**Review**

# Recent advance on the efficiency at maximum power of heat engines*

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This review reports several key advances on the theoretical investigations of efficiency at maximum power of heat engines in the past five years. The analytical results of efficiency at maximum power for the Curzon–Ahlborn heat engine, the stochastic heat engine constructed from a Brownian particle, and Feynman's ratchet as a heat engine are presented. It is found that: the efficiency at maximum power exhibits universal behavior at small relative temperature differences; the lower and the upper bounds might exist under quite general conditions; and the problem of efficiency at maximum power comes down to seeking for the minimum irreversible entropy production in each finite-time isothermal process for a given time.

**Keywords:** efficiency at maximum power, heat engine, universality, bounds

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## 1. Introduction

It is well known that the efficiency of a heat engine operating between two reservoirs at different temperatures can reach a maximum value only for the reversible (quasi-static) cycle. This value is called the Carnot efficiency, $\eta_{\rm C} = 1 - T_{\rm c}/T_{\rm h}$, where $T_{\rm h}$ and $T_{\rm c}$ are the temperatures of the hot and the cold reservoirs, respectively. However, heat engines operating at the Carnot efficiency produce zero power output. To achieve a non-vanishing power output, the Carnot cycle should be speeded up and completed in a finite time. It is of great importance to investigate the problem of the efficiency at maximum power (EMP) of heat engines in these times of soaring oil prices. This problem has attracted much attention.[1−28] In 1975, Curzon and Ahlborn[1] considered a heat engine undergoing a Carnot-like cycle and derived the EMP as

$$\eta_{\rm CA} = 1 - \sqrt{T_{\rm c}/T_{\rm h}} = 1 - \sqrt{1 - \eta_{\rm C}}. \qquad (1)$$

Subsequently, Chen and Yan,[7] Schmiedl and Seifert,[17] and Esposito *et al.*[21] investigated the Curzon–Ahlborn heat engine by using different heat transfer laws, stochastic heat engines, and low-dissipation Carnot-like engines, respectively. They found that the EMP for those engines could be expressed with the same form [see Eq. (56)]. Esposito *et al.* also found that the EMP for low-dissipation Carnot-like engines was bounded between $\eta_- \equiv \eta_{\rm C}/2$ and $\eta_+ \equiv \eta_{\rm C}/(2 - \eta_{\rm C})$.[21] Additionally, based on the minimum irreversible entropy production, Wang and the present author[29] recently have proved that the EMP of the Carnot-like engines are bounded between $\eta_- \equiv \eta_{\rm C}/2$ and $\eta_+ \equiv \eta_{\rm C}/(2 - \eta_{\rm C})$.

In 2008, we investigated Feynman's ratchet as a heat engine and obtained an analytical expression for the EMP of that engine.[18] We also proposed a conjecture that "a universal efficiency at maximum power, $\eta_{\rm C}/2 + \eta_{\rm C}^2/8$, should exist at small relative temperature differences."[18] Subsequently, Esposito *et al.*[20] constructed a general heat engine and verified that the EMP exhibits universality up to a quadratic order term of $\eta_{\rm C}$ for the strong coupling system in the presence of a left–right symmetry.

This review will report the recent advance on the EMP problem. The rest of this article is organized as follows. In Section 2, the critical assumptions and the steps that lead to the Curzon–Ahlborn efficiency (1) are reviewed. In Section 3, the EMP of a stochastic heat engine proposed by Schmiedl and Seifert is presented. In Section 4, Feynman's ratchet is considered

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as a heat engine, and then the EMP of that engine is derived. In Section 5, we discuss the work by Esposito *et al.*,[21] who obtained the conditions for the existence of universality up to a quadratic order term of $\eta_{\rm C}$. In Section 6, the key ideas and the results of low-dissipation Carnot-like engines are presented. In Section 7, we revisit the Carnot-like engines based on the minimum irreversible entropy production. We give a brief summary and envisage some open questions on the present topic in the last section.

## 2. Curzon–Ahlborn heat engine

Now we briefly reinvestigate the EMP problem for the Carnot-like heat engines proposed by Curzon and Ahlborn.

### 2.1. Model

The heat engine considered by Curzon and Ahlborn operates in a Canot-like cycle consisting of the following four processes.

(i) ***Isothermal expansion process***

In this process, the working substance expands in contact with a hot reservoir at temperature $T_{\rm h}$ and absorbs heat $Q_1$ from the hot reservoir during the time interval of $0 < t < t_1$. The effective temperature of the working substance is assumed to be $T_{\rm he}$, which is a constant and lower than $T_{\rm h}$. The total entropy production in this process is

$$\Delta S_1 = Q_1/T_{\rm h} + \Delta S_1^{\rm ir}, \tag{2}$$

where $\Delta S_1^{\rm ir} > 0$ is the irreversible entropy production. Due to the convenient consideration, we take the Boltzmann factor $k_{\rm B} = 1$ in the whole review.

(ii) ***Adiabatic expansion process***

The heat exchange and the variation of entropy in this process are vanishing, i.e., $Q_2 = 0$ and $\Delta S_2 = 0$. The time for completing this process is denoted as $t_2$.

(iii) ***Isothermal compression process***

In this process, the working substance is compressed in contact with a cold reservoir at temperature $T_{\rm c}$ and releases heat $Q_3$ to the cold reservoir. The time for completing this process is assumed to be $t_3$. The effective temperature of the working substance is assumed to be $T_{\rm ce}$, which is a constant and higher than $T_{\rm c}$. The total entropy production in this process is

$$\Delta S_3 = -Q_3/T_{\rm c} + \Delta S_3^{\rm ir}, \tag{3}$$

where $\Delta S_3^{\rm ir} > 0$ is the irreversible entropy production.

(iv) ***Adiabatic compression process***

Similar to the adiabatic expansion process, both the heat exchange and the variation of entropy are vanishing, i.e., $Q_4 = 0$ and $\Delta S_4 = 0$.

After a whole cycle, the working substance recovers its initial state. Thus the changes of the total energy and the entropy (both are state functions) vanish, from which we can easily derive that the entropy productions in the two isothermal processes cancel each other, i.e.,

$$\Delta S_1 = -\Delta S_3 \equiv \Delta S, \tag{4}$$

and that the net work output in the cycle satisfies

$$W = Q_1 - Q_3. \tag{5}$$

### 2.2. Basic assumptions

Three key assumptions were adopted by Curzon and Ahlborn.

i) ***Assumption on heat exchanges***

Because the working substance and the reservoirs have different temperatures during the isothermal processes, heat transfer takes place between them. The working substance absorbs heat

$$Q_1 = \alpha(T_{\rm h} - T_{\rm he})t_1 \tag{6}$$

from the hot reservoir and releases heat

$$Q_3 = \beta(T_{\rm ce} - T_{\rm c})t_3 \tag{7}$$

to the cold reservoir, where $\alpha$ and $\beta$ are the heat conductivities.

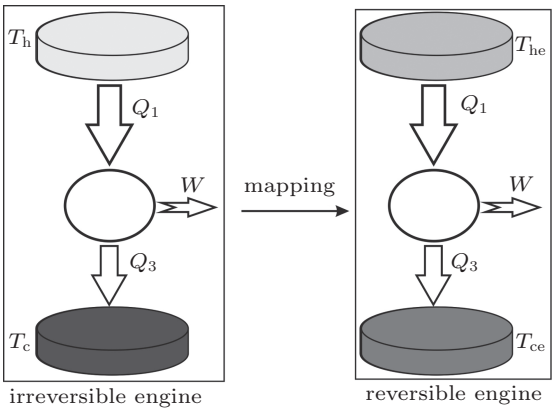



**Fig. 1.** Schematic diagram of endoreversible assumption.

ii) ***Assumption on total time***

The total time for the cycle is proportional to the time for completing the two isothermal processes, i.e.,

$$t_{\rm tot} = \xi(t_1 + t_3), \tag{8}$$

where $\xi$ is a constant.

iii) ***Endoreversible assumption***

Because $\Delta S_1^{\rm ir}$ and $\Delta S_3^{\rm ir}$ are positive, the engine operating between the two reservoirs at temperatures $T_{\rm h}$ and $T_{\rm c}$ is irreversible. However, as shown in Fig. 1, the irreversible engine can be mapped into a reversible one working between two reservoirs at effective temperatures $T_{\rm he}$ and $T_{\rm ce}$. This assumption can be expressed as

$$Q_1/T_{\rm he} - Q_3/T_{\rm ce} = 0, \tag{9}$$

so the Curzon–Ahlborn heat engine is also called the endoreversible engine.

## 2.3. Optimization

The power output for the Curzon–Ahlborn heat engine is defined as $P = W/t_{\rm tot}$. Considering Eqs. (5)–(9), we have

$$P = \frac{T_{\rm he} - T_{\rm ce}}{\xi[T_{\rm he}/\alpha(T_{\rm h} - T_{\rm he}) + T_{\rm ce}/\beta(T_{\rm ce} - T_{\rm c})]}. \tag{10}$$

There are only two unknown parameters $T_{\rm he}$ and $T_{\rm ce}$ in the expression of $P$. Maximizing $P$ with respect to them, i.e., setting $\partial P/\partial T_{\rm he} = \partial P/\partial T_{\rm ce} = 0$, we obtain

$$T_{\rm ce}/T_{\rm he} = \sqrt{T_{\rm c}/T_{\rm h}}. \tag{11}$$

On the other hand, the efficiency is defined as $\eta = W/Q_1$. Considering Eqs. (5) and (9), we have

$$\eta = 1 - T_{\rm ce}/T_{\rm he}. \tag{12}$$

Substituting Eq. (11) into the above equation, we can derive the Curzon–Ahlborn efficiency (1). This relation has in fact been obtained by Novikov[30] and Chambadal[31] when they investigated the efficiency of atomic power stations. For the historic cause, equation (1) is still called the Curzon–Ahlborn efficiency in most literature.

As it is pointed out by Chen and Yan[7], if equations (6) and (7) are replaced with $Q_1 = \alpha(1/T_{\rm he} - 1/T_{\rm h})t_1$ and $Q_3 = \beta(1/T_{\rm c} - 1/T_{\rm ce})t_3$, respectively, through similar calculations, one can derive the EMP to be

$$\eta_{\rm CY} = \frac{\eta_{\rm C}}{2 - \gamma_{\rm CY}\eta_{\rm C}} \tag{13}$$

with $\gamma_{\rm CY} = 1/(1 + \sqrt{\beta/\alpha})$.

# 3. Stochastic heat engine

In 2008, Schmeild and Seifert conceptually constructed a stochastic heat engine by using an optic trap to control a Brownian particle, which will be discussed in this section.

## 3.1. Model

The controlled particle performs a Carnot-like cycle through the following processes.[17]

(I) ***Isothermal expansion process***

In this process, the particle is embedded in a hot medium (reservoir) at temperature $T_{\rm h}$. The governing potential $V(\boldsymbol{r}, \lambda_1(\tau))$ is time-dependently changed by coordinating the intensity of the optic trap during $0 < \tau < t_1$, where $\lambda_1(\tau)$ represents the protocol of coordination. In this process, a certain amount of heat $Q_1$ is absorbed from the hot reservoir. The total entropy production in this process has the form given by Eq. (2).

(II) ***Adiabatic expansion process***

This process takes place due to instantaneously switching the temperature from $T_{\rm h}$ to $T_{\rm c}$ at time $\tau = t_1$. The position distribution of the Brownian particle does not change during this step. To keep the distribution unchanged, the potential also needs a corresponding sharp change. The heat exchange and the variation of entropy in this process are vanishing, i.e., $Q_2 = 0$ and $\Delta S_2 = 0$.

(III) ***Isothermal compression process***

In this process, the particle is embedded in a cold medium (reservoir) at temperature $T_{\rm c}$. The governing potential is time-dependently changed during $t_1 < \tau < t_1 + t_3$. The protocol of coordination is denoted by $\lambda_3(\tau)$. In this process, a certain amount of heat $Q_3$ is released to the cold reservoir. The total entropy production in this process has the form given by Eq. (3).

(IV) ***Adiabatic compression process***

Similar to the adiabatic expansion process, this step is also completed instantaneously. Both the heat exchange and the variation of entropy are vanishing, i.e., $Q_4 = 0$ and $\Delta S_4 = 0$.

After a whole cycle, the position distribution of the Brownian particle returns its initial distribution. Thus the changes of the total energy and the entropy vanish, from which we can easily obtain equations similar to Eqs. (4) and (5).

### 3.2. Basic assumptions

The key assumption is that the evolutional behavior of position distribution $p(\boldsymbol{r},t)$ of the Brownian particle in an isothermal process at temperature $T$ satisfies the stochastic dynamics[17]

$$\frac{\partial p(\boldsymbol{r},\tau)}{\partial \tau} = -\nabla \cdot \boldsymbol{j}, \tag{14}$$

where $\boldsymbol{j} = -\boldsymbol{\mu} \cdot (\nabla V + T\nabla) p(\boldsymbol{r},\tau)$, and $\boldsymbol{\mu}$ is the mobility tensor.

Based on the above dynamics, the stochastic thermodynamics[32–34] can be established. In particular, within this theoretical framework, the irreversible entropy production in the isothermal process can be expressed as

$$\Delta S^{\rm ir} = \frac{1}{T}\int_{t_{\rm i}}^{t_{\rm f}} {\rm d}\tau \times \int {\rm d}^3\boldsymbol{r} \frac{\boldsymbol{j}(\boldsymbol{r},\lambda(\tau)) \cdot \boldsymbol{\mu}^{-1} \cdot \boldsymbol{j}(\boldsymbol{r},\lambda(\tau))}{p(\boldsymbol{r},\tau)}, \tag{15}$$

where $\boldsymbol{\mu}^{-1}$ is the inverse of the mobility tensor, $t_{\rm i}$ and $t_{\rm f}$ represent the start and the end time of the process, respectively, $T$ is the temperature and equals $T_{\rm h}$ and $T_{\rm c}$ for the isothermal expansion process and the isothermal compression process, respectively, $\lambda(\tau)$ represents the protocol and equals $\lambda_1(\tau)$ and $\lambda_3(\tau)$ for the isothermal expansion process and the isothermal compression process, respectively.

By introducing a normalized time $\hat{\tau}(\tau) \equiv (\tau - t_{\rm i})/(t_{\rm f} - t_{\rm i})$ and defining $\mathscr{P}(\boldsymbol{r},\hat{\tau}) \equiv p(\boldsymbol{r},\tau)$, $\mathit{\Lambda}(\hat{\tau}) = \lambda(\tau)$, and $\mathscr{J}(\boldsymbol{r},\mathit{\Lambda}(\hat{\tau})) = (t_{\rm f} - t_{\rm i})\boldsymbol{j}(\boldsymbol{r},\lambda(\tau))$, Schmeild and Seifert proved that the form of Eq. (14) is invariant[17] and that equation (15) can be transformed into

$$\Delta S^{\rm ir} = \frac{1}{T(t_{\rm f} - t_{\rm i})}\int_0^1 {\rm d}\hat{\tau} \times \int {\rm d}^3\boldsymbol{r} \frac{\mathscr{J}(\boldsymbol{r},\mathit{\Lambda}(\hat{\tau})) \cdot \boldsymbol{\mu}^{-1} \cdot \mathscr{J}(\boldsymbol{r},\mathit{\Lambda}(\hat{\tau}))}{\mathscr{P}(\boldsymbol{r},\hat{\tau})}. \tag{16}$$

For a given protocol, particularly the protocol for minimizing $\Delta S^{\rm ir}$, the integral of the above equation does not depend on $(t_{\rm f} - t_{\rm i})$, thus it is not hard to see that the minimum irreversible entropy production in the isothermal process has the form

$$\min\{\Delta S^{\rm ir}\} = \frac{A}{T(t_{\rm f} - t_{\rm i})}, \tag{17}$$

where $A$ is the irreversible action named by Schmeild and Seifert.[17]

### 3.3. Optimization

Similar to the thermodynamics analysis in Section 2, we can derive the expressions of power and efficiency for the stochastic heat engine, which are given respectively as

$$P = \frac{(T_{\rm h} - T_{\rm c})\Delta S - (T_{\rm h}\Delta S_1^{\rm ir} + T_{\rm c}\Delta S_3^{\rm ir})}{t_1 + t_3}, \tag{18}$$

and

$$\eta = \frac{(T_{\rm h} - T_{\rm c})\Delta S - (T_{\rm h}\Delta S_1^{\rm ir} + T_{\rm c}\Delta S_3^{\rm ir})}{T_{\rm h}(\Delta S - \Delta S_1^{\rm ir})}. \tag{19}$$

It is important to notice that $\Delta S_1^{\rm ir}$ and $\Delta S_3^{\rm ir}$ depend on the protocols. In order to maximize the power, we can first minimize $\Delta S_1^{\rm ir}$ and $\mathit{\Delta} S_3^{\rm ir}$ with respect to the protocols for given time $t_1$ and $t_3$, which gives the optimized protocols $\lambda_1^*(\tau)$ and $\lambda_3^*(\tau)$. Then the power can be maximized with respect to time $t_1$ and $t_3$ for the optimized protocols.

In terms of Eq. (17), the minimum irreversible entropy productions in the two isothermal processes can be expressed as $\min\{\Delta S_1^{\rm ir}\} = A_1/t_1$ and $\min\{\Delta S_3^{\rm ir}\} = A_3/t_3$. Substituting them into Eq. (18) and maximizing the power with respect to $t_1$ and $t_3$, we obtain the optimized time as

$$t_l^* = \frac{2\sqrt{A_l}(\sqrt{A_1} + \sqrt{A_3})}{(T_{\rm h} - T_{\rm c})\Delta S}, \qquad (l = 1, 3). \tag{20}$$

Then substituting the above equation and the minimum irreversible entropy productions into Eq. (19), we can derive the EMP as[17]

$$\eta_{\rm SS} = \frac{\eta_{\rm C}}{2 - \gamma_{\rm SS}\eta_{\rm C}}, \tag{21}$$

where $\gamma_{\rm SS} = 1/(1 + \sqrt{A_3/A_1})$. In particular, using the harmonic potential to represent the effect of the optic trap, Schmeild and Seifert[17] proved that $A_1 = A_3$. Therefore, the above equation is reduced to

$$\eta'_{\rm SS} = \frac{2\eta_{\rm C}}{4 - \eta_{\rm C}}. \tag{22}$$

## 4. Feynman's ratchet as heat engine

The ratchet model is well known, since Feynman used it to illustrate the second law of thermodynamics.[35] He proved that the ratchet system can achieve the Carnot efficiency when the detailed balance holds. Here we will present the key ideas and results in our previous work[18] on the EMP of this system.

## 4.1. Model

As shown in Fig. 2, Feynman's ratchet device consists of a ratchet, a pawl and spring, vanes, two thermal reservoirs at temperatures $T_{\rm h}$, $T_{\rm c}$ ($T_{\rm h} > T_{\rm c}$), an axle and wheel, and a load. For simplicity, we assume that the axle and wheel is a rigid and frictionless thermal insulator. The ratchet is so small that its kinetic energy can be neglected.

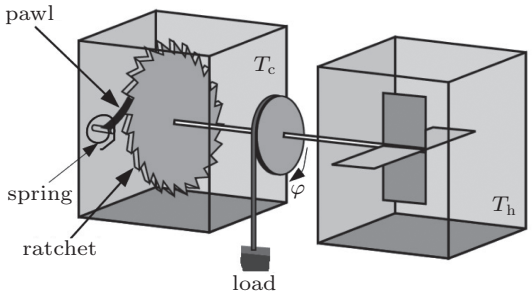



**Fig. 2.** Feynman's ratchet device (revised from Ref. [18]). The arrow above variable $\varphi$ represents the positive direction of rotation.

Let $\varphi$ represent the angular variable and $\theta$ the angle corresponding to a single rotational step. Figure 3 depicts the potential of Feynman's ratchet. The internal variable $\epsilon$ is the height of the energy barrier. The load results in a moment $M$ respect to the axle, and the corresponding potential is $M\varphi$.

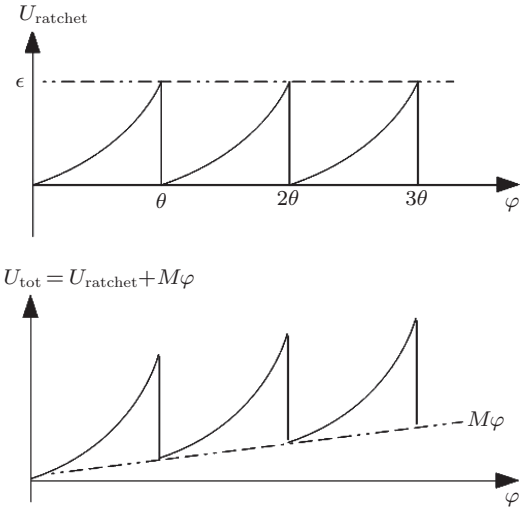



**Fig. 3.** Schematic diagram of potential for Feynman's ratchet device. Panel (a) gives the bare ratchet potential due to lifting of the pawl in the absence of load, panel (b) gives the total potential including the contribution of lifting the pawl and the load.

Now let us analyse the steps of forward and backward rotations following Feynman's idea.[35] In the forward step, the vanes accumulate energy $\epsilon + M\theta$ to lift the pawl and the load at a rate

$$R_{\rm F} = r_0\, {\rm e}^{-(\epsilon + M\theta)/T_{\rm h}}, \tag{23}$$

where $r_0$ is a rate constant with units of $\rm s^{-1}$. In this process, the ratchet absorbs heat $\epsilon + M\theta$ from the hot reservoir. A part of this heat is transduced into work $M\theta$, and the remaining energy is eventually transferred as heat $\epsilon$ to the cold reservoir through the interaction between the ratchet and the pawl. Similarly, energy $\epsilon$ should be accumulated from the cold reservoir to lift the pawl high enough so that the ratchet can rotate backward. The rate to get this energy is

$$R_{\rm B} = r_0\, {\rm e}^{-\epsilon/T_{\rm c}}. \tag{24}$$

In the backward step, the work done by the load is $M\theta$. This energy and the accumulated energy $\epsilon$ are returned to the hot reservoir in the form of heat.

Using the rates of forward and backward rotations, we write the power of Feynman's ratchet system as

$$\begin{aligned} P &= (R_{\rm F} - R_{\rm B}) M\theta \\ &= r_0 M\theta [\, {\rm e}^{-(\epsilon + M\theta)/T_{\rm h}} - {\rm e}^{-\epsilon/T_{\rm c}}]. \end{aligned} \tag{25}$$

Similarly, by assuming that the heat leakage due to the kinetic energy vanishes, we can derive the rate of heat absorbtion from the hot bath

$$\dot{Q}_1 = (R_{\rm F} - R_{\rm B})(\epsilon + M\theta). \tag{26}$$

Therefore, the efficiency can be expressed as

$$\eta = P/\dot{Q}_1 = M\theta/(\epsilon + M\theta). \tag{27}$$

## 4.2. Optimization

In fact, the optimization of power with respect to external load $M$ for a given internal parameter was investigated by serval researchers.[36–38] Here we maximize the power with respect to internal variable $\epsilon$ and external load $M$ simultaneously. By setting $\partial P/\partial\epsilon = \partial P/\partial M = 0$, we have

$$\begin{cases} (1 - \eta_{\rm C}) \exp(\eta_{\rm C}\epsilon/T_{\rm c} - M\theta/T_{\rm h}) = 1, \\ (1 - M\theta/T_{\rm h}) \exp(\eta_{\rm C}\epsilon/T_{\rm c} - M\theta/T_{\rm h}) = 1. \end{cases} \tag{28}$$

The solution to the above equation can be written as

$$\begin{cases} M\theta/T_{\rm h} = \eta_{\rm C}, \\ \epsilon/T_{\rm c} = 1 - (1/\eta_{\rm C}) \ln(1 - \eta_{\rm C}). \end{cases} \tag{29}$$

Substituting Eq. (29) into Eq. (27), we derive the EMP for Feynman's ratchet device

$$\eta_{\rm T} = \frac{\eta_{\rm C}^2}{\eta_{\rm C} - (1 - \eta_{\rm C}) \ln(1 - \eta_{\rm C})}. \tag{30}$$

### 4.3. Conjecture on universality

Expression (30) appears more complicated than the Curzon–Ahlborn efficiency (1) and the Schmiedl–Seifert efficiency (22). However, they are quite close to each other as shown in Fig. 4 for small relative temperature differences (for example, at $\eta_C < 0.5$), which inspires us to investigate the EMP problem at small relative temperature differences.

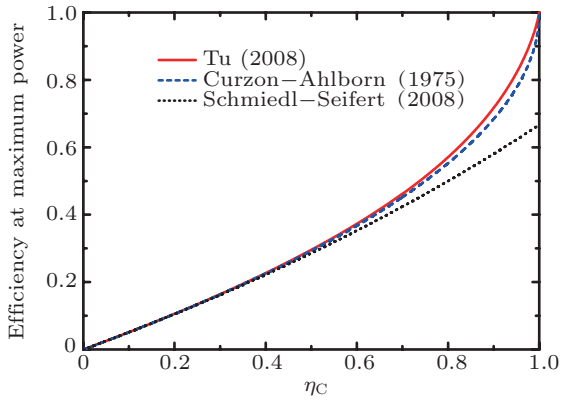



**Fig. 4.** (colour online) Comparison between three typical analytical results $\eta_{CA}$, $\eta'_{SS}$, and $\eta_T$ on the EMP problem.[18]

The above three typical results ($\eta_{CA}$, $\eta'_{SS}$, $\eta_T$) can be expanded into Taylor series with respect to $\eta_C$ as

$$\eta_{CA}, \eta'_{SS}, \eta_T = \frac{\eta_C}{2} + \frac{\eta_C^2}{8} + O(\eta_C^3). \tag{31}$$

The results are the same up to the second order term of $\eta_C$, so we propose a conjecture that a universal efficiency at maximum power, $\eta_C/2+\eta_C^2/8$, should exist at small relative temperature differences.[18] The work on the thermoelectric efficiency at maximum power in a quantum dot by Esposito *et al.* makes this conjecture more clear.[19] It is of some urgency to find the conditions under which this conjecture holds.

## 5. Universality of EMP

Van den Broeck, Esposito, and Lindenberg contributed key excellent investigations to the universality of the EMP. First, Van den Broeck verified that $\eta_U = \eta_C/2$ up to the first order term for small relative temperature differences.[14] Next, Esposito, Lindenberg, and Van den Broeck found the conditions to validate the conjecture on universality.[20] Their key idea will be presented in this section.

### 5.1. Model

A conceptional engine constructed by Esposito *et al.* is schematically shown in Fig. 5. The temperature and the chemical potential of the left (right) reservoir are denoted as $T_L$ ($T_R$) and $\mu_L$ ($\mu_R$), respectively. The heat engine of interest uses the temperature difference to drive particles to move uphill against the chemical potential. Without losing generality, we assume $T_L < T_R$ and $\mu_L > \mu_R$. The matter flux and the energy flux are denoted as $I_M$ and $I_E$, respectively.

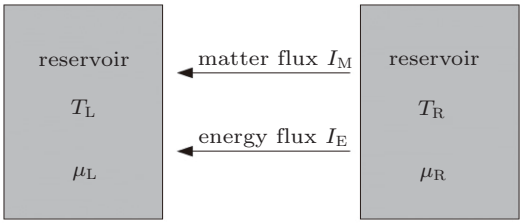



**Fig. 5.** Conceptional engine constructed by Esposito, Lindenberg, and Van den Broeck.

In the nonequilibrium steady state, the irreversible entropy production rate can be phenomenologically expressed as

$$\sigma = F_M I_M + F_E I_E, \tag{32}$$

where the thermodynamic forces can be expressed as

$$F_M = \left(-\frac{\mu_L}{T_L}\right) - \left(-\frac{\mu_R}{T_R}\right),$$
$$F_E = \frac{1}{T_L} - \frac{1}{T_R}. \tag{33}$$

Within this framework, the power of the engine and the rate of the heat absorbed from the hot reservoir can be expressed as[20]

$$P = (\mu_L - \mu_R) I_M, \tag{34}$$

and

$$\dot{Q} = I_E - \mu_R I_M, \tag{35}$$

respectively. Therefore, the efficiency can be defined as

$$\eta = \frac{P}{\dot{Q}} = \frac{\mu_L - \mu_R}{I_E/I_M - \mu_R}. \tag{36}$$

### 5.2. Basic assumptions

It is found that the following two conditions are sufficient to validate the conjecture on the universality of the EMP.[20]

1) ***Strong coupling assumption***

The energy flux is proportional to the matter flux, that is, $I_E = \epsilon I_M \equiv \epsilon I$.

2) ***Left–right symmetry sssumption***

The flux is inverted when the left and the right reservoirs are exchanged.

Now let us discuss the direct consequences of the above assumptions. If we introduce dimensionless parameters $x_{\rm L} = (\epsilon - \mu_{\rm L})/T_{\rm L}$ and $x_{\rm R} = (\epsilon - \mu_{\rm R})/T_{\rm R}$, the irreversible entropy production rate can be simplified to $\sigma = FI$ with $F = x_{\rm L} - x_{\rm R}$ and $I = I(x_{\rm R}, x_{\rm L}) = I(x_{\rm L} - F, x_{\rm L})$. In the equilibrium state, $F = 0$, and the flux should also be zero, which implies

$$I = \chi F + \psi F^2 + O(F^3), \tag{37}$$

where $\chi = -I_1(x_{\rm L}, x_{\rm L})$, $\psi = (1/2)I_{11}(x_{\rm L}, x_{\rm L})$, and subscripts 1 and 11 represent the first and the second order partial derivatives with respect to the first independent variable, respectively.

On the other hand, the assumption of left–right symmetry suggests $I(x_{\rm R}, x_{\rm L}) = -I(x_{\rm L}, x_{\rm R})$, from which we have $I_{12}(x_{\rm L}, x_{\rm L}) = -I_{12}(x_{\rm L}, x_{\rm L})$, i.e., $I_{12}(x_{\rm L}, x_{\rm L}) = 0$, where subscript 1 represents the second order partial derivative with respect to the first and the second independent variables. Then $\chi = -I_1(x_{\rm L}, x_{\rm L})$ leads to $\chi' \equiv {\rm d}\chi/{\rm d}x_{\rm L} = -I_{11}(x_{\rm L}, x_{\rm L}) - I_{12}(x_{\rm L}, x_{\rm L}) = -I_{11}(x_{\rm L}, x_{\rm L})$. Thus we obtain

$$\chi' = -2\psi. \tag{38}$$

### 5.3. Optimization

Substituting the expressions of $x_{\rm L}$ and $x_{\rm R}$ into the expressions of power and efficiency, we have

$$P = T_{\rm R}(\eta_{\rm C} x_{\rm L} - F) I(x_{\rm L} - F, x_{\rm L}), \tag{39}$$

and

$$\eta = 1 - (1 - \eta_{\rm C}) x_{\rm L}/(x_{\rm L} - F). \tag{40}$$

Then optimizing $P$, we obtain

$$I + (F - x_l \eta_{\rm C})(\partial I/\partial F) = 0, \tag{41}$$

and

$$-\eta_{\rm C} I + (F - x_{\rm L}\eta_{\rm C})(\partial I/\partial x_{\rm L}) = 0. \tag{42}$$

Expanding $F$ into the Taylor series with respect to $\eta_{\rm C}$, we assume

$$F = \phi_0 + \phi_1 \eta_{\rm C} + \phi_2 \eta_{\rm C}^2 + O(\eta_{\rm C}^3). \tag{43}$$

Substituting the above equation into Eqs. (41) and (42), we obtain $\phi_0 = 0$, $\phi_1 = x_{\rm L}/2$, $\phi_2 = \psi x_{\rm L}^2/8\chi$, and $x_{\rm L} = -2\chi/\chi'$. Substituting them into Eq. (40), we obtain the EMP of the engine

$$\eta_{\rm U} = \frac{\eta_{\rm C}}{2} + \left(1 + \frac{\psi}{\chi'}\right)\frac{\eta_{\rm C}^2}{4}, \tag{44}$$

up to the second order term of $\eta_{\rm C}$. Finally, by combining with Eq. (38), the above equation is transformed into

$$\eta_{\rm U} = \frac{\eta_{\rm C}}{2} + \frac{\eta_{\rm C}^2}{8} \tag{45}$$

for small relative temperature differences, that is, the strong coupling and the left–right symmetry assumptions ensure the validation of the conjecture on the universality of the EMP. In fact, it should not be difficult to prove that these two assumptions are also the necessary conditions to validate the conjecture.

## 6. EMP and its bounds for low-dissipation Carnot-like engines

Although universality exists for the EMP problem for small relative temperature differences, the behaviors of different kinds of engines deviate from each other at large relative temperature differences. Are there bounds of EMP for heat engines? Esposito *et al.* investigated the Carnot-like engines working in the low-dissipation region and obtained the lower and the upper bounds of EMP.[21] Their model is similar to the one mentioned in Section 2. So equations (2)–(5) still hold. The main difference is that here the effective temperatures need not to be constants in the isothermal processes. Thus we add quotation marks to the word "isothermal" to express the system in contact with a reservoir at constant temperature. The "isothermal" process may not be truely isothermal.

### 6.1. Basic assumptions

Two key assumptions are listed as follows.

a) ***Assumption on total time***

The time for completing the adiabatic processes is much less than that for completing the "isothermal" processes. Thus the total time $t_{\rm tot} = (t_1 + t_3)$.

b) ***Low-dissipation assumption***

The heat engines work in the low-dissipation region so that the minimum irreversible entropy production in each "isothermal" process is inversely proportional to the time required for completing that process. That is, $\min\{\Delta S_l^{\rm ir}\} = \Gamma_l/t_l$, $(l = 1, 3)$, where $\Gamma_l > 0$

is the proportional coefficient. It is necessary to point out that here we correct the presentation of Esposito *et al.,* who assumed that the irreversible entropy production was inversely proportional to the time required for completing the "isothermal" process. In fact, there might not exist a generic relationship between the irreversible entropy production and the time required for completing the "isothermal" process, because the former depends on the protocol. Only the "minimum" irreversible entropy production and the time for completing the "isothermal" process exhibit generic behavior.

### 6.2. Lower and upper bounds of EMP

It is easy to see that the low-dissipation assumption is equivalent to Eq. (17) in the discussion of Schmeild and Seifert. Therefore it is straightforward to transplant their discussion here and derive the EMP for low-dissipation Carnot-like engines

$$\eta_{\rm E} = \frac{\eta_{\rm C}}{2-\gamma_{\rm E}\eta_{\rm C}}, \tag{46}$$

where $\gamma_{\rm E} = 1/(1+\sqrt{T_{\rm c}\Gamma_3/T_{\rm h}\Gamma_1})$.[21] For symmetric proportional coefficients, $\Gamma_1 = \Gamma_3$, the above equation degenerates into the Curzon–Ahlborn efficiency (1).

On the other hand, it is easy to prove $0 < \gamma_{\rm E} < 1$, because $T_{\rm c}$, $T_{\rm h}$, $\Gamma_1$, and $\Gamma_3$ are positive. Then we have

$$\frac{\eta_{\rm C}}{2} \equiv \eta_- < \eta_{\rm E} < \eta_+ \equiv \frac{\eta_{\rm C}}{2-\eta_{\rm C}}, \tag{47}$$

that is, the EMP is bounded between lower bound $\eta_- \equiv \eta_{\rm C}/2$ and upper bound $\eta_+ \equiv \eta_{\rm C}/(2-\eta_{\rm C})$. Of course, the low-dissipation assumption is the sufficient condition for the existence of the lower and the upper bounds. Is it also necessary? This question will be discussed in the next section.

## 7. EMP problem viewed from minimum irreversible entropy production

The low-dissipation assumption is reasonable in the long-time limit, because the "isothermal" process is quasistatic in the long-time limit, and the irreversible entropy production should be vanishing. Thus the simplest estimate is $\Delta S_l^{\rm ir} \propto 1/t_l$, $(l = 1, 3)$. However, it must be convergent in the short-time limit. Thus it is necessary to correct the assumption for finite time. Wang and the present author overcome this shortage in a recent work,[29] which will be briefly discussed in this section.

### 7.1. Basic assumptions

We adopt a similar model to the one mentioned in Section 2 but without any precondition on the effective temperature. Additionally, two key assumptions are as follows.

1 ***Assumption on total time***

The total time is proportional to the time required for completing the two "isothermal" processes, i.e., $t_{\rm tot} = \xi(t_1 + t_3)$. We take $\xi = 1$ if the time required for completing the adiabatic processes is much less than that for the "isothermal" processes.

2 ***Quadratic-form assumption***

There is no heat leakage between the hot and the cold reservoirs. The irreversible entropy is merely produced in the two "isothermal" processes due to the heat transfers between the substance and its surrounding reservoirs. The rate of irreversible entropy production in each "isothermal" process has a quadratic form for the heat exchange rate between the working substance and the reservoir, which can be expressed as

$$\Delta S_l^{\rm ir} = \kappa_l^{-1}\int_0^{t_l}[q_l(\tau)]^2{\rm d}\tau, \quad (l = 1, 3), \tag{48}$$

where $\kappa_l$ can be regarded as the thermal conductivity, and $q_l(\tau)$ represents the heat transfer rate.

### 7.2. Minimum irreversible entropy production

Now we discuss the minimum irreversible entropy production for a given time. Minimizing $\Delta S_l^{\rm ir}$ under the constraint $\int_0^{t_l}[q_l(\tau)]{\rm d}\tau = Q_l$, we obtain

$$q_l = Q_l/t_l, \quad (l = 1, 3), \tag{49}$$

and

$$\min\{\Delta S_l^{\rm ir}\} = \frac{Q_l^2}{k_l t_l}, \quad (l = 1, 3), \tag{50}$$

during time interval $t_l$.

It is necessary to point out that equation (50) does not mean that the low-dissipation assumption holds ($\min\{\Delta S_l^{\rm ir}\} \propto t_l$). In fact, combining this equation and Eqs. (2)–(4), we can derive[29]

$$\min\{\Delta S_1^{\rm ir}\} = \frac{\kappa_1 t_1}{2T_1^2}\left(1+\frac{2\Delta S T_1^2}{\kappa_1 t_1} - \sqrt{1+\frac{4\Delta S T_1^2}{\kappa_1 t_1}}\right), \tag{51}$$

and

$$\min\{\Delta S_3^{\rm ir}\} = \frac{\kappa_3 t_3}{2T_3^2}\left(1 - \frac{2\Delta S T_3^2}{\kappa_3 t_3} - \sqrt{1 - \frac{4\Delta S T_3^2}{\kappa_3 t_3}}\right). \quad (52)$$

It is easy to verify

$$\min\{\Delta S_l^{\rm ir}\} = \begin{cases} (T_l \Delta S)^2/\kappa_l t_l, & (t_l \to \infty), \\ \Delta S, & (t_l \to 0), \end{cases} \quad (53)$$

which suggests that the low-dissipation assumption holds only for our model in the long-time limit.

### 7.3. EMP and its bounds

Now we investigate the EMP problem under the assumption on total time and the quadratic-form assumption. Similar to the analysis in Section 3, we can derive

$$P = \frac{(T_1 - T_3)\Delta S - (T_1 \Delta S_1^{\rm ir} + T_3 \Delta S_3^{\rm ir})}{\xi(t_1 + t_3)}, \quad (54)$$

from which we can see that $\max\{P\}$ is equivalent to $\min\{\Delta S_l^{\rm ir}\}$ for given $t_l$ $(l = 1, 3)$. Then substituting Eqs. (51) and (52) into the above equation and maximizing the power with respect to $t_1$ and $t_3$, we can obtain the optimized ratio

$$\frac{Q_3^{\rm opt}}{Q_1^{\rm opt}} = 1 - \frac{(1 + \sqrt{\kappa_1/\kappa_3})\eta_{\rm C}}{2 + (2 - \eta_{\rm C})\sqrt{\kappa_1/\kappa_3}}. \quad (55)$$

Then we find the EMP to be

$$\eta_{\max P} = 1 - \frac{Q_3^{\rm opt}}{Q_1^{\rm opt}} = \frac{\eta_{\rm C}}{2 - \gamma\eta_{\rm C}} \quad (56)$$

with $\gamma = 1/(1 + \sqrt{\kappa_3/\kappa_1})$. Since $\kappa_1$ and $\kappa_3$ are positive, it is easy to see that $\eta_{\max P}$ is bounded between lower bound $\eta_- \equiv \eta_{\rm C}/2$ and upper bound $\eta_+ \equiv \eta_{\rm C}/(2 - \eta_{\rm C})$. As shown in Fig. 6, several theoretical expressions of the EMP for different kinds of heat engines are indeed located in the bounded region between $\eta_-$ and $\eta_+$.

It is very interesting that the above result has a similar form to those derived by Esposito *et al.* and Chen *et al.*, although we adopt different assumptions. Equation (53) implies that the quadratic-form assumption includes the low-dissipation assumption in the long-time limit, so it is not surprising that equations (56) and (46) have a similar form. A natural question that arises is whether the quadratic-form assumption also includes the endoreversible assumption. Indeed, equation (49) implies that the effective temperatures $T_{\rm he}$ and $T_{\rm ce}$ in the two "isothermal" processes satisfy

$$\kappa_1\left(\frac{1}{T_{\rm he}} - \frac{1}{T_{\rm h}}\right) = \frac{Q_1}{t_1}, \quad (57)$$

and

$$\kappa_3\left(\frac{1}{T_{\rm c}} - \frac{1}{T_{\rm ce}}\right) = \frac{Q_3}{t_3}. \quad (58)$$

Considering Eqs. (2)–(4) and (50), we have $Q_1/T_{\rm he} - Q_3/T_{\rm ce} = 0$, which implies that the heat engines working at maximum power happen to be endoreversible. This is the reason why equations (56) and (13) have the same form.

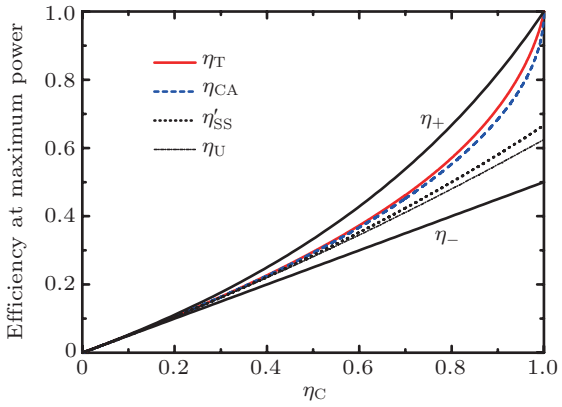



**Fig. 6.** (colour online) Bounds and several typical expressions of EMP for different kinds of heat engines.

## 8. Discussion and prospective

We briefly review the recent advance on the EMP of heat engines. We show the key procedures to derive several typical efficiencies at maximum power, such as $\eta_{\rm CA}$, $\eta_{\rm CY}$, $\eta_{\rm SS}$, $\eta_{\rm T}$, and $\eta_{\rm E}$. The culmination of the EMP problem is to find the existence of the lower and the upper bounds of the EMP. We finally realize that the crucial issue of the EMP problem is the minimum irreversible entropy production in each finite-time "isothermal" process. Although much progress has been achieved, there are still several open questions which need further discussion.

(A) ***Relationship between the minimum irreversible entropy production and the time for completing the "isothermal" process***

To instruct the irreversible entropy production functional is the essential step in nonequilibrium thermodynamics. Although an analytical expression of the minimum irreversible entropy production is derived

under the quadratic-form assumption, it is still necessary to check whether the minimum irreversible entropy production has similar behavior for the stochastic heat engines or Feynman's ratchet.

(B) ***Universality of bounds for the EMP problem***

The discussion in Sections 6 and 7, in particular Fig. 6, suggests that the bounds for the EMP problem might hold under a quite general condition. To what extent are these bounds universal? Recent work by Wang and the present author gives a preliminary answer.[39] However we cannot fully answer this question unless the character of the minimum irreversible entropy production mentioned in question (A) is deeply understood.

(C) ***Possibility to transplant these advances to biomolecular motors***

Molecular motors in biological systems operate in a single thermal bath at physiological temperatures. It is interesting to investigate the EMP of biomolecular motors. There are a few preliminary discussions in earlier literature.[40−42] We will face two difficulties in solving the EMP problem of biomolecular motors: one is controversial definitions[43−45] of the efficiency of biomolecular motors; the other is still the unknown form of the irreversible entropy production functional.

The above problems are involved in the core issue of nonequilibrium thermodynamics, the irreversible entropy production functional or the dissipation functional, which is still a big challenge. We believe that solutions to them will greatly promote the development of nonequilibrium thermodynamics.

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