Application of local polynomial estimation in suppressing strong chaotic noise*

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In this paper, we propose a new method that combines chaotic series phase space reconstruction and local polynomial estimation to solve the problem of suppressing strong chaotic noise. First, chaotic noise time series are reconstructed to obtain multivariate time series according to Takens delay embedding theorem. Then the chaotic noise is estimated accurately using local polynomial estimation method. After chaotic noise is separated from observation signal, we can get the estimation of the useful signal. This local polynomial estimation method can combine the advantages of local and global law. Finally, it makes the estimation more exactly and we can calculate the formula of mean square error theoretically. The simulation results show that the method is effective for the suppression of strong chaotic noise when the signal to interference ratio is low.

Keywords: strong chaotic noise, local polynomial estimation, weak signal detection

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1. Introduction

Chaos exists widely in nature and engineering technology. It is a sort of complex phenomenon and often hides the useful signal in the form of noise, as in biomedical, socio-economic and communication signal processing. The problem of chaotic noise has been analysed in some papers , but they only considered the situation of white noise.[1−3] In this article we establish a model as follows: $y(t) = s(t)+I(t)+n(t), t \in R$, where $s(t)$ is useful and weak cosine signal, $I(t)$ is strong chaotic noise, $n(t)$ is white noise. In actual problems, we expect to acquire the useful weak signal when chaotic noise is subtracted from the total signal. Suppression of chaotic noise algorithms have been investigated in recent years. However, the suggested solutions seem to be not efficient.[4,5]

In this paper, the discrete $y_n, n = 1, 2, \ldots, M$ is considered as a bivariate time sequence (t_n, y_n) $(n = 1, 2, \ldots, M)$, where the noise researched is chaotic. Then we express chaotic noise series as multivariate time series using Takens delay embedding theorem.[6−12] So we estimate the chaotic noise in an accurate form by applying local polynomial estimation theory.[13−17] Furthermore, we can accept the estimation of useful weak signal and white noise if noise is subtracted from the observation signal. That is, this chaotic noise is suppressed successfully.

This local polynomial estimation (LPE) method combines the advantages of local and global laws. It is easy to realize because of its small amount of computation and the estimation of higher precision. We calculate the signal to interference ratio (SIR) in the simulation experiment and obtain the results that the SIR is improved greatly. So this method is effective in the suppression of chaotic noise.

This paper is organized as follows. In Section 2, phase space reconstruction model is introduced. In Section 3, the theory of local polynomial fitting is described, including selection of parameters and fast computing. The performance and analysis are given in Section 4. In Section 5, we show the computer simulation results and the analysis. Conclusions are drawn in Section 6.

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2. Phase space reconstruction model

Suppose a time sequence $y(t), t = 1, 2, \ldots, M$ is given. We restructure the phase space as follows:

$$Y = [Y_1, Y_2, \ldots, Y_L], \tag{1}$$

where $Y_i = (y(i), y(i+\tau), \ldots, y(i+(m-1)\tau))^{\mathrm{T}}$, $i = 1, 2, \ldots, L$, where τ and m are the time delays and embedding dimensions respectively, $L = N-(m-1)\tau$. According to Takens delay embedding theorem, there is a continued vector mapping: $f : R^m \to R^m$, that is: $Y_{i+1} = f(Y_i)$. Or there is a continued function: $f : R^m \to R$. That is: $y_{i+1} = f(Y_i)$.

The evolution from Y_i to y_{i+1} reflects the original unknown system's movement. That is, the geometric property of the strange attractors in the restructured phase space is equivalent to the state of the original system. So the differential and topology of the restructured strange attractors are equivalent to those of the original system.

3. The theory of local polynomial fitting

3.1. Local polynomial fitting

Local polynomial fitting is a kind of parameter technology used in many fields. It has fine statistic features. Using local polynomial technology, we can estimate $m^v(x)$ and return function $m(x) = m^0(x)$ conveniently. The data points far away from x_0 provide less information for $m(x_0)$. So we can only use the local data points near x_0.

Suppose $m(x)$ has a $p+1$ order derivative near x_0. For x near x_0, we expand $m(x)$ as follows according to Taylor's formula

$$\begin{aligned} m(x) = {} & m(x_0) + m'(x_0)(x - x_0) \\ & + \frac{m''(x_0)}{2!}(x - x_0)^2 + \cdots \\ & + \frac{m^{(p)}(x_0)}{p!}(x - x_0)^p \\ & + O\{(x - x_0)^{p+1}\}. \end{aligned} \tag{2}$$

When it is used in statistics modeling, for the local points x near x_0, we establish a model for $m(x)$ as follows:

$$m(x) \approx \sum_{j=0}^{p} \beta_j (x - x_0)^j. \tag{3}$$

Parameter $\{\beta_j\}$ depends on x_0, so it is called local parameter. It is obvious that

$$\beta_v = m^{(v)}(x_0)/v!. \tag{4}$$

Let

$$X = \begin{pmatrix} 1 & (X_1 - x_0) & \ldots & (X_1 - x_0)^P \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (X_T - x_0) & \ldots & (X_T - x_0)^P \end{pmatrix},$$
$$y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_T \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_0 \\ \vdots \\ \beta_P \end{pmatrix}. \tag{5}$$

Then the problem of weighted least squares estimation can be written as follows:

$$\min (y - X\beta)^{\mathrm{T}} W (y - X\beta), \tag{6}$$

where $\beta = (\beta_0, \ldots, \beta_p)^{\mathrm{T}}$. The i-th element of W is $K_h(X_i - x_0)$ and its solution vector is

$$\hat{\beta} = (X^{\mathrm{T}} W X)^{-1} X^{\mathrm{T}} W y. \tag{7}$$

Furthermore, we consider the united vector $e_{v+1} = (0, \ldots, 0, 1, 0, \ldots, 0)$, with 1 on the $(v+1)$-th position, then the estimator for $m(x)$ is $\hat{m}(x) = e_1^{\mathrm{T}} \hat{\beta} = \hat{\beta}_0$.

3.2. Selection of parameters

To implement the local polynomial estimator, we need to choose the order p, the bandwidth h, and the kernel K. These parameters are of course related to each other. Local polynomial fitting becomes the global local polynomial fitting when $h = \infty$ and order p determines the complexity of the model. The complexity of local polynomial fitting is determined by the bandwidth. This is different from the parameter model. So p is usually small and it is not very important to choose p.

The choice of the bandwidth h plays an important part in polynomial fitting. A too large bandwidth causes excessive smooth and this can produce excessive bias of modeling. While a too small bandwidth causes inadequate smoothing so that we can only have the estimation disturbed. The basic idea is to find a bandwidth h that minimizes the estimated mean integrated square error (MISE):

$$\begin{aligned} h_{\mathrm{opt}} = {} & \arg\min \int \{(\mathrm{bias}(\hat{m}(x)|X))^2 \\ & + \mathrm{var}(\hat{m}(x)|X)\} \mathrm{d}x. \end{aligned} \tag{8}$$

Here, we give a practical procedure for selecting the bandwidth parameter. Suppose we want to minimize the MISE over an interval $[h_1, h_2]$. The approximate minimum of MISE can be found as follows. Start from $h = h_1$, keep inflating h by a factor C, and compute MISE at the geometric grid points. When $h > h_2$ we stop, and the minimization of MISE is chosen at the geometric grid point having the smallest computed MISE value.

Another issue in local polynomial fitting is how to choose the order p of the local polynomial. Since the modeling bias is primarily controlled by the bandwidth, the order p is usually small, and the issue of choosing p becomes less critical. For the choice of the core function K, p, and v, the optimal weighted function is $K(z) = 3(1-z^2)_+/4$, which is called Epanechnikov. It is a universal weighted kernel function.

3.3. Fast computing

Due to the high pulse rate, any suppression algorithms should be very simple, while it can be shown that the method of local polynomial fitting can fit in this requirement. Supposing the sampled rate is same, and the kernel is symmetric, we estimate interior point r_n using the local bivariate data $\{(t_{n-s}, y_{n-s}), \ldots, (t_n, y_n), \ldots, (t_{n+s}, y_{n+s})\}$, where $s = \max\{a : a > 0, K_h(a) > 0\}$. Therefore, the problem (6) is equivalence to the problem as follows:

$$L = \sum_{i=n-s}^{i=n+s} \left\{ y_n - \sum_{j=0}^{p} \beta_j (t_n - t_0)^j \right\}^2 K_h(t_n - t_0). \tag{9}$$

Then the solution of problem of Eq. (7) is $\hat{\beta} = ({X_n}^{\mathrm{T}} W_n X_n)^{-1} {X_n}^{\mathrm{T}} W_n y_n$, where

$$X_n = \begin{pmatrix} 1 & (t_{n-s} - t_n) & \ldots & (t_{n-s} - t_n)^p \\ \vdots & \vdots & \ldots & \vdots \\ 1 & (t_{n+s} - t_n) & \ldots & (t_{n+s} - t_n)^p \end{pmatrix},$$

$$y_n = \begin{pmatrix} Y_{n-s} \\ \vdots \\ Y_n \\ \vdots \\ Y_{n+s} \end{pmatrix}. \tag{10}$$

$W_n = \mathrm{diag}(K_h(t_{n-s+i} - t_n))$, $i = 0, 1, \ldots, 2s$, $Y_n = (y_{n-s}, \ldots, y_{n-1}, y_n, y_{n+1}, \ldots, y_{n+s})^{\mathrm{T}}$, and the kernel $K_n = (X_n^{\mathrm{T}} W_n X_n)^{-1} X_n^{\mathrm{T}} W_n$, the estimator is $\hat{m}(x) = \hat{m}^0(x) = \hat{\beta}_0$.

This method also fits in the estimator of the neighbourhood of left or right boundary, which just uses the data $\{y_1, \ldots, y_{n-1}, y_n, y_{n+1}, \ldots, y_{n+s}\}$ to estimate y_n, where $1 \le n < s$, and it also uses the data $\{y_{n-s}, \ldots, y_{n-1}, y_n, y_{n+1}, \ldots, y_M\}$ to estimate y_n, where $M - s < n \le M$.

4. Performance and analysis

The performance of every digital communication system is measured by the system's bit error rate (BER). Usually, computing the system's BER is a very hard task, moreover, in many systems the BER is dominated by rare event which makes it useless for describing the typical system performance. Various performance measures have been suggested over the years, among them the average signal to interference ratio, outage probability, and asymptotic relative efficiency are considered. These measures are similar to taking computation and proof for almost all the information on the system performance. The average SIR, denoted by $\overline{\mathrm{SIR}}$, can be used to estimate the system BER. Thus, we demonstrate the effect of the estimator by the average ratio of SIR at the output of filters to the average SIR at the input, which reduces to

$$\overline{\mathrm{SIR}} = \frac{E\left\{|y_n - s_n|^2\right\}}{E\left\{|\varepsilon_n - s_n|^2\right\}}, \tag{11}$$

where $\varepsilon_n = y_n - \hat{y}_n$, $\hat{y}_n$ is the estimated value at time n. Denote $S = (\mu_{i+j-2})_{(p+1)\times(p+1)}$, $S^* = (v_{i+j-2})_{(p+1)\times(p+1)}$, $c_p = (\mu_{p+2-j})_{(p+1)\times 1}$, where $\mu_i = \int_{-\infty}^{+\infty} t^i K(t)\mathrm{d}t$, $v_i = \int_{-\infty}^{+\infty} t^i K^2(t)\mathrm{d}t$. Then the $\overline{\mathrm{SIR}}$ of the local polynomial fitting is given by the following theorem.[15,16]

Condition 1 Suppose $X_1, \ldots, X_t$ is a stationary sequence. Let F_i^k be the σ-algebra of events generated by the random variables $\{X_j, i \le j \le k\}$, and let $L^2(F_i^k)$ consist of F_i^k-measure random variables with finite second moment.

(i) The kernel $K(\cdot)$ is bounded with a bounded support.

(ii) $\sigma^2(\cdot)$ and $f(\cdot)$ are continuous at the point x and $f(x) > 0$.

(iii) The sequence is ρ-mixing processes, i.e., $\rho(n) = \sup|\mathrm{corr}(X, Y)| \to 0$, as $T \to \infty$. We also assume that $\sum_l \rho(l) < \infty$, $EY_0^2 < \infty$.

(iv) For ρ-mixing processes, there exists a sequence of positive integers satisfying as $T \to \infty$ and $S_T = \circ(\sqrt{Th})$, such that $\sqrt{T/h}\rho(S_T) \to 0$ as $T \to \infty$.

Theorem 1 Under the Condition 1, if $h = \mathrm{O}(T^{1/(2p+1)})$ and $I^{(p+1)}(t)$ is continuous at $[0, T]$, then as $T \to \infty$,

$$\overline{\mathrm{SIR}} = T({\sigma_1}^2 + {\sigma^2}_N)/\int_0^{\mathrm{T}} M(t)\,\mathrm{d}t, \tag{12}$$

where

$$\begin{aligned} M(t) = & \left[e_1^{\mathrm{T}} S^{-1} c_p \frac{I^{(p+1)}(t)}{(p+1)!} h^{p+1-v}\right]^2 \\ & + e_1^{\mathrm{T}} S^{-1} S^* S^{-1} e_1 \frac{\sigma_I^2}{Thf(t)} \\ & + \left(1 - 2s^{11} K_h(0)\right)\sigma_N^2, \end{aligned} \tag{13}$$

where h is a bandwidth, and s^{11} is the $(1,1)$-element of $(X^{\mathrm{T}}WX)^{-1}$. σ_I^2 and σ_N^2 are the variances of the $I(t)$ and $n(t)$ respectively. Here, we give the approximate variance and bias of local polynomial estimators

$$\begin{aligned} \mathrm{var}\left(\hat{\beta}\right) &= \mathrm{var}\left(\left(X^{\mathrm{T}}WX\right)^{-1} X^{\mathrm{T}} WR\right) \\ &= \left(X^{\mathrm{T}}WX\right)^{-1} X^{\mathrm{T}} W\left[\mathrm{var}(R)\right] \\ &\quad \times WX\left(X^{\mathrm{T}}WX\right)^{-1}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} \mathrm{var}(R) &= (R_{ij})_{(s+t+1)\times(s+t+1)}, \\ R_{ij} &= \mathrm{cov}(r_{n-s+i}, r_{n-s+j})i, \\ j &= 0, 1, \ldots, s+t. \end{aligned} \tag{15}$$

We know that $s(t)$, $I(t)$, and $n(t)$ are independent of each other. Let us assume that $\{(t_i, r_i)\}$ are an independent sample from the population. Then we have

$$\begin{aligned} R_{ij} &= \mathrm{cov}(r_{n-s+i}, r_{n-s+j}) \\ &= \mathrm{cov}(s_{n-s+i} + I_{n-s+i} + n_{n-s+i}, s_{n-s+i} \\ &\quad + I_{n-s+i} + n_{n-s+i}) \\ &= (\sigma_S^2 + \sigma_N^2)\delta_{ij}, \end{aligned} \tag{16}$$

where $\delta_{ij} = 0$, if $i \neq j$, and 1, otherwise. σ_S^2 and σ_N^2 are the variances of $s(t)$ and $n(t)$. Denote $\sigma = \sigma_S^2 + \sigma_N^2$. Then it yields

$$\mathrm{var}(\hat{\beta}) = \sigma^2 (X^{\mathrm{T}}WX)^{-1} X^{\mathrm{T}} W^2 X (X^{\mathrm{T}}WX)^{-1}, \tag{17}$$

where σ^2 can be estimated by $\hat{\sigma}^2 = \sum_{i=1}^{M} (r_i - \hat{r}_i)^2/M$. Then we have

$$\begin{aligned} \mathrm{var}(\hat{r}_n) &= \hat{\sigma}^2 e_1^{\mathrm{T}} (X^{\mathrm{T}}WX)^{-1} X^{\mathrm{T}} \\ &\quad \times W^2 X (X^{\mathrm{T}}WX)^{-1} e_1. \end{aligned} \tag{18}$$

The bias of the local polynomial estimator for an independent sample is

$$\mathrm{bais}(t_n) = (X^{\mathrm{T}}WX)^{-1} X^{\mathrm{T}} Wq, \tag{19}$$

where $q = m - T'\beta_p$, $\beta_p = (m(x), \ldots, m^{(p)}(x)/p!)$. By Taylor's expansion the i-th element of q is

$$\begin{aligned} q_i &= m(X_i) - \sum_{j=1}^{p} \frac{m^{(j)}(X_0)}{j!}(X_i - x_0)^j \\ &\approx \frac{m^{(p+1)}(X_0)}{(p+1)!}(X_i - x_0)^{p+1} \\ &\quad + \frac{m^{(p+2)}(X_0)}{(p+2)!}(X_i - x_0)^{p+2}. \end{aligned} \tag{20}$$

This gives an estimator of q and estimated bias of the received signal at time n

$$\mathrm{bais}(\hat{r}_n) = e_1^{\mathrm{T}} (X^{\mathrm{T}}WX)^{-1} X^{\mathrm{T}} W\hat{q}. \tag{21}$$

Under Condition 1, we have the consistent estimates for asymptotic bias and asymptotic variance. Then we obtain the approximate MISE of $\hat{y}_n$ as

$$\begin{aligned} \mathrm{MISE}(\hat{r}_n) &= [\,\mathrm{bais}(\hat{r}_n)\,]^2 + \mathrm{var}(\hat{r}_n) \\ &= e_1^{\mathrm{T}} (X^{\mathrm{T}}WX)^{-1} X^{\mathrm{T}} W(\hat{q}\hat{q}^{\mathrm{T}} + \hat{\sigma}^2 I) \\ &\quad \times WX(X^{\mathrm{T}}WX)^{-1} e_1. \end{aligned} \tag{22}$$

5. Computer simulation results and analyses

A computer program and its subroutines by Matlab7.1 are compiled. First we give the algorithm flow chart of chaotic noise suppression as shown in Fig. 1.

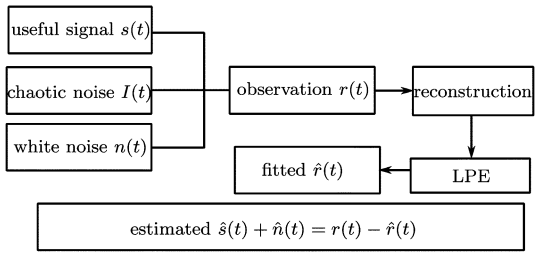


Fig. 1. Algorithm structure diagram of chaotic noise suppression.

In the simulation, we note the weak and useful signal $s(t)$ in the signal $y(t) = s(t) + I(t) + n(t)$ as a cosine signal. And the algorithm structure diagram is plotted in Fig. 1. The strong chaotic noise $I(t)$ is generated by the chaotic model Lorenz as follows:

$$x' = \sigma(y - x),$$

$$y' = -xz + rx - y,$$
$$z' = xy - bz, \tag{23}$$

where $\sigma = 10, r = 28, b = 8/3$. We only consider the estimation of the component x for the model. Supposing the initial conditions as follows: $x_0 = 5$, $y_0 = 5$, $z_0 = 5$, we solve the Lorenz system using the fourth-order Runge–Kutta algorithm. The chaotic time series on x are generated by time sampling interval $\Delta t = 0.01$. We obtain 3000 points by solving it as shown in Fig. 2.

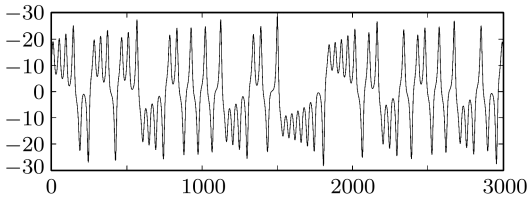


Fig. 2. The X component of Lorenz chaotic time series.

We choose the forward 800 points to put up the experiment in the actual fitting process. Chaotic noise, useful signal, and white noise are shown in Figs. 3–5.

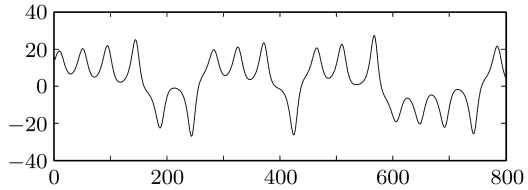


Fig. 3. The former 800 data of X component of Lorenz chaotic time series.

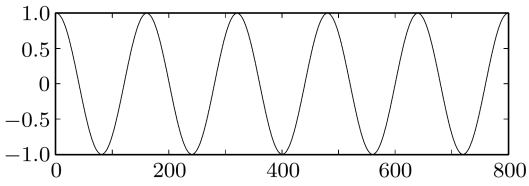


Fig. 4. Cosine signal.

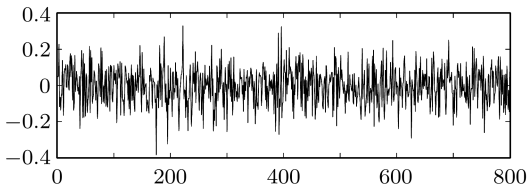


Fig. 5. White noise.

The signal we observed is shown in Fig. 6. It is integrated by the above three signals. The picture shows directly that the signal observed is similar to the signal in Fig. 3 in chaotic characteristic. It is shown below.

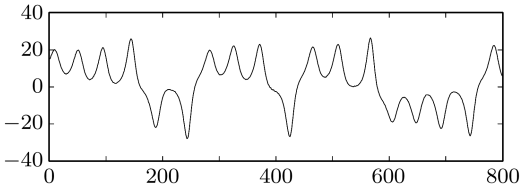


Fig. 6. Observation signal.

It is fitted using the local polynomial method. In the fitting process, we find that the local polynomial method having higher precision is more accurate for small data and smaller relative error. So we use the forward 800 data to simulate in the experiment. In the practical problems, if the quantity of data is large we can operate it one subsection after another. The results we fitted are shown in Fig. 7.

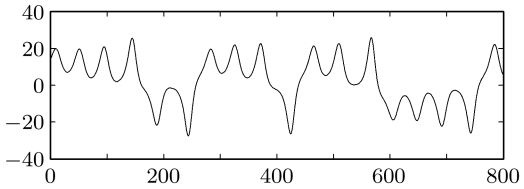


Fig. 7. Fitted signal.

Figure 8 is the estimated cosine signal plot. It is the subtraction between the actual signal observed and the chaotic signal fitted. We calculate the SIR. In the above signal, the SIR= −25.1498 before it is fitted and the SIR = 38.1862 after it is fitted. We can see that the chaotic noise is well restrained.

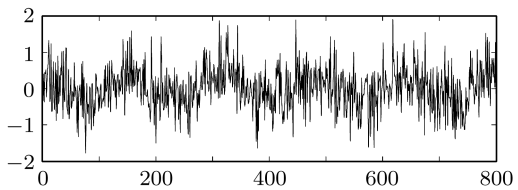


Fig. 8. Estimated cosine signal.

6. Conclusion

Chaotic noise is complex in a practical problem and chaotic noise suppression is not very satisfactory when using traditional methods based on model approximation and conventional smoothing of wavelet transform. To solve this problem, this paper proposes a new scheme based on the local polynomial estimation and the chaotic time series re-construction

method. In the simulation, the SIR= −25.1498 before the estimation and the SIR=38.1862 after fitting. So it is greatly improved. The chaotic noise is predicted accurately with the low SIR. We can see that the new solution combining the local polynomial estimation with the chaotic series re-construction is effective at suppressing chaotic noise.

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