Non-Gaussianity and decoherence of generalized photon-added coherent state as a Hermite-excited coherent state*

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Generalized photon-added coherent state (GPACS) is obtained by repeatedly acting the combination of Bose creation and annihilation operations on the coherent state. It is found that GPACS can be regarded as a Hermite-excited coherent state due to its normalization factor related to a Hermite polynomial. In addition, we adopt the Hilbert–Schmidt distance to quantify the non-Gaussian character of GPACS and discuss the decoherence of GPACS in dissipative channel by studying the loss of nonclassicality in reference of the negativity of Wigner function.

Keywords: generalized photon-added coherent state, Hermite polynomial, non-Gaussianity, the negativity of Wigner function

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1. Introduction

In recent years, much effort has been dedicated to the theoretical investigation and experimental production of highly nonclassical, non-Gaussian states of the radiation field.[1,2] Such states are endowed with enhanced nonclassical properties, such as sub-Poissonian statistics,[3] squeezing,[4] higher-order squeezing,[5] antibunching,[6] negativity of Wigner function (WF),[7] etc., which can be useful for improving the efficiency of quantum information and communication protocols.[8,9] Especially, the non-positive WF is regarded as a clear signature of the highly nonclassical character of the optical fields[10] and is often used to monitor the decoherence of some quantum states.[11,12] Degaussification can be realized in a simple manner through adding (subtracting) photons to (from) a Gaussian state.[13−20] For example, photon-added coherent state (PACS) or excited coherent state, first introduced by Agarwal and Tara,[13] was a new class of non-Gaussian states, obtained by repeated application of photon creation operator ($a^\dagger$) to a coherent state. It was also considered as a nonlinear coherent state[21] and exhibited both higher-order squeezing and higher-order sub-Poissonian character.[22] Later, the single PACS has been generated experimentally and tomographically reconstructed WF for such states has been analysed.[23] Another scheme was proposed by Kalamidas *et al.*[24] to generate a two-PACS. In addition, following the Agarwal's idea, photon-added entangled coherent states (PAECS)[25−28] have been constructed as well. Their amount of entanglement can be increased by repeatedly adding photon number. Ren *et al.*[29] proposed a scheme of optical generation of single- or two-mode PAECS with nonlinear Mach-Zehnder interferometer and a type-I beta-barium borate crystal.

In another development, much attention has been paid to the combination of photon-added and photon-subtracted operation on a given state. Among them, Lee *et al.*[30] investigated the properties of these states obtained by repeatedly applying $(a^\dagger a)^k$ and $(aa^\dagger)^k$ to the coherent state and thermal state, respectively. Yang and Li[31] also analysed the operation $a^l a^{\dagger k}$ and its inverse $a^k a^{\dagger l}$ on an arbitrary state and they studied their nonclassicalities. Fiurášek[32] later proposed a scheme for the approximate probabilistic realization of an arbitrary operation that can be expressed as a function of photon number operator $a^\dagger a$. Very recent-

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ly, Lee and Nha[33] have considered a coherent superposition of photon addition and subtraction $\mu a+\nu a^{\dagger}$ with $|\mu|^2+|\nu|^2=1$ acting on a coherent state and a thermal state to form non-Gaussian states. Presented in Refs. [34] and [35], the generalized photon-added coherent state (GPACS) which is created by repeatedly applying the operator $a^{\dagger}+a$ to coherent state m times, GPACS exhibits highly nonclassical behaviours such as sub-Poissonian distribution, the squeezing property, WF and so on, depending on what the m value is. In this paper, according to the above work we focus our attention on the non-Gaussianity of GPACS based on the Hilbert–Schmidt distance and the decoherence of GPACS in dissipative channel by studying its WF with time evolution. Motivations to study this work lie in the fact that i) the normalization factor of GPACS is a Hermite polynomial, so GPACS is regarded as a Hermite-excited coherent state; ii) as is well known, non-Guassianity is revealing itself as a resource for continuous variable quantum information. We evaluate the non-Guassianity of GPACS, which serves as a guideline to quantify the non-Guassian states; iii) when the nonclassical states propagate in the medium, they inevitably interact with their surrounding environment, which tends to deteriorate the degree of nonclassicality (i.e. to render quantum feature unobservable). It is necessary to further investigate the dynamic behaviour of their nonclassicality; iv) as far as we know, the decoherence of arbitrary number GPACS in dissipative channel has not been derived analytically in the literature before.

The rest of this paper is organized as follows. In the following section we review the general structure of GPACS, whose normalization factor is related to Hermite polynomial function and derive some properties such as scalar products, photon number distribution and fidelity. Then, in Section 3 we adopt the Hilbert–Schmidt distance to quantify the non-Gaussian character of GPACS after calculating the average photon number in GPACS. We devote Sections 4 and 5 to investigating the decoherence of GPACS in a dissipative channel. In Section 4 the normally ordered form of density operator of GPACS is firstly derived in a dissipative channel, whereas in Section 5 we deduce the explicit analytical expression of the GPACS's WF with time evolution in a dissipative channel (see Eq. (35) below) and discuss the loss of nonclassicality in reference about the negativity of WF.

2. GPACS as a Hermite-excited coherent state

Theoretically, GPACS can be obtained by repeatedly operating the coherent superposition of photon addition and subtraction ($\Omega\equiv a^{\dagger}+a$) in a coherent state $|\alpha\rangle$, i.e.,

$$|\alpha,m\rangle=N_{m,\alpha}\Omega^m|\alpha\rangle, \tag{1}$$

where m is an integer and $N_{m,\alpha}$ is the normalization constant. Note that the operator Ω is Hermitian $\Omega=\Omega^{\dagger}$, different from the operator $b\equiv\mu a+\nu a^{\dagger}$. According to the normal ordering form of Ω^m,[25]

$$\Omega^m=\left(\frac{-\mathrm{i}}{\sqrt{2}}\right)^m:\mathrm{H}_m\left(\frac{\mathrm{i}\,\Omega}{\sqrt{2}}\right):, \tag{2}$$

where : : denotes the normally ordered product of Bose operators and $\mathrm{H}_m(x)$ is Hermite polynomial

$$\begin{aligned}\mathrm{H}_m(x)&=\frac{\partial^m}{\partial t^m}\,\mathrm{e}^{2tx-t^2}\Big|_{t=0}\\&=\sum_{l=0}^{[m/2]}\frac{(-1)^l\,m!}{l!(m-2l)!}(2x)^{m-2l},\end{aligned} \tag{3}$$

we see that based on the normalization condition $1=(N_{m,\alpha})^2\langle\alpha|\,\Omega^{2m}|\alpha\rangle$, the compact form of $N_{m,\alpha}$ is found to be

$$(N_{m,\alpha})^{-2}=\left(-\frac{1}{2}\right)^m\mathrm{H}_{2m}\left[\frac{\mathrm{i}\,(\alpha^*+\alpha)}{\sqrt{2}}\right]. \tag{4}$$

Using Eqs. (1) and (2), GPACS is also expressed as

$$|\alpha,m\rangle=N_{m,\alpha}\left(\frac{-\mathrm{i}}{\sqrt{2}}\right)^m\mathrm{H}_m\left[\frac{\mathrm{i}\,(a^{\dagger}+\alpha)}{\sqrt{2}}\right]|\alpha\rangle, \tag{5}$$

which indicates that GPACS is equivalent to a Hermite-excited coherent state, where in the last step we have used the eigen-equation $a|\alpha\rangle=\alpha|\alpha\rangle$. From Eq. (2), we also give the quantity

$$\langle\beta|\,\Omega^m|\alpha\rangle=\left(\frac{-\mathrm{i}}{\sqrt{2}}\right)^m\mathrm{H}_m\left[\frac{\mathrm{i}\,(\beta^*+\alpha)}{\sqrt{2}}\right]\langle\beta|\,\alpha\rangle, \tag{6}$$

where $|\beta\rangle$ is also a coherent state, and

$$\langle\beta|\,\alpha\rangle=\exp\left(-\frac{1}{2}|\beta|^2-\frac{1}{2}|\alpha|^2+\beta^*\alpha\right).$$

Further, the expansion leads to the following results for the scalar products:

$$\begin{aligned}&\langle\beta,m\,|\alpha,m\rangle\\&=\frac{(-1)^m\,N_{m,\alpha}N_{m,\beta}}{2^m}\mathrm{H}_{2m}\left[\frac{\mathrm{i}\,(\beta^*+\alpha)}{\sqrt{2}}\right]\langle\beta|\,\alpha\rangle,\end{aligned} \tag{7}$$

and

$$\langle\alpha, m\,|\alpha, n\rangle = N_{m,\alpha}N_{n,\alpha}\left(\frac{-\mathrm{i}}{\sqrt{2}}\right)^{m+n}\mathrm{H}_{m+n}\left(C\right), \quad (8)$$

where $C \equiv \mathrm{i}\sqrt{2}\left(\alpha+\alpha^{*}\right)/2$.

On the other hand, according to Eq. (3), equation (5) is rewritten as

$$|\alpha, m\rangle = N_{m,\alpha}\sum_{l=0}^{[m/2]}\sum_{k=0}^{m-2l}\frac{m!\alpha^{m-2l-k}}{2^{l}l!k!\left(m-2l-k\right)!}a^{\dagger k}\,|\alpha\rangle\,, \quad (9)$$

which is the superposition of the coherent state $|\alpha\rangle$ and PACS $a^{\dagger k}\,|\alpha\rangle$. Using $a\,|n\rangle = \sqrt{n}\,|n-1\rangle$, we can obtain $a^{k}\,|n\rangle = \sqrt{n!/(n-k)!}\,|n-k\rangle$, and $\langle n|\,\alpha\rangle = (\alpha^{n}/\sqrt{n!})\exp\left(-|\alpha|^{2}/2\right)$. It is seen from Eq. (9) that the photon number distribution formula is given by

$$P(n) = N_{m,\alpha}^{2}\,\mathrm{e}^{-|\alpha|^{2}}n!\left(m!\right)^{2} \times\left|\sum_{l=0}^{[m/2]}\sum_{k=0}^{m-2l}\frac{\alpha^{m+n-2l-2k}}{2^{l}l!k!\left(m-2l-k\right)!\left(n-k\right)!}\right|^{2}. \quad (10)$$

Using Eq. (10), in Fig. 1 we show the plots of photon number distribution $P(n)$ of GPACS as a function of $n+1$ for several different values of parameters α and m. It can be seen apparently from the plots that the position of peak of $P(n)$ depends on excitation number and the value of coherent state initially. Figure 1(a) just describes photon number distribution of the coherent state. By adding photons, we can move the peak with the increase of photon number (see Figs. 1(a)–1(c)).

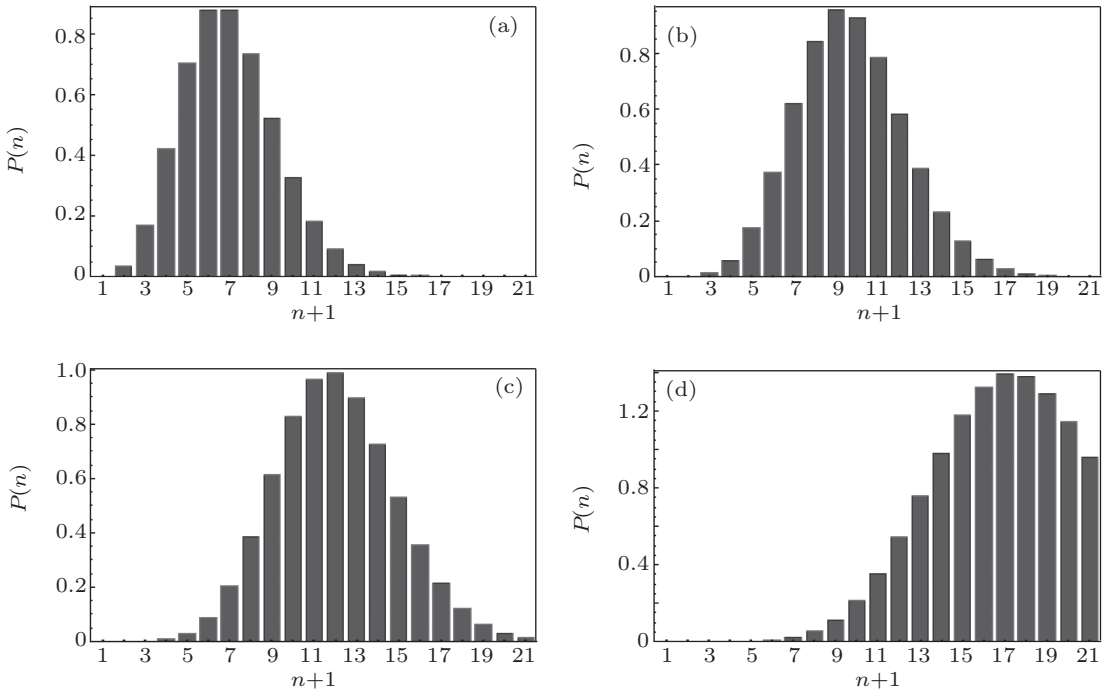


Fig. 1. Plots of photon number distribution $P(n)$ of GPACS as a function of $n+1$ for several different values of parameters α and m: (a) $\alpha = 2+\mathrm{i}$, $m=0$; (b) $\alpha = 2+\mathrm{i}$, $m=3$; (c) $\alpha = 2+\mathrm{i}$, $m=6$; and (d) $\alpha = 2+3\,\mathrm{i}$, $m=3$.

Finally, we examine the fidelity between GPACS and its original coherent state, $F = |\langle\alpha\,|\alpha, m\rangle|^{2}$, which is a measure of how close to the coherent state the GPACS is . Since the coherent state is a standard Gaussian state, the fidelity with coherent state may be seen as a non-Gaussianity measure to exhibit the non-Gaussian character of GPACS. Using Eq. (5), we have

$$F = \frac{(-1)^{m}\left|\mathrm{H}_{m}\left(C\right)\right|^{2}}{\mathrm{H}_{2m}\left(C\right)}. \quad (11)$$

The fidelities of Eq. (11) are plotted in Fig. 2 each as a function of α for several different values of parameter m where we suppose that α is real for simplicity. From Fig. 2 one can clearly see that for the case of $m=0$, the fidelity equals unity as expected and in the limit of high value of $|\alpha|$, the fidelity tends to be unity. In addition, the fidelity becomes smaller with the increase of parameter m for the smaller $|\alpha|$.

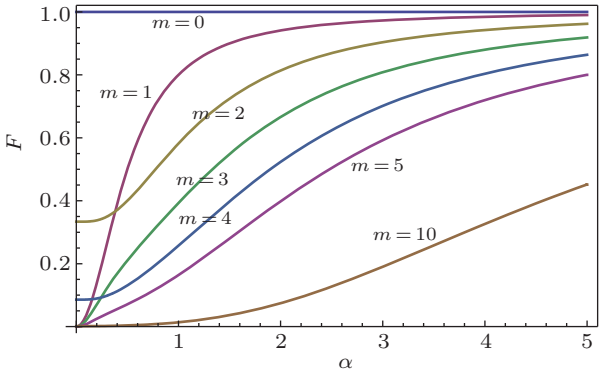


Fig. 2. (colour online) Fidelities between GPACS and its original coherent state as a function of α for several different values of parameter m where we suppose α to be real for simplicity.

3. Non-Gaussianity of GPACS

As is well known, a non-Guassianity measure may serves as a guideline to quantify the non-Guassian states.[36−38] In this section, we adopt the Hilbert–Schmidt distance between a quantum state ρ and a reference Gaussian state τ, which itself depends on ρ, to quantify the non-Guassianity $\delta[\rho]$ of GPACS. The non-Gaussianity $\delta[\rho]$ is defined as[36]

$$\delta[\rho] = \frac{D^2_{\rm HS}[\rho,\tau]}{\mu[\rho]}, \tag{12}$$

where $D^2_{\rm HS}[\rho,\tau]$ denotes the Hilbert–Schmidt distance between ρ and τ, and it is expressed as

$$\begin{aligned} D^2_{\rm HS}[\rho,\tau] &= \frac{1}{2}{\rm Tr}\left[(\rho-\tau)^2\right] \\ &= \frac{\mu[\rho]+\mu[\tau]-2\kappa[\rho,\tau]}{2}, \end{aligned} \tag{13}$$

with $\mu[\rho] = {\rm Tr}\left[\rho^2\right]$ and $\kappa[\rho,\tau] = {\rm Tr}[\rho\tau]$ denoting the purity of ρ and the overlap between ρ and τ, respectively. In fact, the non-Gaussianity $\delta[\rho]$ quantifies how much a state fails to be Gaussian. Especially, $\delta[\rho]=0$ if and only if ρ is a Guassian state.

For GPACS, we readily know $\mu[\rho] = {\rm Tr}\left[\rho^2\right] = 1$. Moreover, as the reference Gaussian state, we choose the thermal state

$$\tau_{\bar n} = \nu(\bar n) = \sum_{l=0}^{\infty} \frac{\bar n^l}{(\bar n+1)^{l+1}} |l\rangle\langle l| \tag{14}$$

with the average photon number $\bar n$ (where $\bar n = \langle\alpha,\ m|a^\dagger a|\alpha,\ m\rangle$), whose purity is

$$\mu[\tau_{\bar n}] = \frac{1}{2\bar n+1}. \tag{15}$$

We now calculate the average photon number $\bar n$ in GPACS. It is obtained from Eq. (5) that

$$\begin{aligned} \bar n = \frac{N^2_{m,\alpha}}{2^m}\langle\alpha|{\rm H}_m\left[-\frac{{\rm i}(\alpha^*+a)}{\sqrt{2}}\right]a^\dagger a \\ \times {\rm H}_m\left[\frac{{\rm i}(a^\dagger+\alpha)}{\sqrt{2}}\right]|\alpha\rangle. \end{aligned} \tag{16}$$

Using Eq. (3) and the completeness relation of coherent state as well as the following integral formula:[39]

$$\int \frac{{\rm d}^2 z}{\pi} {\rm e}^{\zeta|z|^2+\xi z+\eta z^*} = -\frac{1}{\zeta}{\rm e}^{-\xi\eta/\zeta},\ \ {\rm Re}(\zeta)<0, \tag{17}$$

the final result of Eq. (16) is derived by (see Appendix A for details)

$$\begin{aligned} \bar n = \frac{N^2_{m,\alpha}(m!)^2}{2^m}\Bigg\{\sum_{k=0}^{m}\frac{|\alpha|^2 2^k {\rm H}_{m-k}(C^*){\rm H}_{m-k}(C)}{k!\left[(m-k)!\right]^2} + \sum_{k=0}^{m-1}\frac{2^{k+1}{\rm H}_{m-k-1}(C^*){\rm H}_{m-k-1}(C)}{k!\left[(m-k-1)!\right]^2} \\ - {\rm i}\sqrt{2}\sum_{k=0}^{m-1}\frac{2^k\left[\alpha{\rm H}_{m-k-1}(C^*){\rm H}_{m-k}(C) - \alpha^*{\rm H}_{m-k}(C^*){\rm H}_{m-k-1}(C)\right]}{k!(m-k)!\,(m-k-1)!}\Bigg\}. \end{aligned} \tag{18}$$

In order to calculate $\kappa[\rho,\tau_{\bar n}] = {\rm Tr}[\rho\tau_{\bar n}] = \langle\alpha,m|\tau_{\bar n}|\alpha,m\rangle$, using ${\rm e}^{\lambda a^\dagger a} =: \exp\left[\left({\rm e}^\lambda-1\right)a^\dagger a\right]:$ [40] we can reform Eq. (14) as its normal ordering form, i.e.,

$$\tau_{\bar n} = \frac{1}{\bar n+1} :\exp\left[\left(\frac{\bar n}{\bar n+1}-1\right)a^\dagger a\right]:, \tag{19}$$

In a similar approach to deriving Eq. (18), by means of Eqs. (3), (5), and (19), and the complete relation of coherent state, we have

$$\begin{aligned} \kappa[\rho,\tau_{\bar n}] = \frac{N^2_{m,\alpha}{\rm e}^{-|\alpha|^2}}{2^m(\bar n+1)}\frac{\partial^{2m}}{\partial t^m\partial s^m}{\rm e}^{{\rm i}\sqrt{2}\alpha s-{\rm i}\sqrt{2}\alpha^* t-t^2-s^2} \\ \times\int\frac{{\rm d}^2 z_1{\rm d}^2 z_2}{\pi^2}\exp\left[-|z_1|^2+\alpha^* z_1-{\rm i}\sqrt{2}tz_1-|z_2|^2+\frac{\bar n z_1^*}{\bar n+1}z_2+\left(\alpha+{\rm i}\sqrt{2}s\right)z_2^*\right]\Big|_{t=s=0}, \end{aligned} \tag{20}$$

then it follows from Eqs. (17) and (3) that

$$\kappa\left[\rho,\tau_{\bar{n}}\right] = \frac{N_{m,\alpha}^{2}\exp\left(-\frac{|\alpha|^{2}}{\bar{n}+1}\right)}{2^{m}\left(\bar{n}+1\right)} \times\sum_{k=0}^{m}\frac{\left(m!\right)^{2}\left(\frac{2\bar{n}}{\bar{n}+1}\right)^{k}}{k!\left[(m-k)!\right]^{2}}\left|\mathrm{H}_{m-k}(D)\right|^{2}, \quad (21)$$

where

$$D\equiv\frac{\mathrm{i}\sqrt{2}}{2}\left(\alpha+\frac{\bar{n}\alpha^{*}}{\bar{n}+1}\right),$$

and we have used the recurrence relation of $\mathrm{H}_{m}\left(x\right)$

$$\frac{\mathrm{d}^{k}}{\mathrm{d}x^{k}}\mathrm{H}_{m}\left(x\right)=\frac{2^{k}m!}{(m-k)!}\mathrm{H}_{m-k}\left(x\right). \quad (22)$$

Finally, substituting Eqs. (18) and (21) into Eq. (13) and then into Eq. (12), we obtain the analytical expression of non-Guassianity $\delta\left[\rho\right]$ of GPACS as

$$\delta\left[\rho\right]=D_{\mathrm{HS}}^{2}\left[\rho,\tau\right]=\frac{\bar{n}+1}{2\bar{n}+1}-\kappa\left[\rho,\tau_{\bar{n}}\right]. \quad (23)$$

In Fig. 3, we plot the non-Gaussianity $\delta\left[\rho\right]$ of GPACS as a function of parameter α where we suppose α to be real for simplicity. As shown in Fig. 3, for a given α, non-Gaussianity of GPACS is enhanced with m increasing. $\delta\left[\rho\right]$ increases monotonically with parameter α increasing. Especially, $\delta\left[\rho\right]$ tends to be 0.5 for $\alpha\to\infty$. This is sharply different from the coherent state, where as α increases, it becomes more and more classical. It is interesting to verify that applying the operator Ω to coherent state m times transforms a Guassian state into a non-Guassian one.

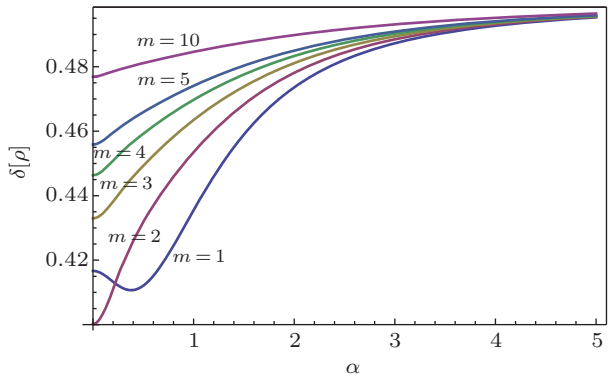


Fig. 3. (colour online) Plots of non-Gaussianity of GPACS as a function of the parameter α for several different values of m where we suppose α to be real for simplicity.

4. Normally ordered form of GPACS's density operator in dissipative channel

When GPACS evolves in a dissipative (photon-loss) channel, the evolution of the density matrix can be described by the following master equation in the interaction picture:[41]

$$\frac{\mathrm{d}\rho\left(t\right)}{\mathrm{d}t}=\kappa\left[2a\rho\left(t\right)a^{\dagger}-a^{\dagger}a\rho\left(t\right)-\rho\left(t\right)a^{\dagger}a\right], \quad (24)$$

where $\left[a,a^{\dagger}\right]=1$ and κ represents the dissipative coefficient. By virtue of the entangled state representation, the density operator $\rho\left(t\right)$ for a lossy channel satisfies the following relation:[42]

$$\rho\left(t\right)=\sum_{n=0}^{\infty}\frac{T^{n}}{n!}\,\mathrm{e}^{-\kappa ta^{\dagger}a}a^{n}\rho_{0}a^{\dagger n}\,\mathrm{e}^{-\kappa ta^{\dagger}a}, \quad (25)$$

with $T=1-\mathrm{e}^{-2\kappa t}$, which is just the Kraus sum representation. From Eq. (25), when $\kappa t=0$ and $T=0$, $\rho\left(0\right)=\rho_{0}$ as expected.

Using Eq. (9) and substituting the initial density matrix

$$\rho_{0}=\left|\alpha,m\right\rangle\left\langle\alpha,m\right|=N_{m,\alpha}^{2}\sum_{l,l'=0}^{[m/2]}\sum_{k=0}^{m-2l}\sum_{k'=0}^{m-2l'} \times\frac{\left(m!\right)^{2}\alpha^{m-2l-k}\alpha^{*m-2l'-k'}a^{\dagger k}\left|\alpha\right\rangle\left\langle\alpha\right|a^{k'}}{2^{l+l'}l!k!l'!k'!\left(m-2l-k\right)!\left(m-2l'-k'\right)!} \quad (26)$$

into Eq. (25), we have

$$\rho\left(t\right)=\sum_{n=0}^{\infty}\frac{T^{n}}{n!}G_{r,m,n}\left(a,a^{\dagger};t\right), \quad (27)$$

where

$$G_{r,m,n}\left(a,a^{\dagger};t\right)=N_{m,\alpha}^{2}\sum_{l,l'=0}^{[m/2]}\sum_{k=0}^{m-2l}\sum_{k'=0}^{m-2l'}\frac{\left(m!\right)^{2}\alpha^{m-2l-k}\alpha^{*m-2l'-k'}\,\mathrm{e}^{-\kappa ta^{\dagger}a}a^{n}a^{\dagger k}\left|\alpha\right\rangle\left\langle\alpha\right|a^{k'}a^{\dagger n}\,\mathrm{e}^{-\kappa ta^{\dagger}a}}{2^{l+l'}l!k!l'!k'!\left(m-2l-k\right)!\left(m-2l'-k'\right)!}. \quad (28)$$

On the other hand, using $a^{n}a^{\dagger m}=\left(-\mathrm{i}\right)^{m+n}:\mathrm{H}_{m,n}\left(\mathrm{i}\,a^{\dagger},\mathrm{i}\,a\right):$,[43] where

$$\mathrm{H}_{m,n}(\xi,\zeta)=\sum_{l=0}^{\min[m,n]}\times\frac{m!n!\,(-1)^l\,\xi^{m-l}\zeta^{n-l}}{l!\,(n-l)!\,(m-l)!}$$

is two-variable Hermite polynomial and

$$\begin{aligned}&\mathrm{e}^{-\kappa t a^\dagger a}a\,\mathrm{e}^{\kappa t a^\dagger a}=a\,\mathrm{e}^{\kappa t},\\&\mathrm{e}^{-\kappa t a^\dagger a}a^\dagger\,\mathrm{e}^{\kappa t a^\dagger a}=a^\dagger\,\mathrm{e}^{-\kappa t},\end{aligned}\tag{29}$$

we obtain that

$$\begin{aligned}&\mathrm{e}^{-\kappa t a^\dagger a}a^n a^{\dagger k}\left|\alpha\right\rangle\left\langle\alpha\right|a^{k'}a^{\dagger n}\,\mathrm{e}^{-\kappa t a^\dagger a}\\&=(-\mathrm{i})^{k+k'+2n}\,\mathrm{e}^{-|\alpha|^2}:\mathrm{H}_{k,n}\left(\mathrm{i}\,a^\dagger\,\mathrm{e}^{-\kappa t},\mathrm{i}\,\alpha\right)\\&\quad\times\mathrm{e}^{\alpha\,\mathrm{e}^{-\kappa t}a^\dagger-a^\dagger a+\alpha^*\,\mathrm{e}^{-\kappa t}a}\mathrm{H}_{n,k'}\left(\mathrm{i}\,\alpha^*,\mathrm{i}\,a\,\mathrm{e}^{-\kappa t}\right):,\end{aligned}\tag{30}$$

where we have considered the coherent state $|\alpha\rangle=\exp(-|\alpha|^2/2+\alpha a^\dagger)\,|0\rangle$ and the normal product form of vacuum projector $|0\rangle\langle 0|=:\exp\left(-a^\dagger a\right):$. Thus substituting Eq. (30) into Eq. (28), equation (27) becomes the normally ordered form of GPACS's density operator in dissipative channel, which is convenient for us to calculate its WF with time evolution.

5. Wigner function evolution of GPACS in dissipative channel

WF is a powerful tool to investigate the nonclassicality of optical fields. Its partial negativity implies the highly nonclassical properties of quantum states and is often used to describe the decoherence of quantum states. In this section, we derive the analytical expression of WF with time evolution for GPACS in dissipative channel. For a single-mode system ρ, WF is given by[7]

$$W(z)=\mathsf{Tr}\left[\rho\Delta_w(z,z^*)\right],\tag{31}$$

where $z=(q+\mathrm{i}\,p)/\sqrt{2}$ and the Wigner operator $\Delta_w(z,z^*)$ in the coherent representation $|z\rangle$ can be expressed as[39,44]

$$\Delta_w(z,z^*)=\frac{\mathrm{e}^{2|z|^2}}{\pi}\int\frac{\mathrm{d}^2\beta}{\pi}\left|\beta\right\rangle\left\langle-\beta\right|\,\mathrm{e}^{-2(\beta z^*-\beta^* z)}.\tag{32}$$

Based on the normally ordered form of $\rho(t)$ for GPACS, its WF is

$$\begin{aligned}W(z,t)=&\frac{N_{m,\alpha}^2\,\mathrm{e}^{2|z|^2-|\alpha|^2}}{\pi}\sum_{n=0}^{\infty}\sum_{l,l'=0}^{[m/2]}\sum_{k=0}^{m-2l}\sum_{k'=0}^{m-2l'}\frac{T^n\,(m!)^2\,\alpha^{m-2l-k}\alpha^{*m-2l'-k'}\,(-\mathrm{i})^{k+k'+2n}}{n!2^{l+l'}l!k!l'!k'!\,(m-2l-k)!\,(m-2l'-k')!}\\&\times\int\frac{\mathrm{d}^2\beta}{\pi}\,\mathrm{e}^{-|\beta|^2+\left(\alpha^*\,\mathrm{e}^{-\kappa t}-2z^*\right)\beta+\left(2z-\alpha\,\mathrm{e}^{-\kappa t}\right)\beta^*}\mathrm{H}_{k,n}\left(-\,\mathrm{i}\,\beta^*\,\mathrm{e}^{-\kappa t},\mathrm{i}\,\alpha\right)\mathrm{H}_{n,k'}\left(\mathrm{i}\,\alpha^*,\mathrm{i}\,\beta\,\mathrm{e}^{-\kappa t}\right).\end{aligned}\tag{33}$$

Then, using Eq. (17) and the generating function of two variable Hermite polynomial $\mathrm{H}_{m,n}(\xi,\zeta)$,

$$\mathrm{H}_{m,n}(\xi,\zeta)=\frac{\partial^{m+n}}{\partial\lambda^m\partial\lambda'^n}\,\mathrm{e}^{-\lambda\lambda'+\lambda\xi+\lambda'\zeta}|_{\lambda=\lambda'=0},\tag{34}$$

we finally obtain the analytical expression of WF of GPACS in dissipative channel (see Appendix B for details) as

$$\begin{aligned}W(z,t)=&\frac{N_{m,\alpha}^2\,(m!)^2\,\mathrm{e}^{2|z|^2-|\alpha|^2-\left|2z-\alpha\,\mathrm{e}^{-\kappa t}\right|^2}}{\pi}\sum_{n=0}^{\infty}\sum_{l,l'=0}^{[m/2]}\sum_{k=0}^{m-2l}\sum_{k'=0}^{m-2l'}\sum_{j=0}^{\min[k,k']}\left\{\frac{T^n}{n!}\right.\\&\left.\times\frac{(-\mathrm{i})^{k+k'+2n}\,\mathrm{e}^{-2j\kappa t}\alpha^{m-2l-k}\alpha^{*m-2l'-k'}\mathrm{H}_{k-j,n}\left(-E^*,\mathrm{i}\,\alpha\right)\mathrm{H}_{n,k'-j}\left(\mathrm{i}\,\alpha^*,E\right)}{2^{l+l'}l!l'!j!\,(m-2l-k)!\,(m-2l'-k')!\,(k-j)!\,(k'-j)!}\right\},\end{aligned}\tag{35}$$

where we have set $E\equiv\mathrm{i}\left(2z\,\mathrm{e}^{-\kappa t}-\alpha\,\mathrm{e}^{-2\kappa t}\right)$. Equation (35) is just the analytical expression of WF for GPACS in dissipative channel.

In particular, at the initial time ($t=0$), noticing $E=\mathrm{i}\,(2z-\alpha)$, $T=0$, and $n=0$, equation (35) becomes

$$W(z,0)=\frac{N_{m,\alpha}^2\,(m!)^2\,\mathrm{e}^{-2|z-\alpha|^2}}{\pi}\sum_{j=0}^{m}\frac{(-1)^j\,2^{j-m}}{j!\left[(m-j)!\right]^2}\left|\mathrm{H}_{m-j}\left(\mathrm{i}\,\frac{2z^*+\alpha-\alpha^*}{\sqrt{2}}\right)\right|^2,\tag{36}$$

where we have used Eq. (3) and considered $\mathrm{H}_{m,0}(\xi,\zeta) = \xi^m, \mathrm{H}_{0,n}(\xi,\zeta) = \zeta^n$. The result of Eq. (36) accords with that of Ref. [35]. On the other hand, when $\kappa t \to \infty$, noting $E \to 0$, $T \to 1$, $W(z,\infty) \to 1/\pi \exp(-2|z|^2)$, which indicates that the WF losses its non-Gaussian nature and becomes Gaussian after a long interaction in the dissipative environment. In Fig. 4, the evolutions of Wigner function $W(q,p)$ of GPACS are depicted in phase space each as a function of q and p with $\alpha = 0.5 + \mathrm{i}\,0.5$ for several different values of m and κt. It is interesting to note that GPACS's WF distribution is sensitive to parameter m. There exists an upward peak and the number of downward minor peaks is equal to the value of m (see Figs. 4(a) and 4(d)). When $m \geq 1$ WF presents the partial negative region, which may become more nonclassical by increasing the value of m. On the other hand, from Fig. 4 it is easily seen that the negative region of WF gradually disappears as κt increases. This implies that WF of GPACS losses its non-Guassian nature and reduces into a Gaussian state after a long interaction in this dissipative channel. Thus the loss of channel causes the partial negativity of WF of GPACS to disappear if the decay time κt exceeds a threshold value.

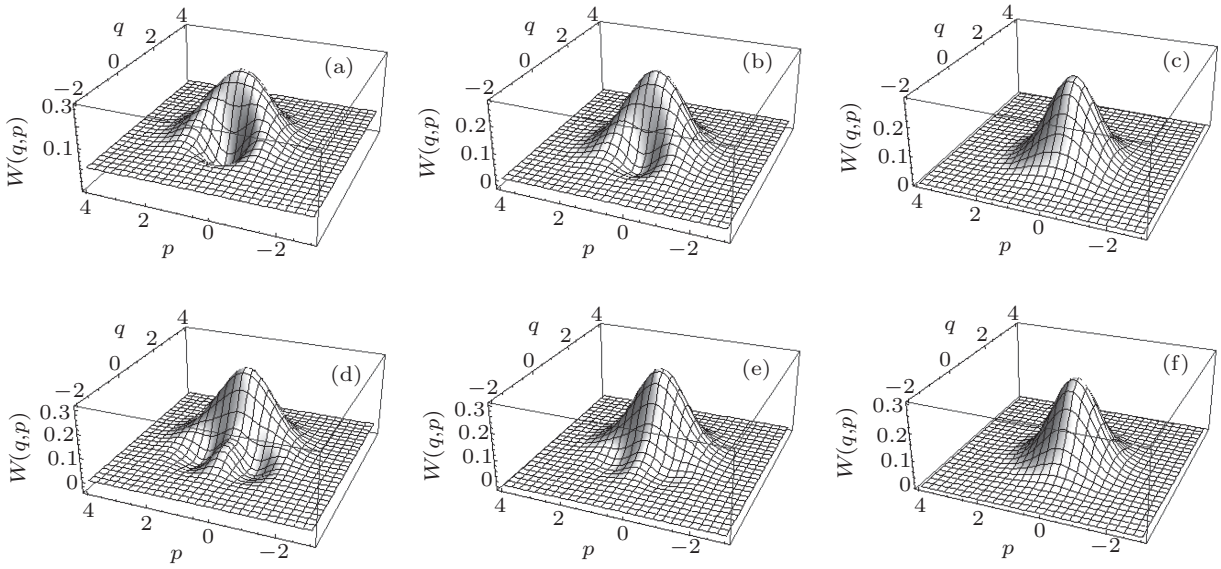


Fig. 4. Wigner function evolution of GPACS in phase space as a function of q and p with $\alpha = 0.5 + \mathrm{i}\,0.5$ for several different values of m: (a) $m = 1$, $\kappa t = 0$; (b) $m = 1$, $\kappa t = 0.2$; (c) $m = 1$, $\kappa t = 0.8$; (d) $m = 2$, $\kappa t = 0$; (e) $m = 2$, $\kappa t = 0.2$; and (f) $m = 2$, $\kappa t = 0.8$.

6. Conclusion

Because the normalization factor of GPACS is related to a Hermite polynomial function, whose form is much more compact than that of Ref. [34], GPACS is regarded as a Hermite-excited coherent state. Next, we calculate the photon-number distribution and the fidelity between GPACS and its original coherent state, which exhibits its highly nonclassical behaviours depending on what the m value is. Moreover, we have successfully adopted the Hilbert–Schmidt distance to quantify the Non-Gaussian characteristics of GPACS where we choose the thermal state with an average photon number as the reference Gaussian state. In addition, the decoherence of GPACS in dissipative channel is also discussed by studying the loss of nonclassicality in reference of the negativity of WF. As expected, WF presents the partial negative region, which may become more nonclassical by increasing m value and in dissipative channel it losses its non-Guassian nature and reduces into a Gaussian state after a long time interaction. Finally, it is worth noting that in Ref. [33] an experimental scheme was proposed to implement the coherent superposition operator $\mu a + \nu a^\dagger$, which can generate GPACS.

Appendix A: Derivation of Eq. (18)

Using Eq. (3) and the completeness relation of coherent state, i.e.,

$$\int \frac{\mathrm{d}^2 z_k}{\pi} |z_k\rangle \langle z_k| = 1, \quad \text{and} \quad \langle z_k| \alpha\rangle = \exp\left(-\frac{1}{2}|z_k|^2 - \frac{1}{2}|\alpha|^2 + z_k^* \alpha\right),$$

equation (16) becomes

$$\begin{aligned}
\bar{n} &= \frac{N_{m,\alpha}^2}{2^m} \int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2} \langle\alpha| \mathrm{H}_m\left[-\frac{\mathrm{i}(\alpha^* + a)}{\sqrt{2}}\right] |z_1\rangle \langle z_1| a^\dagger a |z_2\rangle \langle z_2| \mathrm{H}_m\left[\frac{\mathrm{i}(a^\dagger + \alpha)}{\sqrt{2}}\right] |\alpha\rangle \\
&= \frac{N_{m,\alpha}^2}{2^m} \frac{\partial^{2m}}{\partial t^m \partial s^m} \mathrm{e}^{-|\alpha|^2 - \mathrm{i}\sqrt{2}\alpha^* t + \mathrm{i}\sqrt{2}\alpha s - t^2 - s^2} \int \frac{\mathrm{d}^2 z_1}{\pi} z_1^* \mathrm{e}^{-|z_1|^2 - \mathrm{i}\sqrt{2} z_1 t + \alpha^* z_1} \\
&\quad \times \int \frac{\mathrm{d}^2 z_2}{\pi} z_2 \mathrm{e}^{-|z_2|^2 + z_1^* z_2 + z_2^*(\alpha + \mathrm{i}\sqrt{2} s)}|_{t=s=0}. \qquad \text{(A1)}
\end{aligned}$$

It follows from Eq. (17) that

$$\begin{aligned}
\bar{n} &= \frac{N_{m,\alpha}^2}{2^m} \frac{\partial^{2m}}{\partial t^m \partial s^m} \mathrm{e}^{-|\alpha|^2 - \mathrm{i}\sqrt{2}\alpha^* t + \mathrm{i}\sqrt{2}\alpha s - t^2 - s^2} \frac{\partial}{\partial \mu} \int \frac{\mathrm{d}^2 z_1}{\pi} \mathrm{e}^{-|z_1|^2 + (\alpha^* - \mathrm{i}\sqrt{2} t) z_1 + \mu z_1^*} \\
&\quad \times \frac{\partial}{\partial \nu} \int \frac{\mathrm{d}^2 z_2}{\pi} \mathrm{e}^{-|z_2|^2 + (z_1^* + \nu) z_2 + (\alpha + \mathrm{i}\sqrt{2} s) z_2^*}|_{\mu=\nu=t=s=0} \\
&= \frac{N_{m,\alpha}^2}{2^m} \frac{\partial^2}{\partial \mu \partial \nu} \mathrm{e}^{\alpha\nu + \mu\alpha^*} \frac{\partial^{2m}}{\partial t^m \partial s^m} \mathrm{e}^{2st - t^2 - s^2 - \mathrm{i}\sqrt{2}(\alpha^* + \alpha + \mu)t + \mathrm{i}\sqrt{2}(\alpha + \alpha^* + \nu)s}|_{\mu=\nu=t=s=0}. \qquad \text{(A2)}
\end{aligned}$$

Then, expanding the term e^{2st} into the form of the series, using Eq. (3) again and letting $A \equiv -\mathrm{i}\sqrt{2}(\alpha^* + \alpha + \mu)/2$ and $B \equiv \mathrm{i}\sqrt{2}(\alpha + \alpha^* + \nu)/2$, we finally obtain

$$\begin{aligned}
\bar{n} &= \frac{N_{m,\alpha}^2}{2^m} \frac{\partial^2}{\partial \mu \partial \nu} \mathrm{e}^{\alpha\nu + \mu\alpha^*} \sum_{k=0}^{\infty} \frac{\partial^{2k}}{k! 2^k \partial A^k \partial B^k} \mathrm{H}_m(A) \mathrm{H}_m(B)|_{\mu=\nu=0} \\
&= \frac{N_{m,\alpha}^2 (m!)^2}{2^m} \sum_{k=0}^{m} \frac{2^k \frac{\partial}{\partial \mu}\left[\mathrm{e}^{\mu\alpha^*} \mathrm{H}_{m-k}(A)\right]|_{\mu=0} \frac{\partial}{\partial \nu}\left[\mathrm{e}^{\alpha\nu} \mathrm{H}_{m-k}(B)\right]|_{\nu=0}}{k! \left[(m-k)!\right]^2}, \qquad \text{(A3)}
\end{aligned}$$

where in the last step we have used Eq. (22). Further, we can calculate

$$\frac{\partial}{\partial \mu}\left[\mathrm{e}^{\mu\alpha^*} \mathrm{H}_{m-k}(A)\right]|_{\mu=0} = \alpha^* \mathrm{H}_{m-k}(C^*) - \frac{\mathrm{i}\sqrt{2}(m-k)!}{(m-k-1)!} \mathrm{H}_{m-k-1}(C^*), \qquad \text{(A4)}$$

and

$$\frac{\partial}{\partial \nu}\left[\mathrm{e}^{\alpha\nu} \mathrm{H}_{m-k}(B)\right]|_{\nu=0} = \alpha \mathrm{H}_{m-k}(C) + \frac{\mathrm{i}\sqrt{2}(m-k)!}{(m-k-1)!} \mathrm{H}_{m-k-1}(C), \qquad \text{(A5)}$$

where $C \equiv \mathrm{i}\sqrt{2}(\alpha + \alpha^*)/2$. Substituting Eqs. (A4) and (A5) into Eq. (A3) leads to the result of Eq. (18).

Appendix B: Derivation of Eq. (35)

Using Eqs. (34) and (17), we have

$$\begin{aligned}
\mathcal{F} &\equiv \int \frac{\mathrm{d}^2 \beta}{\pi} \mathrm{e}^{-|\beta|^2 + (\alpha^* \mathrm{e}^{-\kappa t} - 2z^*)\beta + (2z - \alpha \mathrm{e}^{-\kappa t})\beta^*} \mathrm{H}_{k,n}\left(-\mathrm{i}\beta^* \mathrm{e}^{-\kappa t}, \mathrm{i}\alpha\right) \mathrm{H}_{n,k'}\left(\mathrm{i}\alpha^*, \mathrm{i}\beta \mathrm{e}^{-\kappa t}\right) \\
&= \frac{\partial^{k+n}}{\partial s^k \partial s'^n} \frac{\partial^{k'+n}}{\partial \lambda^n \partial \lambda'^{k'}} \mathrm{e}^{-ss' - \lambda\lambda' + \mathrm{i}\alpha s' + \mathrm{i}\alpha^* \lambda} \\
&\quad \times \int \frac{\mathrm{d}^2 \beta}{\pi} \mathrm{e}^{-|\beta|^2 + (\alpha^* \mathrm{e}^{-\kappa t} - 2z^* + \mathrm{i}\,\mathrm{e}^{-\kappa t}\lambda')\beta + (2z - \alpha \mathrm{e}^{-\kappa t} - \mathrm{i}\,\mathrm{e}^{-\kappa t} s)\beta^*}|_{s=s'=\lambda=\lambda'=0} \\
&= \mathrm{e}^{-|2z - \alpha \mathrm{e}^{-\kappa t}|^2} \frac{\partial^{k+n}}{\partial s^k \partial s'^n} \frac{\partial^{k'+n}}{\partial \lambda^n \partial \lambda'^{k'}} \mathrm{e}^{s\lambda' \mathrm{e}^{-2\kappa t} - ss' - E^* s + \mathrm{i}\alpha s' - \lambda\lambda' + \mathrm{i}\alpha^* \lambda + E\lambda'}|_{s=s'=\lambda=\lambda'=0}, \qquad \text{(B1)}
\end{aligned}$$

where $E \equiv 2\,\mathrm{i}\,z\,\mathrm{e}^{-\kappa t} - \mathrm{i}\,\alpha\,\mathrm{e}^{-2\kappa t}$. Expanding the exponential term $\mathrm{e}^{s\lambda'\,\mathrm{e}^{-2\kappa t}}$, and again using Eq. (34), it follows that

$$\begin{aligned}\mathcal{F} &= \mathrm{e}^{-\left|2z-\alpha\,\mathrm{e}^{-\kappa t}\right|^2}\sum_{j=0}^{\infty}\frac{\mathrm{e}^{-2j\kappa t}\partial^{2j}}{j!\partial\left(-E^*\right)^j\partial E^j}\frac{\partial^{k+n}}{\partial s^k\partial s'^n}\frac{\partial^{k'+n}}{\partial\lambda^n\partial\lambda'^{k'}}\,\mathrm{e}^{-ss'-E^*s+\mathrm{i}\,\alpha s'-\lambda\lambda'+\mathrm{i}\,\alpha^*\lambda+E\lambda'}\big|_{s=s'=\lambda=\lambda'=0}\\ &= \mathrm{e}^{-\left|2z-\alpha\,\mathrm{e}^{-\kappa t}\right|^2}\sum_{j=0}^{\infty}\frac{\mathrm{e}^{-2j\kappa t}}{j!}\frac{\partial^j}{\partial\left(-E^*\right)^j}\mathrm{H}_{k,n}\left(-E^*,\mathrm{i}\,\alpha\right)\frac{\partial^j}{\partial E^j}\mathrm{H}_{n,k'}\left(\mathrm{i}\,\alpha^*,E\right). \qquad \text{(B2)}\end{aligned}$$

Noticing the well-known differential relation of $\mathrm{H}_{m,n}\left(\xi,\zeta\right)$

$$\begin{aligned}&\frac{\partial}{\partial\xi}\mathrm{H}_{m,n}\left(\xi,\zeta\right) = m\mathrm{H}_{m-1,n}\left(\xi,\zeta\right), \quad \frac{\partial}{\partial\zeta}\mathrm{H}_{m,n}\left(\xi,\zeta\right) = n\mathrm{H}_{m,n-1}\left(\xi,\zeta\right),\\ &\frac{\partial^{l+k}}{\partial\xi^l\partial\zeta^k}\mathrm{H}_{m,n}\left(\xi,\zeta\right) = \frac{m!n!}{(m-l)!\,(n-k)!}\mathrm{H}_{m-l,n-k}\left(\xi,\zeta\right), \qquad \text{(B3)}\end{aligned}$$

we have

$$\mathcal{F} = \mathrm{e}^{-\left|2z-\alpha\,\mathrm{e}^{-\kappa t}\right|^2}\sum_{j=0}^{\min[k,k']}\frac{\mathrm{e}^{-2j\kappa t}k!k'!}{j!\,(k-j)!\,(k'-j)!}\mathrm{H}_{k-j,n}\left(-E^*,\mathrm{i}\,\alpha\right)\mathrm{H}_{n,k'-j}\left(\mathrm{i}\,\alpha^*,E\right). \qquad \text{(B4)}$$

Substitution of Eq. (B4) into Eq. (33) just leads to the result of Eq. (35).

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