Number-phase quantization of a mesoscopic RLC circuit

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With the help of the time-dependent Lagrangian for a damped harmonic oscillator, the quantization of mesoscopic RLC circuit in the context of a number-phase quantization scheme is realized and the corresponding Hamiltonian operator is obtained. Then the evolution of the charge number and phase difference across the capacity are obtained. It is shown that the number-phase analysis is useful to tackle the quantization of some mesoscopic circuits and dynamical equations of the corresponding operators.

Keywords: mesoscopic RLC circuit, number-phase quantization, Lagrangian function

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1. Introduction

In recent years, research work on mesoscopic circuits has attracted great interest. In 1973, Louisell first quantized a mesoscopic LC circuit.[1] In his scheme, electric charge q is quantized as the coordinate operator Q and electric current I multiplied by L is quantized as the momentum operator P, and then the LC circuit is quantized as a quantum harmonic oscillator. The equation of motion for an RLC circuit with a time-dependent power source has been quantized[2] and the quantum fluctuations of charge and current in the vacuum state can be obtained when the circuit has no power. After that, much work had been done on the complicated electric circuit quantization.[3−10]

Instead of the coordinate–momentum quantum variable,[1] references [11]–[13] adopted a number-phase quantization scheme to construct the Hamiltonian operator for an LC circuit and two LC circuits with mutual-inductance, where electric charge takes discrete values. In this paper, we are motivated to adopt this new quantization scheme to further study the RLC circuit. This motivation also arises from a previous number-phase quantization scheme for a Josephson junction.[14,15] According to Feynman,[16] the Josephson current crossing an insulating barrier is caused by the phase difference between two superconductors. Vourdas[17] and Fan[18] considered this phase difference as an operator. Therefore, we expect that quantization of an RLC circuit can be recast into the formalism based on a number-phase commutative relation. With the help of the time-dependent Lagrangian and using the number-phase quantization, we obtain the dynamical equations of charge number and phase difference crossing the inductance and capacity of the RLC circuit, and then derive their evolution equations with time.

2. Number-phase quantization of a mesoscopic RLC circuit

Now, we first adopt a number-phase quantization scheme to construct the Hamiltonian operator for RLC circuits. Considering the discreteness of electric charge, the electric charge is written as $q = en$, where n is the number of electrons across the inductance L. The equation of motion for an RLC circuit with a power source is

$$L\frac{\mathrm{d}^2(en)}{\mathrm{d}t^2} + R\frac{\mathrm{d}(en)}{\mathrm{d}t} + \frac{en}{C} = \varepsilon(t), \tag{1}$$

where L, R, and C stand for the inductance, resistance, and capacity respectively, $\varepsilon(t)$ represents the power source. It shows that the RLC circuit can be regarded as a damped harmonic oscillator. For the damped harmonic oscillator, a time-dependent Lagrangian has been constructed[19,20]

$$\mathscr{L} = \left[\frac{1}{2}Le^2\dot{n}^2 - (2C)^{-1}e^2n^2 + en\varepsilon(t)\right] \times \exp\left(\frac{Rt}{L}\right). \tag{2}$$

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Therefore, Eq. (1) can be obtained by the following Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathscr{L}}{\partial \dot{n}}\right)-\frac{\partial \mathscr{L}}{\partial n}=0. \tag{3}$$

When the circuit has no power source, i.e. $\varepsilon(t)=0$, from Eq. (2) the Lagrange function of the circuit system is just[21]

$$\mathscr{L}=\left[\frac{1}{2}Le^2\dot{n}^2-(2C)^{-1}e^2n^2\right]\exp\left(\frac{Rt}{L}\right). \tag{4}$$

In the number-phase quantization scheme, n is regarded as canonical coordinate. Correspondingly, the canonical momentum conjugated to n is

$$p=\frac{\partial \mathscr{L}}{\partial \dot{n}}=Le^2\dot{n}\exp\left(\frac{Rt}{L}\right). \tag{5}$$

Since

$$e\dot{n}=\frac{\Phi}{L}, \tag{6}$$

where Φ is the magnetic flux caused by self-inductance, then substituting Eq. (6) into Eq. (5) yields

$$p=e\Phi\exp\left(\frac{Rt}{L}\right). \tag{7}$$

Thus we naturally obtain the relation between the canonical momentum p and the total magnetic flux Φ. Now, by the Faraday theorem, the total voltage caused by inductance is

$$U=-\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tag{8}$$

which is equal to the sum of the voltage across the capacitor and the resistance. That is, the total voltage is

$$U=U_C+U_R, \tag{9}$$

where $U_R=Re\dot{n}$, the voltage of the resistance. The voltage U_C, from the quantum mechanical wave function's point, is related to the phase difference between two plates of the capacitor within a time interval $\mathrm{d}t$, i.e., assuming the wave functions on the first plate, the second plate of the capacitor is

$$\begin{aligned}\psi_i &= \phi_i\exp\left(\frac{\mathrm{i}E_it}{\hbar}\right)\\ &= \phi_i\exp\left(\mathrm{i}\omega_it\right)\\ &= \phi_i\exp\left(\mathrm{i}\theta_i\right),\ i=1,2.\end{aligned} \tag{10}$$

Noting that

$$\mathrm{d}\theta_{\mathrm{i}}=\frac{E_i}{\hbar}\mathrm{d}t,\ \mathrm{d}\theta=\mathrm{d}\theta_2-\mathrm{d}\theta_1=\frac{E_2-E_1}{\hbar}\mathrm{d}t. \tag{11}$$

Then the energy gap between two plates in the capacitor is

$$E_1-E_2=eU_C=\hbar\left(\mathrm{d}\theta_1-\mathrm{d}\theta_2\right)/\mathrm{d}t=-\hbar\,\mathrm{d}\theta/\mathrm{d}t. \tag{12}$$

So we can obtain from Eqs. (8)–(12)

$$U=-\frac{\hbar\,\mathrm{d}\theta}{e\mathrm{d}t}+Re\frac{\mathrm{d}n}{\mathrm{d}t}. \tag{13}$$

After integration, we can obtain from Eqs. (8) and (13)

$$\Phi=\frac{\hbar\,\theta}{e}-Ren. \tag{14}$$

Substituting Eq. (14) into Eq. (7), the canonical momentum can be rewritten as

$$p=\left(\hbar\,\theta-Re^2n\right)\exp\left(\frac{Rt}{L}\right). \tag{15}$$

This equation indicates that $\left(\hbar\,\theta-Re^2n\right)\exp\left(Rt/L\right)$ is quantized as the canonical momentum, while the number n is quantized as the canonical coordinate. Therefore the canonical quantization condition is

$$\begin{aligned}&\left[\hat{n},\left(\hbar\,\hat{\theta}-Re^2\hat{n}\right)\right]=\mathrm{i}\hbar\exp\left(-\frac{Rt}{L}\right),\\ &\text{or}\quad\left[\hat{n},\hat{\theta}\right]=\mathrm{i}\exp\left(-\frac{Rt}{L}\right).\end{aligned} \tag{16}$$

Then the uncertain relation is just $\Delta n\Delta\theta\geq\exp\left(-Rt/L\right)/2$, which means quantum fluctuation. When $R=0$, we can obtain the same result as that in Ref. [11], which is also consistent with the result of Ref. [12] theoretically.

Now based on the operators $\hat{n}$, $\hat{\theta}$ and with the help of the time-dependent Lagrangian, the Hamiltonian operator of this circuit system can be written as

$$\begin{aligned}h &= \dot{n}p-\mathscr{L}\rightarrow\hat{H}\\ &= \frac{1}{2Le^2}\hat{p}^2\exp\left(-\frac{Rt}{L}\right)+\frac{1}{2C}e^2\hat{n}^2\exp\left(\frac{Rt}{L}\right)\\ &= \left[\frac{1}{2Le^2}\left(\hbar\,\hat{\theta}-Re^2\hat{n}\right)^2+\frac{1}{2C}e^2\hat{n}^2\right]\exp\left(\frac{Rt}{L}\right),\end{aligned} \tag{17}$$

which is consistent with that in Ref. [22]. We introduce the following time-dependent annihilator and creator operators

$$\begin{aligned}\hat{a} &= \sqrt{\frac{e^2L\omega_0}{2\hbar}}\left[\left(1-\mathrm{i}\frac{R}{L\omega_0}\right)\hat{n}+\mathrm{i}\frac{\hbar}{e^2L\omega_0}\hat{\theta}\right]\\ &\quad\times\exp\left(Rt/2L\right),\end{aligned} \tag{18}$$

$$\begin{aligned}\hat{a}^+ &= \sqrt{\frac{e^2L\omega_0}{2\hbar}}\left[\left(1+\mathrm{i}\frac{R}{L\omega_0}\right)\hat{n}-\mathrm{i}\frac{\hbar}{e^2L\omega_0}\hat{\theta}\right]\\ &\quad\times\exp\left(Rt/2L\right),\end{aligned} \tag{19}$$

where $\omega_0 = \sqrt{1/LC}$ is the resonant frequency of an LC circuit in the absence of the resistance. Then the Hamiltonian of the circuit system can be rewritten in the simple form

$$\hat{H} = (\hat{a}^+\hat{a} + \frac{1}{2})\hbar\omega_0. \tag{20}$$

So with the help of the time-dependent Lagrangian for a damped harmonic oscillator and using the number-phase quantization, the RLC circuit can be quantized as a quantum harmonic oscillator too.

3. Evolutions of charge number and phase difference

In quantum mechanics the Heisenberg equation is written as

$$\frac{\mathrm{d}\hat{A}(t)}{\mathrm{d}t} = \frac{1}{\mathrm{i}\hbar}\left[\hat{A}(t), \hat{H}\right]. \tag{21}$$

Substituting Eq. (17) into Eq. (21), we obtain the following equations

$$\frac{\mathrm{d}\hat{n}}{\mathrm{d}t} = \frac{\left(\hbar\hat{\theta} - Re^2\hat{n}\right)}{Le^2}, \tag{22}$$

$$\hbar\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} = \frac{R\hbar}{L}\hat{\theta} - \left(\frac{1}{2L}Re^2 + \frac{e^2}{C}\right)\hat{n}, \tag{23}$$

which represent the current equation and Faraday theorem, respectively. From Eqs. (22) and (23), we can obtain

$$\frac{\mathrm{d}^2\hat{n}}{\mathrm{d}t^2} = -\left(\frac{1}{LC} - \frac{R^2}{2L^2}\right)\hat{n}, \tag{24}$$

$$\frac{\mathrm{d}^2\hat{\theta}}{\mathrm{d}t^2} = -\left(\frac{1}{LC} - \frac{R^2}{2L^2}\right)\hat{\theta}. \tag{25}$$

Then we derive the evolution of charge number by Eq. (24)

$$\hat{n} = \hat{n}(t=0)\exp(\mathrm{i}\omega t), \tag{26}$$

where $\omega = \sqrt{1/(LC) - R^2/(2L^2)}$ is the resonant frequency of the RLC circuit. This equation indicates the behaviour of the electromagnetically damped oscillation of the RLC circuit. On the other hand, the evolution of phase difference on capacity can be derived from Eq. (25)

$$\hat{\theta} = \hat{\theta}(t=0)\exp(\mathrm{i}\omega t). \tag{27}$$

When $R = 0$, we obtain the same result as that of Ref. [11], which is also consistent with the result of Ref. [12] theoretically. Therefore, quantization of the RLC mesoscopic circuits can also be explained in the context of number-phase quantization method, in which the canonical observables are the electron number and the phase difference across the capacitor.

4. Conclusion

In summary, with the help of the time-dependent Lagrangian, we have quantized the RLC circuit by means of the new number-phase quantization scheme, which is a complementary view of coordinate–momentum quantization scheme. Then we have derived the evolution of the charge number and phase difference, separately. It is shown that number-phase analysis is useful for dealing with the quantization of some mesoscopic circuits and dynamical equations of the corresponding operators.

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