\pi} (-z^*)^m z^m \exp\left(-|z|^2 - 2\alpha^* z + 2\alpha z^* + \frac{\tanh r}{2} z^2 + \frac{\tanh r}{2} z^{*2}\right) \\ &= \frac{N_{am}^2 (-1)^m}{\pi} \exp\left(-2|\alpha|^2 \cosh 2r + \alpha^2 \sinh 2r + \alpha^{*2} \sinh 2r\right) \frac{\partial^{2m}}{\partial t^m \partial \tau^m} \\ &\quad \times \exp\left(\frac{\sinh 2r}{4} t^2 + \frac{\sinh 2r}{4} \tau^2 + 2\cosh r\bar{\alpha} t - 2\cosh r\bar{\alpha}^* \tau + t\tau \cosh^2 r\right)\Big|_{t=\tau=0}, \end{aligned} \tag{B1}$$

Noticing the generating function of Hermite polynomial

$$\mathrm{H}_m(x) = \frac{\partial^m}{\partial t^m} \exp(2tx - t^2)|_{t=0}, \tag{B2}$$

and its recurrence relation,

$$\frac{\mathrm{d}^l}{\mathrm{d}x^l} \mathrm{H}_n(x) = \frac{2^l n!}{(n-l)!} \mathrm{H}_{n-l}(x), \tag{B3}$$

we have

$$\begin{aligned} W_{am}(\alpha, \alpha^*) &= N_{am}^2 \frac{1}{\pi} \left(\frac{\sinh 2r}{4}\right)^m \mathrm{e}^{-2|\alpha|^2 \cosh 2r + \alpha^2 \sinh 2r + \alpha^{*2} \sinh 2r} \\ &\quad \times \sum_{k=0}^{m} \frac{m! m! (-2\coth r)^k}{k! [(m-k)!]^2} |\mathrm{H}_{m-k}(-\mathrm{i}\sqrt{2\coth r}\bar{\alpha})|^2. \end{aligned} \tag{B4}$$

(II) The photon-subtracted case

In a similar way, substituting $\rho_{sn} = |r\rangle_{sn\,sn}\langle r|$ into Eq. (5) and inserting the completeness relation for two times, we have

$$\begin{aligned} W_{sn}(\alpha, \alpha^*) &= N_{sn}^2 \mathrm{sech}r \frac{\mathrm{e}^{2|\alpha|^2}}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 z}{\pi} \mathrm{e}^{-|z|^2 - 2\alpha^* z + 2\alpha z^*} \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 z_1}{\pi} z_1^n \exp\left(-|z_1|^2 + \frac{\tanh r}{2} z_1^{*2} - z^* z_1\right) \\ &\quad \times \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 z_2}{\pi} z_2^{*n} \exp\left(-|z_2|^2 + \frac{\tanh r}{2} z_2^2 + z_2^* z\right) \\ &= N_{sn}^2 \mathrm{sech}r \frac{\mathrm{e}^{2|\alpha|^2}}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 z}{\pi} \mathrm{e}^{-|z|^2 - 2\alpha^* z + 2\alpha z^*} \frac{\partial^{2n}}{\partial t^n \partial \tau^n} \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 z_1}{\pi} \exp\left(-|z_1|^2 - z^* z_1 + t z_1 + \frac{\tanh r}{2} z_1^{*2}\right) \\ &\quad \times \int_{-\infty}^{\infty} \frac{\mathrm{d}^2 z_2}{\pi} \exp\left(-|z_2|^2 + z z_2^* + \tau z_2^* + \frac{\tanh r}{2} z_2^2\right)\Big|_{t=\tau=0}. \end{aligned} \tag{B5}$$

After employing the integration, we have

$$W_{sn}(\alpha,\alpha^*) = \frac{N_{sn}^2}{\pi}\,\mathrm{e}^{-2|\alpha|^2\cosh 2r+\alpha^2\sinh 2r+\alpha^{*2}\sinh 2r} \times \frac{\partial^{2n}}{\partial t^n\partial\tau^n}\exp\left(\frac{\sinh 2r}{4}t^2+\frac{\sinh 2r}{4}\tau^2+2\sinh r\bar{\alpha}^*t+2\sinh r\bar{\alpha}\tau-t\tau\sinh^2 r\right)\Big|_{t=\tau=0}$$
$$= \frac{N_{sn}^2\sinh^n 2r}{4^n\pi}\,\mathrm{e}^{-2|\alpha|^2\cosh 2r+\alpha^2\sinh 2r+\alpha^{*2}\sinh 2r}\sum_{l=0}^{n}\frac{n!n!(-2\tanh r)^l}{l![(n-l)!]^2}|\mathrm{H}_{n-l}(i\bar{\alpha}\sqrt{2\tanh r})|^2. \quad \text{(B6)}$$

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