Stochastic synchronization for time-varying complex dynamical networks*

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This paper studies the stochastic synchronization problem for time-varying complex dynamical networks. This model is totally different from some existing network models. Based on the Lyapunov stability theory, inequality techniques, and the properties of the Weiner process, some controllers and adaptive laws are designed to ensure achieving stochastic synchronization of a complex dynamical network model. A sufficient synchronization condition is given to ensure that the proposed network model is mean-square stable. Theoretical analysis and numerical simulation fully verify the main results.

Keywords: stochastic dynamical networks, synchronization, time-varying coupling strength, adaptive control

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1. Introduction

Complex networks[1−6] exist in all fields of science, nature, and human societies. Over the past decade, complex networks have been studied intensively in various disciplines, such as computer networks, the World Wide Web, telephone call graphs, food webs, neural networks, electrical power grids, and citation networks of scientists, among many others, even reputation computation in virtual organizations.[7] A complex network is a large set of interconnected nodes, where the nodes and connections can be anything. A node is a fundamental unit having specific contents and exhibiting dynamical behaviour. There are two types of connection between nodes: directed connection and undirected connection, the connection relationship can be unweighted and weighted. According to the different types of connection and whether there are weights or not between nodes, we obtain some different kinds of complex networks, such as undirected unweighted networks, directed weighted networks, etc. Complex networks can exhibit complicated dynamics which may be absolutely different from those of a single node. Hence, the investigation of complex dynamical networks is of great importance, and many systems in science and technology can be modeled as complex networks.

For coupled complex networks, one prominent phenomenon is the synchrony of all dynamical nodes, which can well demonstrate many natural phenomena. Therefore, synchronization of complex networks of dynamical nodes has received a great deal of attention.[8−12] By using the Lyapunov functional method, or taking the synchronization manifold and linear matrix inequality approach, several sufficient conditions have been derived for ensuring the synchronization of various complex networks.[8,9] Various controllers have been commonly used, such as feedback and delayed feedback controllers,[13,14] nonlinear adaptive feedback controllers,[15−17] and so on. In Ref. [18], the authors proposed a new adaptive law to ensure synchronization of a complex network.

Recently, however, stochastic systems have received increasing attention and stochastic modeling has become to play an important role in many scientific and engineering applications. In fact, signals transmitted between subsystems of complex networks are unavoidably subjected to stochastic perturbations from the environment, which may cause information contained in these signals to be lost.[19] Therefore, transmitted signals may not be fully detected and received by other subsystems. This can have a great influence on the behaviour of complex networks. Therefore, to investigate and simulate more realistic and

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complex networks, the noise's effect should be taken into account in modeling the networks. Many fundamental results about deterministic systems have been extended to stochastic systems. Stability of stochastic delayed systems has been a focal topic for research due to the uncertainties existing in the real-world systems.[14,20−22] In particular, synchronization in coupled stochastic networks has also been carefully studied.[13,23,24] Recently, the authors of Ref. [19] introduced the stochastic coupling term to model linearly coupled neural networks and studied its complete synchronization. However, the stochastic disturbances in Ref. [19] are scalar Weiner processes, which means that every node is influenced by the same one noise. Such an assumption might not be realistic in the true environment of networks because each node is always influenced by different multidimensional noises. In Ref. [25], the authors studied the exponential stochastic synchronization problem for coupled neural networks with stochastic noise perturbations. The authors of Ref. [26] studied adaptive synchronization of uncertain coupled complex networks, and designed some controllers and adaptive laws to ensure achieving synchronization of a general complex network model. However, the authors of Ref. [26] did not consider time-varying complex dynamic networks. Recently, the authors of Ref. [27] studied the finite-time stochastic synchronization problem for complex networks with stochastic noise perturbations. In Ref. [28], the authors studied stochastic synchronization for a new array of coupled discrete time stochastic complex networks with randomly occurring nonlinearities and time delays. In Ref. [29], the pinning exponential synchronization of stochastic delayed complex networks with coupling delays in mean square was investigated.

Motivated by the aforementioned discussions, in this paper, we introduce a new model of complex dynamical networks, which includes time-varying coupling strength and stochastic noise perturbations. It is well known that adaptive control can effectively deal with constant parametric uncertainties for systems. It is still an open problem how to control a system with unknown time-varying parameters. Recently, according to the synchronization problem of complex dynamic networks with time-varying parameters there has been a few research results.[12,17] However, the existing studies show that the time-varying parameters were processed by the bounding technique. To the best of the authors' knowledge, no work has been reported in the literature at present. The main design difficulty is how to deal with the unknown time-varying parameters. Fortunately, this paper shows that this difficulty can be successfully overcome by constructing a Lyapunov–Krasovskii-like composite energy function (CEF). By virtue of the properties of the Weiner process and inequality techniques, suitable adaptive controllers are designed to ensure stochastic synchronization for the complex dynamical networks with stochastic perturbations. The main contributions of this paper are highlighted as follows. 1) The CEF-based adaptive control strategy, developed for nonlinear time-varying systems, is extended to complex dynamical networks. Thus, complex dynamical networks are further enlarged so that a CEF-based adaptive control scheme can be designed for them. As far as we know, until now this extension has not been investigated in the existing literature on adaptive control. 2) From the viewpoint of technical innovation, we introduce a novel approach to handle unknown time-varying parameters in time-varying complex dynamical networks. Some existing studies showed that the system with periodic time-varying parameters is stable in L_T^2 norm. As far as the control design is concerned, our control law ensures that the system is asymptotically stable.

The rest of this paper is organized as follows. In Section 2, a new model of complex dynamical networks with time-varying coupling strength is presented, some necessary assumptions, definitions, and lemmas are also given in this section. Our main results and their rigorous proofs are described in Section 3. In Section 4, an illustrative example is provided to demonstrate the effectiveness of our results. Finally, some conclusions are given in Section 5.

2. Problem formulation

Consider a stochastic complex dynamical network consisting of N coupled identical nodes, with each node being an n-dimensional dynamical system, in the following form:

$$\begin{aligned} \mathrm{d}x_i(t) = & \Big[f(x_i(t)) + \sum_{j=1}^{N} \varphi_i(t) a_{ij} \boldsymbol{\Gamma} x_j(t)\Big]\mathrm{d}t \\ & + g_i(x_1, x_2, \ldots, x_N)\mathrm{d}\omega_i(t), \\ & i = 1, 2, \ldots, N, \end{aligned} \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^{\mathrm{T}} \in R^n$ is the state vector representing the state variable of node i, $f : R^n \times R \to R^n$ is a smooth nonlinear vector-valued function, $\varphi_i(t)$ represents the unknown time-varying coupling strength, $a_{ij} = a_{ji}$ is the coupling

element, if there exists a connection between node i and node j ($j \neq i$), then $a_{ij} = a_{ji} > 0$, else $a_{ij} = a_{ji} = 0$. The inner coupling matrix $\boldsymbol{\Gamma} = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$ is non-negative definite. The constant matrix $\boldsymbol{A} = (a_{ij})_{N\times N}$ describes the linear coupling configuration of the network, which satisfies $a_{ij} \geq 0$, for $i \neq j$, and $a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}, i = 1, 2, \ldots, N$, and $g_i \in C(R^n \times \cdots \times R^n, R^{n\times n})$ is the noise intensity function matrix, $\omega_i(t) = (\omega_{i1}(t), \omega_{i2}(t), \ldots, \omega_{in}(t))^{\mathrm{T}}$ is an n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ with a natural filtration $\{\mathscr{F}_t\}_{t\geq 0}$ generated by $\{\omega_i(s) : 0 \leq s \leq t\}$, where Ω is associated with the canonical space generated by $\omega_i(t)$, and $\mathscr{F}$ is the associated σ-algebra generated by $\{\omega_i(t)\}$ with the probability measure P. The white noise $\mathrm{d}\omega_i(t)$ is assumed to be independent of $\mathrm{d}\omega_j(t)$ for all $i \neq j$. This type of random perturbation can be regarded as a result of the occurrence of uncertainties in the nodes.

In this paper, the synchronization of complex network model (1) is investigated. A solution $s(t)$ of an isolated node satisfies

$$\mathrm{d}s(t) = f(s(t))\mathrm{d}t, \tag{2}$$

where $s(t)$ can be any desired state: equilibrium point, a nontrivial periodic orbit, or even a chaotic orbit.

Next, we investigate the synchronization problem of the stochastic network (1) by adding a simple controller to each node of the complex network (1). The controlled network is described by

$$\begin{aligned}\mathrm{d}x_i(t) = &\Big[f(x_i(t)) + \sum_{j=1}^{N} \varphi_i(t) a_{ij} \boldsymbol{\Gamma} x_j(t) + u_i(t)\Big]\mathrm{d}t \\ &+ g_i(x_1, x_2, \ldots, x_N)\mathrm{d}\omega_i(t),\end{aligned} \tag{3}$$

where $u_i(t) \in R^n$ are the feedback controllers.

Next, the following definition is needed:[30]

Definition 1 Network (1) is said to achieve asymptotic synchronization in the mean square sense if for any given initial $x_i(0)$, the solution of Eq. (1) satisfies

$$\lim_{t\to\infty} E\{||x_i(t) - s(t)||^2\} = 0, \quad i = 1, 2, \ldots, N, \tag{4}$$

where $||\cdot||$ stands for the Euclidean vector norm, $E\{\cdot\}$ is the mathematical expectation.

In order to derive the main results, the following assumptions and lemmas are needed:

Assumption 1 In network (1), suppose that there exists $l_i > 0$, satisfying

$$\begin{aligned}&(x_i(t) - s(t))^{\mathrm{T}}(f(x_i(t)) - f(s(t))) \\ &\leq l_i(x_i(t) - s(t))^{\mathrm{T}}(x_i(t) - s(t)),\end{aligned} \tag{5}$$

where $x_i(t)$ and $s(t)$ are time varying vectors.

Assumption 2 In network (1), time-varying coupling strength $\varphi_i(t)$ satisfies $\varphi_i(t) = \theta_i(t) + \phi_i$, and $\theta_i(t)$ is unknown periodic time-varying parameter, that is $\theta_i(t) = \theta_i(t-T)$. ϕ_i is unknown time-invariant nonnegative parameter.

Remark 1 Since $\varphi_i(t) = \theta_i(t) + \phi_i$, obviously the vector $\theta_i(t)$ is an unknown continuous periodic function with a known period T, i.e., for any time instant $lT \leq t \leq (l+1)T$ where $l \in N$, we have $\theta_i(t) = \theta_i(t-T) = \cdots = \theta_i(t-lT)$, $\theta_i(t)$ also has a finite unknown upper bound θ_M such that $\theta_M \geq \theta_i(t)$, thus $\varphi_i(t)$ is bounded with upper bound φ_M. For convenience, we denote $\theta(t) = [\theta_1(t), \ldots, \theta_N(t)]^{\mathrm{T}}$, $\phi = [\phi_1, \ldots, \phi_N]^{\mathrm{T}}$, obviously the vector $\theta(t)$ is unknown continuous periodic function with a known period T.

Assumption 3 In network (1), the inner coupling matrix $\boldsymbol{\Gamma}$ and coupling matrix $\boldsymbol{A}$ satisfy

$$||\boldsymbol{\Gamma}|| = \gamma, \tag{6}$$

$$|a_{ij}| \leq a, \forall i, j = 1, 2, \ldots, N, \tag{7}$$

where γ and a are positive constants.

Assumption 4 There exist nonnegative constants $\rho_{ij}, i, j = 1, 2, \ldots, N$, such that

$$\begin{aligned}&\mathrm{tr}[g_i(x_1, \ldots, x_N) - g_i(s, \ldots, s)]^{\mathrm{T}} \\ &\times [g_i(x_1, \ldots, x_N) - g_i(s, \ldots, s)] \\ &\leq \sum_{j=1}^{N} \rho_{ij} ||x_j(t) - s(t)||^2.\end{aligned} \tag{8}$$

Lemma 1 (Young's inequality) For any vectors $x, y \in R^n$, and any constant $c > 0$, the following matrix inequality holds:

$$x^{\mathrm{T}}y \leq cx^{\mathrm{T}}x + \frac{1}{4c}y^{\mathrm{T}}y. \tag{9}$$

Lemma 2[31] Consider system $\mathrm{d}x = f(x)\mathrm{d}t + g(x)\mathrm{d}\omega$ and suppose that there exist a C^2 function $V : R^n \to R_+$ and class $\mathscr{K}$ functions α_1, α_2, and α_3 such that

$$\begin{aligned}&\alpha_1(||x||) \leq V(x,t) \leq \alpha_2(||x||), \quad \text{and} \\ &\mathscr{L}V(t) \leq -\alpha_3(||x||),\end{aligned} \tag{10}$$

then the equilibrium $x = 0$ is stochastically asymptotically stable. Especially, if α_1, α_2, and α_3 are class $\mathscr{K}_\infty$, then, the equilibrium $x = 0$ is globally stochastically asymptotically stable.

In this paper, $\theta_i(t)$ is an unknown periodic time-varying parameter, the objective is to design an adaptive controller under which the solutions of network

(1) asymptotically synchronize in the mean square sense. If this is satisfied, then the solution $s(t)$ of Eq. (2) is also a solution of the synchronized coupled network. When network (3) achieves synchronization, the coupling functions and the control inputs should vanish.

3. Stochastic synchronization of complex dynamical networks

In this section, we consider the stochastic synchronization of coupled complex network (1). In order to control the states of all nodes to the synchronization state $s(t)$, we apply an adaptive controller $u_i(t) \in R^n, i = 1, 2, \ldots, N$, to the network (1), and choose the nonlinear controller $u_i(t)$ as follows:

$$\begin{aligned} u_i(t) = & -\frac{1}{2}Na\gamma(\hat{\theta}_i(t) + \hat{\theta}_i(t-T))(x_i(t) - s(t)) \\ & - Na\gamma\hat{\phi}_i(t)(x_i(t) - s(t)), \end{aligned} \tag{11}$$

where $\gamma = ||\boldsymbol{\Gamma}||, a \geq |a_{ij}|$ are positive constants, N is the node number of network (1), and $\hat{\theta}_i(t), \hat{\phi}_i(t)$ are estimations to $\theta_i(t), \phi_i$ respectively. For convenience, we denote $\tilde{\theta}_i(t) = \theta_i(t) - \hat{\theta}_i(t)$, $\tilde{\phi}_i(t) = \phi_i - \hat{\phi}_i(t)$.

Letting $e_i(t) = x_i(t) - s(t), i = 1, 2, \ldots, N$, we have the following dynamical error equations:

$$\begin{aligned} \mathrm{d}e_i(t) = & \Big[f(x_i(t)) - f(s(t)) \\ & + \sum_{j=1}^{N} \varphi_i(t) a_{ij} \boldsymbol{\Gamma} e_j(t) + u_i(t)\Big]\mathrm{d}t \\ & + [g_i(x_1, \ldots, x_N) - g_i(s, \ldots, s)]\mathrm{d}\omega_i(t), \\ & i = 1, 2, \ldots, N. \end{aligned} \tag{12}$$

Substituting Eq. (11) into Eq. (12), we obtain the following error dynamical network:

$$\begin{aligned} \mathrm{d}e_i(t) = & \Big[f(x_i(t)) - f(s(t)) + \sum_{j=1}^{N} \varphi_i(t) a_{ij} \boldsymbol{\Gamma} e_j(t) \\ & - \frac{1}{2}Na\gamma(\hat{\theta}_i(t) + \hat{\theta}_i(t-T))e_i(t) \\ & - Na\gamma\hat{\phi}_i(t)e_i(t)\Big]\mathrm{d}t \\ & + [g_i(x_1, \ldots, x_N) - g_i(s, \ldots, s)]\mathrm{d}\omega_i(t), \\ & i = 1, 2, \ldots, N. \end{aligned} \tag{13}$$

The time-varying periodic adaptive gain is designed as follows:

$$\hat{\theta}_i(t) = \begin{cases} \hat{\theta}_i(t-T) + \lambda_i(t) e_i^{\mathrm{T}}(t) e_i(t), & t \in [0, +\infty), \\ 0, & t \in [-T, 0), \end{cases} \tag{14}$$

where

$$\lambda_i(t) = \begin{cases} 0, & 0 \leq t < \frac{1}{3}T, \\ \lambda_{i0}(t), & \frac{1}{3}T \leq t \leq \frac{2}{3}T, \\ 1, & \frac{2}{3}T < t, \end{cases}$$

we can choose $\lambda_{i0}(t), 0 \leq \lambda_{i0}(t) \leq 1$, such that $\lambda_i(t)$ is continuous. Therefore, the periodic adaptive gain is continuous.

The time-invariance update law is designed as follows:

$$\dot{\hat{\phi}}_i(t) = r_i e_i^{\mathrm{T}}(t) e_i(t), \tag{15}$$

where r_i is a positive constant.

The following theorem characterizes a sufficient condition for controlled network (3) to be asymptotical synchronized.

Theorem 1 Let Assumptions 1–4 hold. Then the adaptive control mechanism Eqs. (11), (14), and (15) warrant that controlled network (3) is globally asymptotically synchronized in the mean square sense.

Proof Define a Lyapunov–Krasovskii-like CEF as follows:

$$V(t) = V_1(t) + V_2(t) + V_3(t), \tag{16}$$

where

$$\begin{aligned} V_1(t) = & \frac{1}{2}\sum_{i=1}^{N} e_i^{\mathrm{T}}(t) e_i(t), \\ V_2(t) = & \frac{1}{2}Na\gamma\sum_{i=1}^{N}\int_{t-T}^{t} \tilde{\theta}_i^2(\tau)\mathrm{d}\tau, \\ V_3(t) = & \frac{1}{2}Na\gamma\sum_{i=1}^{N} r_i^{-1}[\tilde{\phi}_i(t) + L_i]^2, \end{aligned}$$

in which $t \geq T$ and L_i is a sufficiently large positive constant which will be determined later.

$$\begin{aligned} \dot{V}_1(t) = & \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\dot{e}_i(t) \\ = & \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\Big[f(x_i(t)) - f(s(t)) \\ & + \sum_{j=1}^{N}\varphi_i(t)a_{ij}\boldsymbol{\Gamma}e_j(t) - \frac{1}{2}Na\gamma(\hat{\theta}_i(t) \\ & + \hat{\theta}_i(t-T))e_i(t) - Na\gamma\hat{\phi}_i(t)e_i(t)\Big] \\ \leq & \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\Big[l_i e_i(t) + \sum_{j=1}^{N}\eta_i(t)a_{ij}\boldsymbol{\Gamma}e_j(t) \\ & - \frac{1}{2}Na\gamma(2\hat{\theta}_i(t) - e_i^{\mathrm{T}}(t)e_i(t))e_i(t) \\ & - Na\gamma\hat{\phi}_i(t)e_i(t)\Big] \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{N}\Big[l_i e_i^{\mathrm{T}}(t)e_i(t)+Na^2c\varphi_i(t)e_i^{\mathrm{T}}(t)e_i(t)\\
&\quad -Na\gamma\hat{\phi}_i(t)e_i^{\mathrm{T}}(t)e_i(t)+\frac{N\gamma^2}{4c}\varphi_i(t)e_i^{\mathrm{T}}(t)e_i(t)\\
&\quad -\frac{1}{2}Na\gamma(2\hat{\theta}_i(t)-e_i^{\mathrm{T}}(t)e_i(t))e_i^{\mathrm{T}}(t)e_i(t)\Big]\\
&= \sum_{i=1}^{N}\Big[l_i e_i^{\mathrm{T}}(t)e_i(t)+Na^2c\varphi_i(t)e_i^{\mathrm{T}}(t)e_i(t)\\
&\quad -Na\gamma\hat{\phi}_i(t)e_i^{\mathrm{T}}(t)e_i(t)+\frac{N\gamma^2}{4c}\varphi_i(t)e_i^{\mathrm{T}}(t)e_i(t)\\
&\quad -\frac{1}{2}Na\gamma(2\hat{\theta}_i(t)-e_i^{\mathrm{T}}(t)e_i(t))e_i^{\mathrm{T}}(t)e_i(t)\Big].
\end{aligned}
$$

If we choose $c=\frac{1}{2}a^{-1}\gamma$, we have

$$
\begin{aligned}
\dot{V}_1(t) \leq & \sum_{i=1}^{N}\Big[l_i e_i^{\mathrm{T}}(t)e_i(t)+Na\gamma\varphi_i(t)e_i^{\mathrm{T}}(t)e_i(t)\\
&-Na\gamma\hat{\phi}_i(t)e_i^{\mathrm{T}}(t)e_i(t)\\
&-\frac{1}{2}Na\gamma(2\hat{\theta}_i(t)-e_i^{\mathrm{T}}(t)e_i(t))e_i^{\mathrm{T}}(t)e_i(t)\Big], \qquad (17)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{V}_2(t) &= \frac{1}{2}Na\gamma\sum_{i=1}^{N}[\tilde{\theta}_i^2(t)-\tilde{\theta}_i^2(t-T)]\\
&= -\frac{1}{2}Na\gamma\sum_{i=1}^{N}[e_i^{\mathrm{T}}(t)e_i(t)(2\tilde{\theta}_i(t)\\
&\quad +e_i^{\mathrm{T}}(t)e_i(t))], \qquad (18)\\
\dot{V}_3(t) &= Na\gamma\sum_{i=1}^{N}r_i^{-1}[\tilde{\phi}_i(t)+L_i]\dot{\tilde{\phi}}_i(t)\\
&= -Na\gamma\sum_{i=1}^{N}r_i^{-1}[\tilde{\phi}_i(t)+L_i]r_i e_i^{\mathrm{T}}(t)e_i(t)\\
&= -Na\gamma\sum_{i=1}^{N}[\tilde{\phi}_i(t)+L_i]e_i^{\mathrm{T}}(t)e_i(t). \qquad (19)
\end{aligned}
$$

So, we can obtain

$$
\begin{aligned}
\dot{V}(t) \leq & \sum_{i=1}^{N}\Big[l_i e_i^{\mathrm{T}}(t)e_i(t)+Na\gamma\varphi_i(t)e_i^{\mathrm{T}}(t)e_i(t)\\
&-Na\gamma\hat{\phi}_i(t)e_i^{\mathrm{T}}(t)e_i(t)\\
&-\frac{1}{2}Na\gamma(2\hat{\theta}_i(t)-e_i^{\mathrm{T}}(t)e_i(t))e_i^{\mathrm{T}}(t)e_i(t)\Big]\\
&-\frac{1}{2}Na\gamma\sum_{i=1}^{N}[e_i^{\mathrm{T}}(t)e_i(t)(2\tilde{\theta}_i(t)+e_i^{\mathrm{T}}(t)e_i(t))]\\
&-Na\gamma\sum_{i=1}^{N}[\tilde{\phi}_i(t)+L_i]e_i^{\mathrm{T}}(t)e_i(t)\\
=& \sum_{i=1}^{N}[l_i-Na\gamma L_i]e_i^{\mathrm{T}}(t)e_i(t).
\end{aligned}
$$

It is obvious that there exists a sufficiently large positive constant L_i such that

$$
l_i - Na\gamma L_i < 0. \qquad (20)
$$

We can choose L_i satisfying inequality (20), so we have

$$
\dot{V}(t) \leq \sum_{i=1}^{N}[l_i-Na\gamma L_i]e_i^{\mathrm{T}}(t)e_i(t) < 0. \qquad (21)
$$

By inequality (21), we obtain

$$
\begin{aligned}
V(t) \leq\ & V(T)-\sum_{i=1}^{N}\int_{T}^{t}[l_i-Na\gamma L_i]e_i^{\mathrm{T}}(\tau)e_i(\tau)\,\mathrm{d}\tau,\\
& t\geq T, \qquad (22)
\end{aligned}
$$

where

$$
\begin{aligned}
V(t) =\ & \frac{1}{2}\sum_{i=1}^{N}e_i^{\mathrm{T}}(t)e_i(t)+\frac{1}{2}Na\gamma\sum_{i=1}^{N}\int_{t-T}^{t}\tilde{\theta}_i^2(\tau)\,\mathrm{d}\tau\\
&+\frac{1}{2}Na\gamma\sum_{i=1}^{N}r_i^{-1}[\tilde{\phi}_i(t)+L_i]^2.
\end{aligned}
$$

We can show that $\tilde{\theta}_i(t),\tilde{\phi}_i(t)$ are all bounded in $[T,+\infty)$.

From Assumption 2 and Eq. (11), one obtains

$$
\begin{aligned}
&\mathscr{L}V(t)\\
&= \sum_{i=1}^{N}e_i^{\mathrm{T}}(t)\Big[f(x_i(t))-f(s(t))+\sum_{j=1}^{N}\varphi_i(t)a_{ij}\boldsymbol{\Gamma}e_j(t)\\
&\quad -\frac{1}{2}Na\gamma(\hat{\phi}_i(t)+\hat{\phi}_i(t-T))e_i(t)-Na\gamma\hat{\theta}_i(t)e_i(t)\Big]\\
&\quad +\frac{1}{2}\sum_{j=1}^{N}\mathrm{tr}\{[g_i(x_1,\ldots,x_N)-g_i(s,\ldots,s)]^{\mathrm{T}}\\
&\quad \times[g_i(x_1,\ldots,x_N)-g_i(s,\ldots,s)]\}\\
&\quad -\frac{1}{2}Na\gamma\sum_{i=1}^{N}[e_i^{\mathrm{T}}(t)e_i(t)(2\tilde{\phi}_i(t)+e_i^{\mathrm{T}}(t)e_i(t))]\\
&\quad -Na\gamma\sum_{i=1}^{N}[\tilde{\theta}_i(t)+L_i]e_i^{\mathrm{T}}(t)e_i(t). \qquad (23)
\end{aligned}
$$

Let $\rho=\max_{1\leq i\leq N,1\leq j\leq N}\{\rho_{ij}\}$. From Assumptions 1–4 and inequality (21), one obtains

$$
\begin{aligned}
&\mathscr{L}V(t)\\
&\leq \sum_{i=1}^{N}[l_i-Na\gamma L_i]e_i^{\mathrm{T}}(t)e_i(t)\\
&\quad +\frac{1}{2}\sum_{j=1}^{N}\mathrm{tr}\{[g_i(x_1,\ldots,x_N)-g_i(s,\ldots,s)]^{\mathrm{T}}\\
&\quad \times[g_i(x_1,\ldots,x_N)-g_i(s,\ldots,s)]\}\\
&\leq \sum_{i=1}^{N}[l_i-Na\gamma L_i]e_i^{\mathrm{T}}(t)e_i(t)+\sum_{i=1}^{N}N\rho e_i^{\mathrm{T}}(t)e_i(t)
\end{aligned}
$$

$$= \sum_{i=1}^{N}[l_i + N\rho - Na\gamma L_i]e_i^{\mathrm{T}}(t)e_i(t).$$

It is obvious that there exists a sufficiently large positive constant L_i such that

$$l_i + N\rho - Na\gamma L_i < 0. \tag{24}$$

We can choose L_i satisfying inequality (24), and choose $k = \min_{1\leq i\leq N}(Na\gamma L_i - l_i - N\rho)$, so we have

$$\mathscr{L}V(t) \leq -k\sum_{i=1}^{N}||e_i(t)||^2. \tag{25}$$

From inequality (25) and Itô's formula, it is obvious that

$$EV(t) - EV(t_0) = E\int_{t_0}^{t}\mathscr{L}V(\tau)\mathrm{d}\tau. \tag{26}$$

From the definition of $V(t)$ in Eq. (16), there exists a positive constant m such that

$$\begin{aligned} &Em\sum_{i=1}^{N}||e_i(t)||^2 \\ &\leq EV(t) = EV(t_0) + E\int_{t_0}^{t}\mathscr{L}V(\tau)\mathrm{d}\tau \leq EV(t_0) \\ &\quad - E\int_{t_0}^{t}k\sum_{i=1}^{N}||e_i(\tau)||^2\mathrm{d}\tau. \end{aligned} \tag{27}$$

From Lemma 2, it is easy to see that dynamical system (13) is globally asymptotically stable in the mean square. Thus, network (1) achieves asymptotic synchronization in the mean square sense.

Remark 2 We can choose a sufficiently large positive constant L_i such that inequalities (20) and (24) hold. To the best of our knowledge, there are very few works about coupled stochastic networks investigated in the literature. Our designed controller (11) only depends on the state $x_i(t)$ and $s(t)$, known as state feedback, but they can effectively control stochastic networks as discussed above. We can choose some other Lyapunov functional candidates rather than those in Eq. (16) to achieve synchronization. However, under the simple adaptive laws in Theorem 1 and without satisfying any additional conditions, synchronization can be reached.

4. Simulation examples

To illustrate the above results, we consider complex networks with vector-form stochastic perturbations, in which each node is a Lorenz system. The dynamics of a single Lorenz system is described as

$$\begin{cases} \dot{x}_1 = -px_1 + x_2x_3, \\ \dot{x}_2 = q(x_3 - x_2), \\ \dot{x}_3 = -x_1x_2 + rx_2 - x_3, \end{cases} \tag{28}$$

where p, q, r are parameters. When $p = \frac{8}{3}$, $q = 10$, $r = 28$, system (28) has chaotic behaviour. Figure 1 depicts the chaotic trajectory of system (28) with initial value $s(0) = (1, -1, 2)^{\mathrm{T}}$.

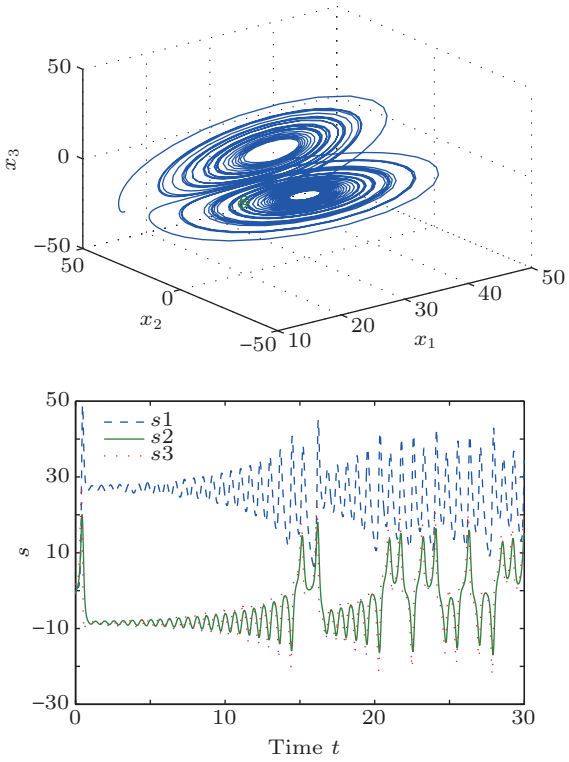


Fig. 1. (colour online) Trajectory of a single node in the Lorenz system with initial conditions $(1, -1, 2)^{\mathrm{T}}$.

Consider the following stochastic complex dynamical network model:

$$\begin{aligned} \mathrm{d}x_i(t) = &\Big[f(x_i(t)) + \sum_{j=1}^{N}\varphi_i(t)a_{ij}\boldsymbol{\Gamma}x_j(t) + u_i(t)\Big]\mathrm{d}t \\ &+ g_i(x_1, x_2, \ldots, x_N)\mathrm{d}\omega_i(t), \\ &i = 1, 2, \ldots, N, \end{aligned} \tag{29}$$

where

$$x_i(t) = \begin{pmatrix} x_{i1}(t) \\ x_{i2}(t) \\ x_{i3}(t) \end{pmatrix},$$

$$f(x_i(t)) = \begin{pmatrix} -px_{i1} + x_{i2}x_{i3} \\ q(x_{i3} - x_{i2}) \\ -x_{i1}x_{i2} + rx_{i2} - x_{i3} \end{pmatrix},$$

$$g_i(x_1, x_2, \ldots, x_N) = \begin{pmatrix} x_{i1}(t) & 0 & 0 \\ 0 & x_{i2}(t) & 0 \\ 0 & 0 & x_{i3}(t) \end{pmatrix},$$

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 2 & 0 \\ 0 & 0 & 2 & -4 & 2 \\ 0 & 0 & 0 & 2 & -2 \end{pmatrix},$$

$$\boldsymbol{\Gamma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The parameters are selected as follows: $\phi(t) = (0.2\sin(2\pi t/3),\ 2\cos \pi t,\ -\sin(2\pi t/3),\ -0.5\cos(2\pi t/3),\ \sin(\pi t/3))^{\mathrm{T}}$, $T = 6$, $\theta = (2,\ 10,\ 8,\ 5,\ 3)^{\mathrm{T}}$. According to Theorem 1 in Section 3, one can synchronize each other by designing the following controller

$$u_i(t) = -\frac{1}{2}Na\gamma(\hat{\theta}_i(t) + \hat{\theta}_i(t-T))(x_i(t) - s(t)) - Na\gamma\hat{\phi}_i(t)(x_i(t) - s(t)), \tag{30}$$

where $\hat{\theta}_i(t), \hat{\phi}_i(t)$ are chosen as in Eqs. (14) and (15).

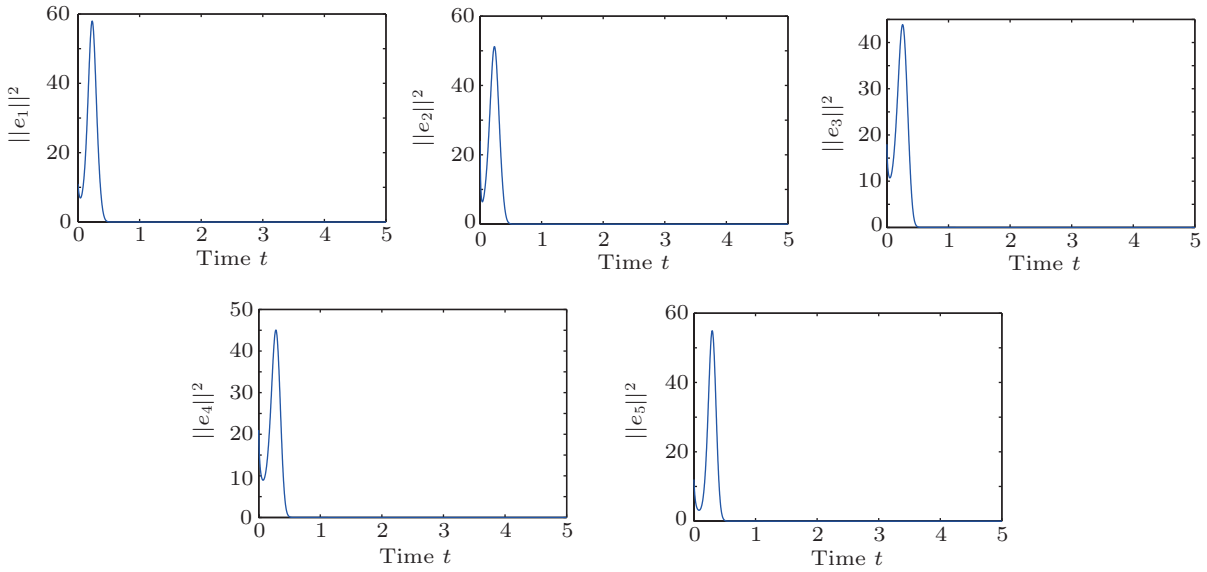


Fig. 2. (colour online) Time responses of the errors $e_i(t)$ under control (24) with $i = 1, 2, 3, 4, 5$.

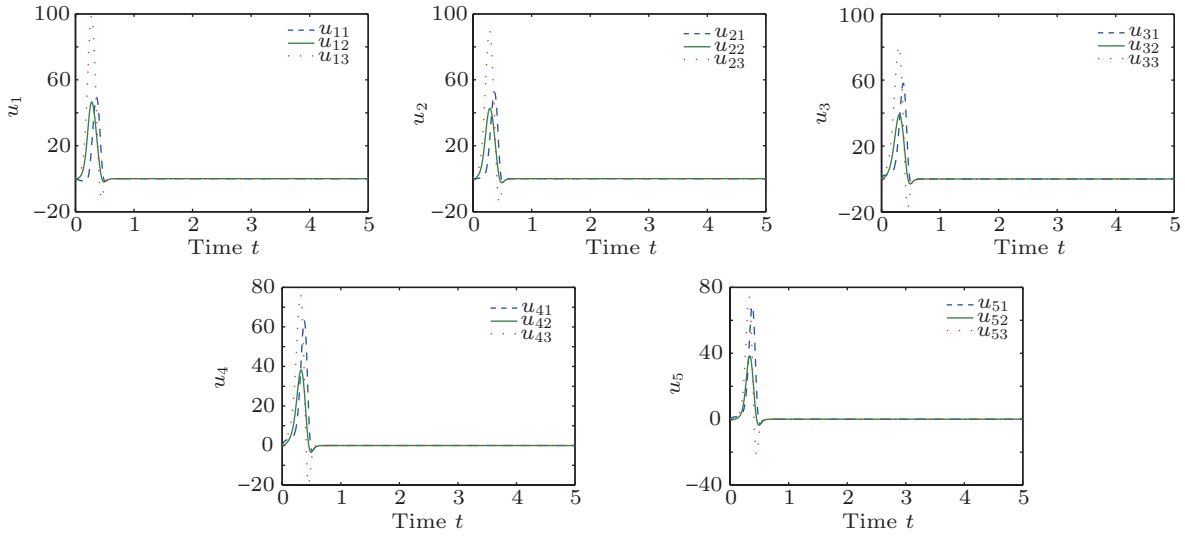


Fig. 3. (colour online) The time evolution of the controller $u_i(t)$ with $i = 1, 2, 3, 4, 5$.

In the following simulations, we choose $N = 5$, $r_i = 0.02$, $\lambda_{i0}(t) = \frac{1}{2}t - 1$, the initially estimated value of the unknown parameter $\phi(t)$ is $\hat{\phi}(0) = (0,0,0,0,0)^{\mathrm{T}}$, the initial states are chosen as $x_1(0) = (2,0,-1)^{\mathrm{T}}$, $x_2(0) = (3,1,-2)^{\mathrm{T}}$, $x_3(0) = (-3,-2,1)^{\mathrm{T}}$, $x_4(0) = (-1,0,-2)^{\mathrm{T}}$, $x_5(0) = (-1,1,0)^{\mathrm{T}}$, respectively. According to Theorem 1, the weighted network (29) can be synchronized by applying the adaptive controllers (11), (14), and (15). Figure 2 shows the error evolutions under the designed controllers. We clearly see that the states of network (29) asymptotically synchronize with the states of the desired orbit (28). Furthermore, figure 3 depicts the time evolution of the controllers.

5. Conclusion

In this paper, we studied the stochastic synchronization problem for time-varying complex dynamical networks. This model is totally different from some existing network models. Some controllers and adaptive laws were designed to ensure achieving stochastic synchronization of the proposed network model. A sufficient synchronization condition was given to ensure that the proposed network model is mean-square synchronized. An example was simulated for complex dynamical networks, which demonstrates the feasibility and effectiveness of the proposed adaptive control method.

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